Modular symbols

Bachelor's thesis defence

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A historical perspective

A modular form? What's that when it's at home?

- Riemann noticed that the function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (\mathrm{Re}(s) > 1)$$

carries information about the distribution of primes.

- This function can be given (formally) by

$$2\pi^{-s/2}\Gamma(s/2)\zeta(s) = \int_0^{\infty} \theta(it) t^{s/2} \frac{dt}{t}, \quad \text{where } \theta(t) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 t}.$$

- By studying $\theta(t)$, Riemann obtained the functional equation

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

and constructed an analytic continuation of $\zeta(s)$.

First definitions

To modular forms...

- The Poincaré upper half-plane is

$$\mathbb{H} = \{ z \in \mathbb{C} : \mathrm{Im}(z) > 0 \}.$$

- It is useful to extend it adding the cusps:

$$\mathbb{H}^* = \mathbb{H} \cup \mathbb{P}^1_{\mathbb{Q}}, \quad \text{where } \mathbb{P}^1_{\mathbb{Q}} = \mathbb{Q} \cup \{ \infty \}.$$

- Fix $N \in \mathbb{N}$ and consider the group of matrices

$$\Gamma_0(N) = \left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N} \right\}.$$

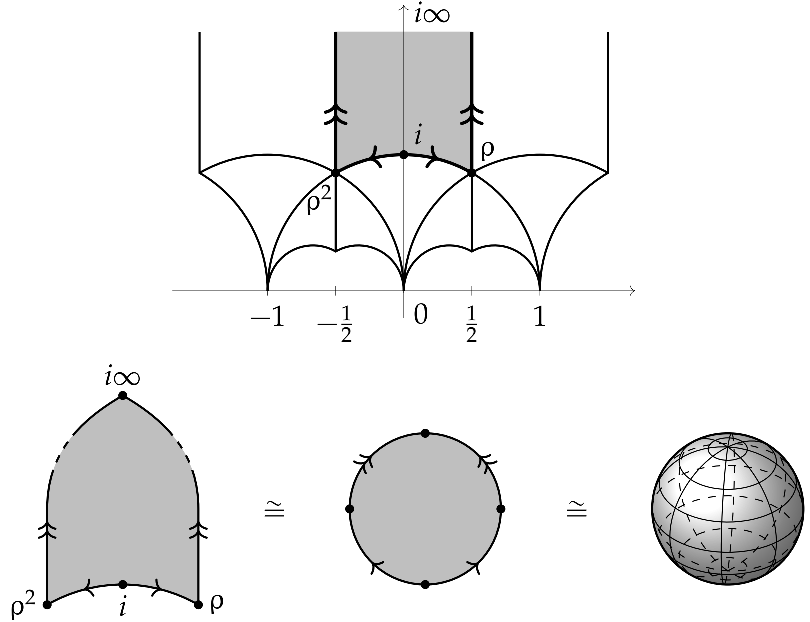
First definitions

... and beyond!

- $\mathrm{GL}_2^+(\mathbb{Q})$ acts on $\mathbb{H}^*$ by linear fractional transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

- The modular curve $X_0(N)$ is the orbit space $\Gamma_0(N) \backslash \mathbb{H}^*$.
- $X_0(N)$ admits a natural structure of Riemann surface.
- Let $\pi \colon \mathbb{H}^* \longrightarrow\!\!\!\!\rightarrow \Gamma_0(N) \backslash \mathbb{H}^*$ be the quotient map.
- We also write $C_0(N) = \pi(\mathbb{P}^1_{\mathbb{Q}})$.



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Modular forms

The object of desire

- If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $d\gamma(z) = \dfrac{\det(\gamma)}{(cz+d)^2}\, dz$.
- A modular form is a holomorphic function $f\colon \mathbb{H}^* \longrightarrow \mathbb{C}$ satisfying that

$$f(\gamma(z))\, d\gamma(z) = f(z)\, dz \qquad \text{for all } \gamma \in \Gamma_0(N)\,.$$

So, if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, $f(z) = (cz+d)^{-2} f\left(\dfrac{az+b}{cz+d}\right)$.

- If, moreover, $f(z) = 0$ for all $z \in \mathbb{P}^1_{\mathbb{Q}}$, f is called a cusp form.
- We write $M_2(\Gamma_0(N))$ for the space of modular forms and $S_2(\Gamma_0(N))$ for the space of cusp forms.

Fourier expansions

The way to represent modular forms

- Let f be a modular form.
- Since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_0(N)$, $f(z+1) = f(z)$.
- The q–expansion of f at infinity is the Fourier series

$$f(z) = \sum_{n=0}^{\infty} a_n(f) q^n \,, \qquad \text{where } q = q(z) = e^{2\pi i z} \,.$$

- Similarly, one defines Fourier expansions at the other cusps after a translation by an element of $\mathrm{SL}_2(\mathbb{Z})$.

- The change of variables $q = e^{2\pi i z}$ is essentially the chart of $X_0(N)$ at ∞.
- Observe that

$$dq = 2\pi i e^{2\pi i z}\, dz \quad \text{or, equivalently,} \quad dz = \frac{1}{2\pi i}\frac{dq}{q}\,.$$

- Therefore, there is a correspondence between cusp forms and holomorphic differentials on $X_0(N)$: we can identify $S_2(\Gamma_0(N))$ with $\Omega^1(X_0(N))$.

- We define a family of operators $T(p) \colon S_2(\Gamma_0(N)) \longrightarrow S_2(\Gamma_0(N))$ for every prime p:

$$T(p)f(z) = p\chi_p(N) f\left(\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} z \right) + p^{-1} \sum_{b=0}^{p-1} f\left(\begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix} z \right)$$

$$= p\chi_p(N) f(pz) + p^{-1} \sum_{b=0}^{p-1} f\left(\frac{z+b}{p} \right),$$

where

$$\chi_p(N) = \begin{cases} 0 & \text{if } p \mid N, \\ 1 & \text{if } p \nmid N. \end{cases}$$

- One can define $T(n)$ for all $n \in \mathbb{N}$ so that
  - $T(m)T(n) = T(mn)$ if $(m,n) = 1$, and
  - $T(p)T(p^n) = T(p^{n+1}) + p\chi_p(N)\,T(p^{n-1})$ for all prime p.
- In particular, the Hecke operators commute.
- In addition, there is a scalar product $\langle\,\cdot\,,\,\cdot\,\rangle$ such that $\langle T(n)f, g\rangle = \langle f, T(n)\,g\rangle$ for all $f, g \in S_2(\Gamma_0(N))$ if $(n,N) = 1$.
- Consequently, there exists a basis of eigenforms.

Lemma

$a_1(T(n)f) = a_n(f)$.

- Define the Hecke algebra $\mathbb{T} = \mathbb{Z}[T(n) : n \in \mathbb{N}]$ and $\mathbb{T}_{\mathbb{C}} = \mathbb{T} \otimes_{\mathbb{Z}} \mathbb{C}$.
- There is a perfect pairing

$$\begin{aligned} S_2(\Gamma_0(N)) \times \mathbb{T}_{\mathbb{C}} &\longrightarrow \mathbb{C} \\ (f, T) &\longmapsto a_1(Tf) \end{aligned}$$

which induces an isomorphism $S_2(\Gamma_0(N)) \cong \mathrm{Hom}_{\mathbb{C}}(\mathbb{T}_{\mathbb{C}}, \mathbb{C})$.

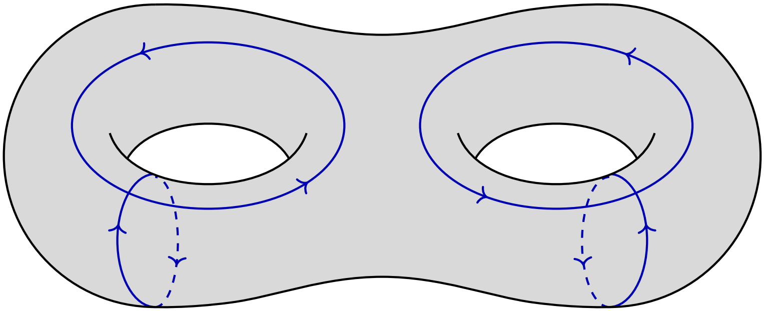
Another duality

The first homology group comes to the rescue

- We need another description of the Hecke operators.
- The theory of compact Riemann surfaces provides a pairing

$$\langle\,\cdot\,,\,\cdot\,\rangle\colon H_1(X_0(N),\mathbb{Z})\times\Omega^1(X_0(N))\longrightarrow\mathbb{C}$$

  defined by $\langle[\sigma],\omega\rangle=\int_\sigma\omega$.
- If g is the genus of $X_0(N)$, then $\dim_{\mathbb{C}}(\Omega^1(X_0(N)))=g$ and $\operatorname{rank}(H_1(X_0(N),\mathbb{Z}))=2g$.
- $\langle\,\cdot\,,\,\cdot\,\rangle$ induces an isomorphism $H_1(X_0(N),\mathbb{R})\cong\operatorname{Hom}_{\mathbb{C}}(\Omega^1(X_0(N)),\mathbb{C})$ as $\mathbb{R}$–vector spaces.



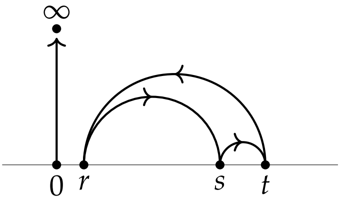
Modular symbols

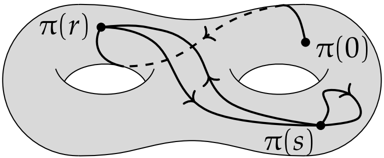
At long last!

- The space $\mathbb{M}$ of (formal) modular symbols is the free abelian group generated by the formal symbols $\{r, s\}$ for $r, s \in \mathbb{P}^1_{\mathbb{Q}}$ modulo the relations

$$\{r, s\} + \{s, t\} + \{t, r\} = 0$$

  and modulo any torsion.
- We think of (formal) modular symbols as paths on $\mathbb{H}^*$.





Modular symbols

The key to describe homology

- $\mathrm{GL}_2^+(\mathbb{Q})$ acts on $\mathbb{M}$:

$$\alpha\{r,s\} = \{\alpha(r), \alpha(s)\}.$$

- The group $\mathbb{M}(\Gamma_0(N))$ of modular symbols for $\Gamma_0(N)$ is $\mathbb{M}$ modulo the relations $\{r,s\} = \gamma\{r,s\}$ if $\gamma \in \Gamma_0(N)$ and modulo any torsion.
- There is a natural morphism

$$\phi\colon \mathbb{M}(\Gamma_0(N)) \longrightarrow H_1(X_0(N), C_0(N), \mathbb{Z})$$

  mapping $\{r,s\}$ to the image under π of a path from r to s.
- $\{r,s\}$ corresponds to

$$\left(\omega \longmapsto \int_r^s \pi^*(\omega) \right) \in \mathrm{Hom}_{\mathbb{C}}(\Omega^1(X_0(N)), \mathbb{C}) \cong H_1(X_0(N), \mathbb{R}).$$

Hecke operators

The return

- We define a family of operators $T(p)\colon \mathbb{M}(\Gamma_0(N)) \longrightarrow \mathbb{M}(\Gamma_0(N))$ for every prime p:

$$T(p)\{r,s\} = \chi_p(N)\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\{r,s\} + \sum_{b=0}^{p-1}\begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix}\{r,s\}.$$

- One can define $T(n)$ for all $n \in \mathbb{N}$ analogously.
- Therefore, by definition,

$$\langle T(n)\{r,s\}, f\rangle = \langle \{r,s\}, T(n)f\rangle .$$

Manin's trick

An important result with a funny name

Theorem (Manin's trick)

Every $\{r,s\} \in \mathbb{M}(\Gamma_0(N))$ *can be expressed as a* $\mathbb{Z}$*–linear combination of elements of the form* $\alpha\{0,\infty\}$ *with* $\alpha \in \mathrm{SL}_2(\mathbb{Z})$.

Idea of the proof.

- Write $\{r,s\} = \{0,s\} - \{0,r\}$.
- Expand $s = \frac{p}{q}$ as a continued fraction

$$\frac{p}{q} = x_0 + \cfrac{1}{x_1 + \cfrac{1}{\ddots + \cfrac{1}{x_n}}}.$$

Manin's trick

An important result with a funny name

Theorem (Manin's trick)

Every $\{r, s\} \in \mathbb{M}(\Gamma_0(N))$ *can be expressed as a* $\mathbb{Z}$*–linear combination of elements of the form* $\alpha\{0, \infty\}$ *with* $\alpha \in \mathrm{SL}_2(\mathbb{Z})$.

Idea of the proof.

- Consider the successive convergents

$$\frac{p_{-2}}{q_{-2}} = \frac{0}{1}, \quad \frac{p_{-1}}{q_{-1}} = \frac{1}{0}, \quad \frac{p_0}{q_0} = \frac{x_0}{1}, \quad \dots, \quad \frac{p_n}{q_n} = \frac{p}{q}.$$

- We can write

$$\left\{0, \frac{p}{q}\right\} = \sum_{k=-1}^{n} \left\{\frac{p_{k-1}}{q_{k-1}}, \frac{p_k}{q_k}\right\} = \sum_{k=-1}^{n} \begin{pmatrix} (-1)^{k-1} p_k & p_{k-1} \\ (-1)^{k-1} q_k & q_{k-1} \end{pmatrix} \{0, \infty\}. \qquad \square$$

Manin symbols

Another way to represent modular symbols

- The image of the map

$$\begin{array}{ccc} \Gamma_0(N) \backslash \mathrm{SL}_2(\mathbb{Z}) & \longrightarrow & \mathbb{M}(\Gamma_0(N)) \\ \Gamma_0(N)\begin{pmatrix} a & b \\ c & d \end{pmatrix} & \longmapsto & \begin{pmatrix} a & b \\ c & d \end{pmatrix}\{0, \infty\} \end{array}$$

  generates $\mathbb{M}(\Gamma_0(N))$ (as a $\mathbb{Z}$–module).

- There is a bijection between $\Gamma_0(N) \backslash \mathrm{SL}_2(\mathbb{Z})$ and $\mathbb{P}^1_{\mathbb{Z}/N\mathbb{Z}}$ given by

$$\Gamma_0(N)\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto (c : d)\,.$$

- The space $\mathrm{Man}(\Gamma_0(N))$ is the free abelian group generated by $\mathbb{P}^1_{\mathbb{Z}/N\mathbb{Z}}$.

Manin symbols

Towards the main result

- The matrices $\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ generate $\mathrm{SL}_2(\mathbb{Z})$.
- For all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$, we have that $\alpha\{0,\infty\} + \alpha\sigma\{0,\infty\} = 0$ and $\alpha\{0,\infty\} + \alpha\tau\{0,\infty\} + \alpha\tau^2\{0,\infty\} = 0$.
- The group $C(\mathrm{Man}(\Gamma_0(N)))$ of chains is $\mathrm{Man}(\Gamma_0(N))$ modulo the relations

  $$(c:d) + (-d:c) = 0$$

  and modulo any torsion.
- The group $B(\mathrm{Man}(\Gamma_0(N)))$ of boundaries is the subgroup of $C(\mathrm{Man}(\Gamma_0(N)))$ generated by the elements

  $$(c:d) + (c+d:-c) + (d:-c-d)$$

  and by the elements $(c:d)$ such that $(c:d) = (c+d:-c)$.

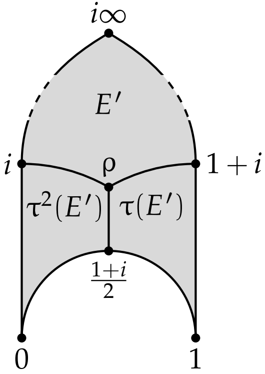
The main result

Modular symbols are what we hoped

Theorem (Manin)

$\mathbb{M}(\Gamma_0(N)) \cong C(\mathrm{Man}(\Gamma_0(N))) \,/\, B(\mathrm{Man}(\Gamma_0(N))) \cong H_1(X_0(N), C_0(N), \mathbb{Z})$.

- The proof is a rather involved topological argument.
- A triangulation of $X_0(N)$ is used.



An exact sequence

Identifying the first homology group

- The long exact sequence of relative homology induces a commutative diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathbb{S}(\Gamma_0(N)) & \longrightarrow & \mathbb{M}(\Gamma_0(N)) & \xrightarrow{\delta} & \mathbb{B}(\Gamma_0(N)) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\ \| & & \downarrow\wr\| & & \phi\downarrow\wr\| & & \updownarrow\wr\| & & \| & & \| \\ 0 & \longrightarrow & H_1(X_0(N),\mathbb{Z}) & \longrightarrow & H_1(X_0(N),C_0(N),\mathbb{Z}) & \xrightarrow{\Delta} & \mathbb{Z}^{|C_0(N)|} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \end{array}$$

  with exact rows.

- The space $\mathbb{S}(\Gamma_0(N))$ of cuspidal modular symbols for $\Gamma_0(N)$ is the kernel of the boundary map δ, defined by

$$\delta(\{r,s\}) = \{s\} - \{r\} \quad \text{(as a divisor of } X_0(N)).$$

An example

Making sense of everything

- Take $N = 23$.
- $\mathbb{M}(\Gamma_0(23)) = \langle (1:0), (1:17), (1:19), (1:20), (1:21) \rangle$ (this is a basis).
- $\delta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}$.
- $\mathbb{S}(\Gamma_0(23)) = \langle (1:17), (1:19), (1:20), (1:21) \rangle$ (this is a basis).
- $\dim_{\mathbb{C}} S_2(\Gamma_0(23)) = 2$.

An example

What do the Hecke operators look like?

- The Hecke operators are represented as 4×4 matrices.
- $T(2) = \begin{pmatrix} 0 & 0 & -1 & -1 \\ 1 & 1 & 2 & 1 \\ -1 & -1 & -2 & 0 \\ 0 & 1 & 1 & -1 \end{pmatrix}.$
- $T(3) = \begin{pmatrix} -1 & 0 & 2 & 2 \\ -2 & -3 & -4 & -2 \\ 2 & 2 & 3 & 0 \\ 0 & -2 & -2 & 1 \end{pmatrix}.$

An example

A basis of $S_2(23)$

- The maps $a_{ij}(\,\cdot\,)$ (for $1 \leqslant i, j \leqslant 4$) generate $\mathrm{Hom}_{\mathbb{C}}(\mathbb{T}_{\mathbb{C}}, \mathbb{C})$.
- Consequently, the cusp forms

$$f_{ij}(z) = \sum_{n=1}^{\infty} a_{ij}(T(n))q^n\,, \qquad \text{where } q = e^{2\pi i z}\,,$$

  generate $S_2(\Gamma_0(N))$.
- $f_{11}(z) = q - q^3 - q^4 + O(q^6)$.
- $f_{12}(z) = O(q^6)$.
- $f_{21}(z) = q^2 - 2q^3 - q^4 + 2q^5 + O(q^6)$.

Final comments

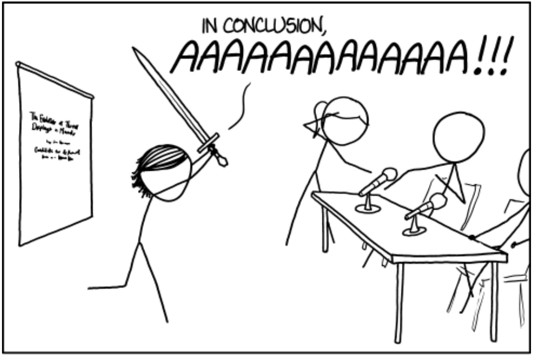
So what? Why should I care about modular forms?

- A cusp form f has an associated Dirichlet series

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a_n(f)}{n^s}\,.$$

- Series of this kind are very common in number theory. For instance, in the theories of elliptic curves and Galois representations.
- $L(s,f)$ has many "nice" properties: a functional equation, analytic continuation…

Thank you for your attention.



THE BEST THESIS DEFENSE IS A GOOD THESIS OFFENSE.

`http://xkcd.com/1403/`

Any questions?