Andreatta–Iovita's p–adic L–functions

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These are my informal notes of a seminar organized by Ju-Feng Wu and myself. The original goal was to study two articles of Fabrizio Andreatta and Adrian Iovita which give geometric constructions of p–adic L–functions. However, the seminar was interrupted midway due to lockdowns in Quebec.

When the speaker was someone other than me, I live-TeXed the notes and edited them afterwards. Ju-Feng's notes of his own talks might be better than mine. Many corrections and additions in comparison to the original article we study come from long discussions between us. Also Giovanni Rosso and Lennart Gehrmann helped us.

The introduction is my typing of a talk by Giovanni (since I was not able to keep up with his pace, there are a couple of incomplete paragraphs which I could not fill out later). The next two sections correspond to talks given by Ju-Feng (except for a few computations that I added later) and the last section corresponds to my talks.

Ju-Feng and I did our best to write a more gentle version of the latest preprint available at the time, which we all found difficult to read. Nevertheless, we were left with several questions, written here mostly in footnotes, because Adrian was away from Montreal at the time. In summer 2020 I sent this document and some questions to Adrian and Fabrizio. I was told that they took it into consideration to improve the exposition of some parts of their paper, but I never checked the next version of the preprint nor asked for more details.

WARNING: This document was written for my own use and not meant for publication. Moreover, Adrian told me that he thought we misunderstood some things. I decided to share these notes publicly, incomplete as they are, because I had already sent them to a couple of colleagues who apparently found them useful.

Contents

1 Introduction

The idea of this seminar is to understand the constructions of Andreatta–Iovita of their new p–adic L–functions, namely:

- the triple product L–functions (their article "Triple product…" [2]) and
- the Katz-type L–functions associated with CM twists of modular forms by Hecke characters (their article "Katz-type…" [1]).

Many other people had constructed similar kind of p–adic L–functions before. The novelty of Andreatta–Iovita's work is that they treat certain finite slope cases that were not know via new geometric methods. In particular, they found a way to compute $\delta^s G$, where δ is the Maass–Shimura operator, G is some (overconvergent) modular form and s is a p–adic variable.

1.1 Damerell's formula

Let $K/\mathbb{Q}$ be a quadratic imaginary field. We fix an embedding $\overline{K} \hookrightarrow \mathbb{C}$. Let $\chi\colon K^\times\backslash\mathbb{A}_K^\times \to \mathbb{C}^\times$ denote an algebraic Hecke character of K of conductor $\mathfrak{m}$. Let $\chi_{\mathfrak{f}}\colon (\mathcal{O}_K/\mathfrak{m})^\times \to \mathbb{C}^\times$ and $\chi_\infty\colon K\otimes_{\mathbb{Q}}\mathbb{R} = \mathbb{C}^\times \to \mathbb{C}^\times$ be the characters defining the infinity type, so that

$$\chi\big((\alpha)\big) = \chi_{\mathfrak{f}}(\alpha)\chi_\infty^{-1}(\alpha\otimes 1).$$

Suppose moreover that $\chi_\infty(\alpha\otimes 1) = \alpha^k$ for some positive integer k.

Define the L–function

$$L(\chi,s) = \sum_{(\mathfrak{a},\mathfrak{m})=1} \frac{\chi(\mathfrak{a})}{\mathbf{N}(\mathfrak{a})^s},$$

where the sum is over all ideals $\mathfrak{a}$ of $\mathcal{O}_K$ that are prime to the conductor.

Theorem 1. *There exists a period $\Omega_K \in \mathbb{C}^\times$ such that*

$$\frac{L(\chi,s_0)}{\Omega_K^k} \in \overline{K}$$

for all integers $0 \le s_0 \le \frac{k}{2}$.

Choose a set of (integral) representatives $\mathfrak{a}_1,\dots,\mathfrak{a}_r$ of the class group of K and assume that they are prime to $\mathfrak{m}$. Every (integral) ideal is of the form $\mathfrak{a} = (\alpha)\mathfrak{a}_i^{-1}$ for some $\alpha \in \mathfrak{a}_i$ and some i. Thus, we can rewrite

$$L(\chi,s) = \frac{1}{|\mathcal{O}_K^\times|}\sum_{i=1}^{n}\frac{\chi(\mathfrak{a}_i^{-1})}{\mathbf{N}(\mathfrak{a}_i)^{-s}}\sum_{0\neq\alpha\in\mathfrak{a}_i}\frac{\chi_{\mathfrak{f}}(\alpha)}{\alpha^k\,\mathbf{N}(\alpha)^s}$$

$$= \frac{1}{|\mathcal{O}_K^\times|} \sum_{i=1}^{n} \frac{\chi(\mathfrak{a}_i^{-1})}{\mathbf{N}(\mathfrak{a}_i)^{-s}} \sum_{\beta \bmod \mathfrak{m}} \chi_{\mathfrak{f}}(\beta) \sum_{u \in \mathfrak{b}} \frac{\mathbf{N}(\beta+u)^{-s}}{(\beta+u)^k},$$

where in the last step we decomposed $\alpha = \beta + u$ with $u \in \mathfrak{b} = \mathfrak{m}\mathfrak{a}_i$.

For example, if $K = \mathbb{Q}(i)$ and $\mathfrak{m} = \mathcal{O}_K = \mathbb{Z}[i]$, then

$$L(\chi, 0) = \frac{1}{n} \sum_{(m,n) \neq (0,0)} \frac{1}{(m+in)^k} = \frac{1}{n} E_k(i),$$

where E_k is the Eisenstein series of weight k.

More generally, we can define

$$E_k(s,z) = \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \frac{y^s}{j(\gamma,z)^k |j(\gamma,z)|^{2s}}$$

(where Γ_∞ is the stabilizer of ∞) and then

$$L(\chi, s) \doteq E_k(s,i) \quad \text{(i.e., for } y = 1).$$

For $0 \leq s_0 \leq k/2$, $E_k(s_0,z)$ is nearly holomorphic and can be expressed as

$$E_k(s_0,z) = \sum_{i=0}^{s_0} \sum_{n=0}^{\infty} a_n^{(i)} y^{-i} q^n.$$

Fact. These forms $E_k(s_0,z)$ are algebraic.

Moreover,

$$E_k(s_0,z) = \delta_{k-2s_0}^{s_0} E_{k-2s_0}(z),$$

where

$$\delta_{k-2s_0}^{s_0} = \frac{1}{2\pi i}\Big(\frac{d}{dz} + \frac{k-2s_0}{y}\Big)$$

and $\delta_{k-2s_0}^{s_0} = \delta_{k-2} \circ \cdots \circ \delta_{k-2s_0}$. (This shows that indeed $E_k(s_0,z)$ is a nearly holomorphic modular form: it is the image of a holomorphic modular form under iterations of the Maass–Shimura operator.)

The upshot of the first computation is that we can express

$$L(\chi, s_0) = \sum_{x_i \text{ CM points of } \mathbb{H}} c_i E_k(s_0, x_i)$$

(i.e., an explicit linear combination of the values of $E_k(s_0,z)$ at CM points).

Theorem 2 (Damerell–Shimura). *Let F be a weight k nearly holomorphic modular form and let $x \in \mathbb{H} \cap \mathcal{O}_K$. Then*

$$\frac{F(x)}{\Omega_K^k} \in \overline{K}.$$

Let us study this period more carefully. Let $\mathfrak{a}$ be a fractional ideal of $\mathcal{O}_K$. Via the embedding $K \hookrightarrow \mathbb{C}$, we can see $\mathfrak{a}$ as a lattice in $\mathbb{C}$. Let $E_\mathfrak{a}$ be the elliptic curve whose $\mathbb{C}$–points correspond to $\mathbb{C}/\mathfrak{a}$, which has CM by $\mathcal{O}_K$. Consider the invariant differential $\omega = \frac{dx}{y} \in \mathrm{H}^0(E_\mathfrak{a}, \Omega^1_{E_\mathfrak{a}})$ (defined algebraically). If τ is the variable on $\mathbb{C}$, we can write $d\tau = \Omega_K \omega$ for some (transcendental) Ω_K. On the other hand, the action of $\mathcal{O}_K$ gives a decomposition $\mathrm{H}^1_{\mathrm{dR}}(E_\mathfrak{a}) \cong K \oplus K^\sigma$ (where σ is the complex conjugation) that, after tensoring with $\mathbb{C}$, coincides with the Hodge decomposition $\mathrm{H}^1_{\mathrm{dR}}(E_\mathfrak{a}) \otimes \mathbb{C} = \mathrm{H}^{1,0}_{\mathrm{dR}}(E_{\mathfrak{a},\mathbb{C}}) \oplus \mathrm{H}^{0,1}_{\mathrm{dR}}(E_{\mathfrak{a},\mathbb{C}})$. Consider a point $x \in \mathbb{H}$ corresponding to $(E_\mathfrak{a}, d\tau)$. Then

$$F(x) = F(E_\mathfrak{a}, d\tau) = F(E_\mathfrak{a}, \Omega_K \omega) = \Omega_K^k F(E_\mathfrak{a}, \omega)$$

and $F(E_\mathfrak{a}, \omega)$ is algebraic.

Remark. This is a pull-back formula for

$$U(1) \times U(1) \to U\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right),$$

where $U(1)$ is the algebraic group over $\mathbb{Z}$ whose points are

$$U(1)(R) = \{\, g \in (R \otimes_\mathbb{Z} \mathcal{O}_K)^\times : g\overline{g} = 1 \,\}$$

and $U\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \sim \mathrm{SL}_2(\mathbb{Z})$. Then $E_k(s,z)$ is a form on $U(1,1)$. Take a level

$$K^\times \backslash \mathbb{A}_K^\times / (1 + m\mathcal{O}_K)(\mathbb{R}^\times) \to$$

Let $X^0(K,m)$ be the Shimura variety of this level, that is a finite set of points corresponding to a ray class group.

$E_k(s,z)$ restricts to the image of

$$X^0(K,m) \times X^0(K,m) \hookrightarrow \mathbb{H}/\Gamma$$

Evaluating $E_k(s,z)$ on the image of this embedding,

$$\begin{aligned} U(n) \times U(n) &\hookrightarrow U(n,n) \\ (f, \overline{f}) &\hookleftarrow E(z,s) \end{aligned}$$

and

$$\langle f \otimes f, E(s, z_{\mathrm{res}}) \rangle \sim L(f,s)\langle f, f\rangle.$$

Here $\langle -,-\rangle$ is a Petersson inner product given by

$$\langle f, g\rangle = \sum_{x\in G} f(x)\overline{g}(x).$$

Remark. In Andreatta–Iovita's paper, they consider Hecke characters χ of type $(k_1,k_2) = (k+j,-j)$, which means that

$$\chi\big((\alpha)\big) = \alpha^{k_1}\overline{\alpha}^{k_2}\chi_{\mathfrak{f}}(\alpha).$$

Then we get

$$\sum \frac{\alpha^{k_1}\overline{\alpha}^{k_2}}{\mathbf{N}(\alpha)^s} = \sum \frac{\alpha^{k+2j}}{\mathbf{N}(\alpha)^{s+j}}$$

The question is how to construct p–adic L–functions interpolating

$$L(\chi, s_0) = \sum_x c_x \chi(x)\delta^s E(x).$$

We want to vary p–adically the pair (k,s) or, equivalently, $(k+j,-j)$.

1.2 Andreatta–Iovita's work

For a Hecke character $\chi\colon K^\times\backslash\mathbb{A}_K^\times \to \mathbb{C}$ of infinity type (k_1,k_2), we consider its p–adic avatar $\chi_p\colon K^\times\backslash\mathbb{A}_K^\times \to \mathbb{C}_p$ given by

$$\chi_p(a) = \chi(a)\chi_\infty^{-1}(a_\infty)a_p^{-k_1}\sigma(a_p^{-k_2}).$$

By class field theory, $\mathbb{A}_{K,p}^\times$ is *almost* the Galois group of the maximal abelian extension of K that is unramified outside p. Therefore, we can regard the characters in consideration as Galois representations $\chi_p\colon \operatorname{Gal}(K_\infty/K) \to \mathbb{C}_p$, where K_∞/K is the $\mathbb{Z}_p^2$–extension of K. Thus, we can define a p–adic L–function by the formula

$$L_p(\chi, s) = \sum_{x \text{ CM points}} c_x \chi(\mathfrak{a}_x)\delta_{k-2s}^s E_{k-2s}(x).$$

For such a formula to make sense, we need some observations. On the one hand, the Eisenstein series E_k are *kind of good* p–adic analytic form of K. On the other hand, it is unclear what the iterates of the Maass–Shimura differential operator should be in the p–adic situation. Recall that δ_k can be constructed from the Gauss–Manin connection ∇_k over $\mathbb{C}$ as follows:

$$\begin{tikzcd}
\omega_{\mathbb{E}}^{k} \arrow[r, hook] \arrow[rdddd, dashed, bend right, "\delta_k"'] & \mathrm{Sym}^k \mathrm{H}^1_{\mathrm{dR}}(\mathbb{E}/Y) \cong \omega^k_{\mathbb{E}/Y} \oplus \cdots \arrow[d, "\nabla_k"] \\
 & \mathrm{Sym}^k \mathrm{H}^1_{\mathrm{dR}}(\mathbb{E}/Y) \otimes \Omega^1_{Y/\mathbb{C}} \arrow[d, "\mathrm{KS}^{-1}", "\wr"'] \\
 & \mathrm{Sym}^k \mathrm{H}^1_{\mathrm{dR}}(\mathbb{E}/Y) \otimes \omega^2_{\mathbb{E}} \cong \omega^{k+2}_{\mathbb{E}} \oplus \cdots \arrow[d, two heads] \\
 & \omega^{k+2}_{\mathbb{E}}
\end{tikzcd}$$

In the last vertical arrow, we use the splitting of the Hodge–de Rham filtration over $\mathbb{C}$ as analytic forms (i.e., the Hodge decomposition of de Rham cohomology).

We can do something similar p–adically. If E is a p–ordinary elliptic curve, we have the unit-root splitting $\mathrm{H}^1_{\mathrm{dR}}(E) = U \oplus C$ (i.e., here U is the Frobenius stable line). Thus, we can similarly define

$$\begin{tikzcd}
\omega_{\mathbb{E}}^{k} \arrow[r, hook] \arrow[rdddd, dashed, bend right, "\theta"'] & \mathrm{Sym}^k \mathrm{H}^1_{\mathrm{dR}}(\mathbb{E}/Y) \cong \omega^k_{\mathbb{E}/Y} \oplus \cdots \arrow[d, "\nabla_k"] \\
 & \mathrm{Sym}^k \mathrm{H}^1_{\mathrm{dR}}(\mathbb{E}/Y) \otimes \Omega^1_{Y} \arrow[d, "\mathrm{KS}^{-1}", "\wr"'] \\
 & \mathrm{Sym}^k \mathrm{H}^1_{\mathrm{dR}}(\mathbb{E}/Y) \otimes \omega^2_{\mathbb{E}} \cong \omega^{k+2}_{\mathbb{E}} \oplus \cdots \arrow[d, two heads] \\
 & \omega^{k+2}_{\mathbb{E}}
\end{tikzcd}$$

The first important problem that we encounter is that not all elliptic curves are ordinary. That is, θE_k can only be evaluated on ordinary points.

Fact. The CM elliptic curve $E_{\mathfrak{a}}$ is p–ordinary if and only if p is split in K.

The second main problem is that the iterate θ^s might not make sense for a p–adic variable s. On q–expansions, we should have

$$\theta^s \left(\sum_n a_n q^n \right) \text{ “ $=$ ” } \sum_n n^s a_n q_n$$

But p^s is not analytic, so we should restrict to n coprime with p. Moreover, we should add $\omega(n)^{-s}$, so we will have to restrict to $s \equiv k \mod p-1$. (This is a minor issue, as this set is still p–adically dense.)

Therefore, one of the ingredients that we need is the p–depletion operator: if

$$F = \sum_n a_n q^n,$$

then we set

$$F^{[p]} = \sum_{(n,p)=1} a_n q^n.$$

We can finally define

$$L_p(\chi) = \sum_{\mathfrak{a}_i} c_i \chi(\mathfrak{a}_i) \theta^s E^{[p]}_{k-2s}(x_i)$$

(where x_i is a CM point corresponding to the ideal $\mathfrak{a}_i$).

We will have to use overconvergent p–adic modular forms of weight k (i.e., sections of $\omega^k_{\mathbb{E}}$ on X^{ord} that extend a bit into the supersingular disks).

Here, we encounter another problem: θ^s makes sense on p–adic modular forms because

$$\|f\|_{X^{\text{ord}}} = \sup_n |a_n|_p$$

but, if f overconverges, then it has some poles in the supersingular locus and θf might not be overconvergent. That is, if we approximate s with integers s_i, it might happen that the poles of $\theta^{s_i}(f)$ get closer and closer to the ordinary locus as $i \to \infty$.

Theorem 3 (Andreatta–Iovita). *There exist operators*

$$\delta^s_k \colon N_k^{\mathrm{U}_p=0,\text{oc}} \to N_{k+2s}^{\mathrm{U}_p=0,\text{oc}}$$

that interpolate $\delta_\kappa \colon N_\kappa \to N_{\kappa+2}$. (Meaning that δ^s is a power series in s and $\delta^s(f)$ converges if $\mathrm{U}_p(f) = 0$.)

Remark. On X^{ord},

$$\theta^{(p-1)p^n} f \equiv f \mod p^{n+1}$$

if f is p–depleted.

The other trick that Andreatta–Iovita use is that $\mathrm{U}_p(f)(x) = 0$ if f is p–depleted, and this is a sum

$$\sum_{y \in \mathrm{U}_p(x)} f(y) = 0$$

If x is not in the overconvergence locus of f, it turns out that only one of the points $y \in \mathrm{U}_p(x)$, say y_0, is outside this locus, so we can define

$$f(y_0) = -\sum_{y \neq y_0} f(y).$$

Remark. Let f be a Hecke eigenform of weight k and consider a Hecke character χ of K of weight

(1) either $(k-1-j, 1+j)$ with $0 \leq j \leq k-2$
(2) or $(k+j, -j)$ with $j \geq 0$.

Bertolini–Darmon–Prasanna define

$$L(f, \chi, s) = L(f, g_\chi, s)$$

where g_χ is the θ–series

$$\begin{aligned} g_\chi &= \sum_{\mathfrak{a} \in \mathcal{O}_K} \chi(\mathfrak{a}) q^{\mathbf{N}(\mathfrak{a})} = \sum_n \Big(\sum_{\mathbf{N}(\mathfrak{a})=n} \chi(\mathfrak{a}) \Big) q^n \\ &= \prod_{\mathfrak{p} \in \mathcal{O}_K} \big(1 - a_{\mathbf{N}(\mathfrak{p})}(f)\, \mathbf{N}(\mathfrak{p})^{-s} + \chi(\mathfrak{p})\, \mathbf{N}(\mathfrak{p})^{-2s+k-1}\big). \end{aligned}$$

Using Waldspurger's formula, they prove that

$$L(f, \chi^{-1}, 0) = \sum_{\mathfrak{a}_i} \chi^{-1}(\mathfrak{a}_i) \delta_k^j(f)(x_i) \in \Omega_K^{k+2j} \overline{K}$$

When p is split in K, g_χ lives in a Hida family and this L–function can be interpolated p–adically. If p is inert, then g_χ does not live in a CM family because the slopes of the roots α_p, β_p of $x^2 + p^{k+j-1}$ are $(k+j-1)/2$.

2 Vector bundles with marked sections

Let $\mathfrak{S}$ be a formal scheme with invertible ideal of definition $\mathscr{I}$. We are going to consider a locally free $\mathscr{O}_{\mathfrak{S}}$–module $\mathscr{E}$ of rank n together with global sections $s_1, \ldots, s_m$ of $\bar{\mathscr{E}} = \mathscr{E}/\mathscr{I}\mathscr{E}$ (for some $m \leq n$) that generate a locally free direct summand of rank m.

2.1 The (formal) vector bundles

Let FSch$/\mathfrak{S}$ denote the category of formal schemes over $\mathfrak{S}$, $f\colon \mathfrak{T} \to \mathfrak{S}$, with invertible ideal of definition $f^*\mathscr{I}$. We consider the following contravariant functors:

- $\mathbb{V}(\mathscr{E})\colon (\mathrm{FSch}/\mathfrak{S})^{\mathrm{op}} \to \mathrm{Set}$ sends $f\colon \mathfrak{T} \to \mathfrak{S}$ to $\mathrm{Hom}_{\mathscr{O}_{\mathfrak{T}}}(f^*\mathscr{E}, \mathscr{O}_{\mathfrak{T}})$.
- $\mathbb{V}_0(\mathscr{E}, s_1, \ldots, s_m)\colon (\mathrm{FSch}/\mathfrak{S})^{\mathrm{op}} \to \mathrm{Set}$ sends $f\colon \mathfrak{T} \to \mathfrak{S}$ to the subset

$$\{\, h \in \mathrm{Hom}_{\mathscr{O}_{\mathfrak{T}}}(f^*\mathscr{E}, \mathscr{O}_{\mathfrak{T}}) : (h \bmod f^*\mathscr{I})(f^*s_i) = 1 \bmod f^*\mathscr{I} \text{ for } 1 \leq i \leq m \,\}.$$

Theorem 4. *The functors $\mathbb{V}(\mathscr{E})$ and $\mathbb{V}_0(\mathscr{E}, s_1, \ldots, s_m)$ are representable in the category* FSch$/S$.

Sketch of the proof. The proof is similar to that of the analogous statement for vector bundles in the category of schemes. In particular, $\mathbb{V}(\mathscr{E}) = \mathrm{Spf}_{\mathfrak{S}}(\widehat{\mathrm{Sym}}\,\mathscr{E})$, where the hat denotes the completion with respect to the $\mathscr{I}$–adic topology. Then the subfunctor $\mathbb{V}_0(\mathscr{E}, s_1, \ldots, s_m)$ corresponds to an open formal subscheme of the blow-up of $\mathbb{V}(\mathscr{E})$ with respect to the ideal sheaf $\mathscr{J}$ of $\widehat{\mathrm{Sym}}\,\mathscr{E}$ generated by $\mathscr{I}$ and the lifts of $s_1 - 1, \ldots, s_m - 1$. Namely, $\mathbb{V}_0(\mathscr{E}, s_1, \ldots, s_m)$ is the open locus where $\mathscr{J}$ is generated by $\mathscr{I}$.

Locally, we can assume that $\mathfrak{S} = \mathrm{Spf}(R)$, $\mathscr{I}$ is given by a principal ideal $I = \alpha R$ and $\mathscr{E}$ corresponds to a free R–module E of rank n. We can choose a basis $e_1, \ldots, e_n$ of E with the property that $(e_i \bmod \alpha) = s_i$ for $1 \leq i \leq m$ and $e_{m+1}, \ldots, e_n \bmod \alpha$ give a basis of Q, where we have the short exact sequence of free (R/I)–modules

$$0 \longrightarrow \bigoplus_{i=1}^{m} (R/I)s_i \longrightarrow \bigoplus_{i=1}^{n} (R/I)e_i \longrightarrow Q \longrightarrow 0.$$

We obtain an induced map $\mathrm{Sym}\,\bar{E} \to \mathrm{Sym}\,Q$ with kernel $(s_1 - 1, \ldots, s_m - 1)$. Now we can express $\mathbb{V}(\mathscr{E}) = \mathrm{Spf}(R\langle X_1, \ldots, X_n\rangle)$, where each variable X_i corresponds to e_i. Then $J = (\alpha, X_1 - 1, \ldots, X_m - 1)$ (this is the ideal generated by α and the lifts of the kernel described above). Hence, the formal open of the blow-up with respect to J where (the inverse image of) J is generated by α is

$$\mathrm{Spf}\left(R\langle X_1,\dots,X_n\rangle\left\langle\frac{X_1-1}{\alpha},\dots,\frac{X_m-1}{\alpha}\right\rangle\right)=\mathrm{Spf}\langle Z_1,\dots,Z_m,X_{m+1},\dots,X_n\rangle$$

(where we made the change of variables $Z_i=\frac{X_i-1}{\alpha}$). One checks easily that this formal scheme represents the desired functor, as $Z_i=1+\alpha X_i\equiv 1 \bmod I$. $\square$

2.1.1 Filtrations

Suppose that $\mathscr{F}\subset\mathscr{E}$ is a locally free $\mathscr{O}_{\mathfrak{S}}$–submodule of rank m such that $s_1,\dots,s_m$ is a basis of $\bar{\mathscr{F}}=\mathscr{F}/\mathscr{I}\mathscr{F}$. By functoriality, we get natural morphisms

$$\begin{array}{ccc}\mathbb{V}_0(\mathscr{E},s_1,\dots,s_m) & \longrightarrow & \mathbb{V}(\mathscr{E})\\ \downarrow & & \downarrow\\ \mathbb{V}_0(\mathscr{F},s_1,\dots,s_m) & \longrightarrow & \mathbb{V}(\mathscr{F})\end{array}$$

making the diagram commutative.

Proposition 5. *Let $f_0\colon\mathbb{V}_0(\mathscr{E},s_1,\dots,s_m)\to\mathfrak{S}$ be the structure morphism. The sheaf $f_{0,*}\mathscr{O}_{\mathbb{V}_0(\mathscr{E},s_1,\dots,s_m)}$ admits an increasing filtration $\mathrm{Fil}_j\, f_{0,*}\mathscr{O}_{\mathbb{V}_0(\mathscr{E},s_1,\dots,s_m)}$ with graded pieces $\mathrm{Gr}_j\, f_{0,*}\mathscr{O}_{\mathbb{V}_0(\mathscr{E},s_1,\dots,s_m)}=f_{0,*}\mathscr{O}_{\mathbb{V}_0(\mathscr{F},s_1,\dots,s_m)}\hat{\otimes}_{\mathscr{O}_{\mathfrak{S}}}\mathrm{Sym}^j(\mathscr{E}/\mathscr{F})$.*

Sketch of the proof. Working locally with the same notation as in the previous proof (but choosing a basis $f_1,\dots,f_m,e_{m+1},\dots,e_n$ of E such that $f_1,\dots,f_m$ is a basis of F), $\mathrm{Fil}_j\, f_{0,*}\mathscr{O}_{\mathbb{V}_0(\mathscr{E},s_1,\dots,s_m)}$ corresponds to polynomials with coefficients in $R\langle Z_1,\dots,Z_m\rangle$ of degree $\leq j$ in the variables $X_{m+1},\dots,X_n$. $\square$

Corollary 6. *Let $(\mathscr{E}',s_1',\dots,s_m')$ be another tuple with the same conditions and admitting $\mathscr{F}'\subset\mathscr{E}'$ as above and suppose that there is a morphism $g\colon\mathscr{E}'\to\mathscr{E}$ that restricts to $g|_{\mathscr{F}'}\colon\mathscr{F}'\to\mathscr{F}$ and such that $(g \bmod \mathscr{I})(s_i')=s_i$. Then, the morphism of vector bundles $\mathbb{V}_0(\mathscr{E},s_1,\dots,s_n)\to\mathbb{V}_0(\mathscr{E}',s_0',\dots,s_n')$ induced by g is compatible with the filtrations.*

2.1.2 Connections

Now consider a p–adically complete $\mathbb{Z}_p$–algebra A_0 and suppose that $\mathfrak{S}$ is locally of finite type over $\mathrm{Spf}(A_0)$. Write $\mathfrak{P}_{\mathfrak{S}/A_0}=\mathfrak{S}\times_{A_0}\mathfrak{S}$ and let $\Delta\colon\mathfrak{S}\to\mathfrak{P}_{\mathfrak{S}/A_0}$ be the diagonal map. Let $\mathfrak{P}^{(1)}_{\mathfrak{S}/A_0}$ denote the first infinitesimal neighbourhood of $\Delta(\mathfrak{S})$ in $\mathfrak{P}_{\mathfrak{S}/A_0}$. We obtain two natural morphisms (corresponding to the two projections) $j_1,j_2\colon\mathfrak{P}^{(1)}_{\mathfrak{S}/A_0}\to\mathfrak{S}$. We want to use this formalism to describe connections.

Let $\mathscr{M}$ be a locally free $\mathscr{O}_{\mathfrak{S}}$–module of finite rank. Grothendieck showed that giving an integrable connection $\nabla\colon \mathscr{M} \to \mathscr{M} \hat{\otimes}_{\mathscr{O}_{\mathfrak{S}}} \Omega^1_{\mathfrak{S}/A_0}$ is equivalent to giving an isomorphism

$$\varepsilon\colon j_2^*\mathscr{M} \to j_1^*\mathscr{M}$$

such that $\Delta^*\varepsilon = \mathrm{id}_{\mathscr{M}}$ and satisfying a suitable cocycle condition with respect to the three possible pull-backs $\mathfrak{S} \times_{A_0} \mathfrak{S} \times_{A_0} \mathfrak{S} \to \mathfrak{S} \times_{A_0} \mathfrak{S}$. Indeed, the relation between ε and ∇ is given by

$$\varepsilon(1 \otimes x) = x \otimes 1 + \nabla(x).$$

Suppose that the locally free $\mathscr{O}_{\mathfrak{S}}$–module $\mathscr{E}$ is endowed with an integrable connection $\nabla\colon \mathscr{E} \to \mathscr{E} \hat{\otimes}_{\mathscr{O}_{\mathfrak{S}}} \Omega^1_{\mathfrak{S}/A_0}$ with respect to which the sections $s_1, \ldots, s_m$ are horizontal. Consider the corresponding isomorphism $\varepsilon\colon j_2^*\mathscr{E} \to j_1^*\mathscr{E}$ as above. By functoriality, we obtain a commutative diagram

$$\begin{tikzcd}
\mathbb{V}_0\big(j_1^*(\mathscr{E}, s_1, \ldots, s_m)\big) \arrow[r] \arrow[d, "\widetilde{\varepsilon}_0"'] & \mathbb{V}(j_1^*\mathscr{E}) \arrow[d, "\widetilde{\varepsilon}"] \\
\mathbb{V}_0\big(j_2^*(\mathscr{E}, s_1, \ldots, s_m)\big) \arrow[r] & \mathbb{V}(j_2^*\mathscr{E})
\end{tikzcd}$$

giving rise to

$$\begin{tikzcd}
j_1^* f_{0,*}\mathscr{O}_{\mathbb{V}_0(\mathscr{E}, s_1, \ldots, s_m)} & j_1^* f_*\mathscr{O}_{\mathbb{V}(\mathscr{E})} \arrow[l] \\
j_2^* f_{0,*}\mathscr{O}_{\mathbb{V}_0(\mathscr{E}, s_1, \ldots, s_m)} \arrow[u, "\widetilde{\varepsilon}_0^\sharp"] & j_2^* f_*\mathscr{O}_{\mathbb{V}(\mathscr{E})} \arrow[l] \arrow[u, "\widetilde{\varepsilon}^\sharp"']
\end{tikzcd}$$

at the level of sheaves. (Here, f and f_0 denote the structure morphisms from $\mathbb{V}(\mathscr{E})$ and $\mathbb{V}_0(\mathscr{E}, s_0, \ldots, s_n)$ to $\mathfrak{S}$, respectively.) It turns out that $\widetilde{\varepsilon}^\sharp$ and $\widetilde{\varepsilon}_0^\sharp$ satisfy Grothendieck's criteria and so define integrable connections $\widetilde{\nabla}$ and $\widetilde{\nabla}_0$ making

$$\begin{tikzcd}
\mathscr{E} \arrow[r, "\nabla"] \arrow[d] & \mathscr{E} \hat{\otimes}_{\mathscr{O}_{\mathfrak{S}}} \Omega^1_{\mathfrak{S}/A_0} \arrow[d] \\
f_*\mathscr{O}_{\mathbb{V}(\mathscr{E})} \arrow[r, "\widetilde{\nabla}"] \arrow[d] & f_*\mathscr{O}_{\mathbb{V}(\mathscr{E})} \hat{\otimes}_{\mathscr{O}_{\mathfrak{S}}} \Omega^1_{\mathfrak{S}/A_0} \arrow[d] \\
f_{0,*}\mathscr{O}_{\mathbb{V}_0(\mathscr{E}, s_1, \ldots, s_m)} \arrow[r, "\widetilde{\nabla}_0"] & f_{0,*}\mathscr{O}_{\mathbb{V}_0(\mathscr{E}, s_1, \ldots, s_m)} \hat{\otimes}_{\mathscr{O}_{\mathfrak{S}}} \Omega^1_{\mathfrak{S}/A_0}
\end{tikzcd}$$

commutative. From now on, write $\nabla_0 = \widetilde{\nabla}_0$.

Lemma 7. *The connection ∇_0 satisfies Griffiths's transversality condition (with respect to the filtration defined in proposition 5).*

Proof. Affine locally, we can assume that $\mathfrak{S} = \mathrm{Spf}(R)$ and we can choose a basis $f_1, \ldots, f_m, e_{m+1}, \ldots, e_n$ as before. Since the sections $s_1, \ldots, s_m$ (the reductions of $f_1, \ldots, f_m$) are horizontal, we can express

$$\nabla f_k = \sum_{i=1}^{m} \alpha f_i \otimes \omega_{k,i} + \sum_{j=m+1}^{n} \alpha e_j \otimes \nu_{k,j}$$

and

$$\nabla e_k = \sum_{i=1}^{m} f_i \otimes \tau_{k,i} + \sum_{j=m+1}^{n} e_j \otimes \sigma_{k,j}$$

for some sections $\omega_{k,i}, \nu_{k,j}, \tau_{k,i}, \sigma_{k,j}$ of $\Omega^1_{\mathfrak{S}/A_0}$. Therefore,

$$\widetilde{\nabla} X_k = \begin{cases} \sum_{i=1}^{m} \alpha X_i \otimes \omega_{k,i} + \sum_{j=m+1}^{n} \alpha X_j \otimes \nu_{k,j} & \text{if } k \leq m, \\ \sum_{i=1}^{m} X_i \otimes \tau_{k,i} + \sum_{j=m+1}^{n} X_j \otimes \sigma_{k,j} & \text{if } k > m. \end{cases}$$

Since $X_k = 1 + \alpha Z_k$, we find that $\widetilde{\nabla} X_k = \nabla_0(\alpha Z_k) = \alpha \nabla_0 Z_k + Z_k \otimes d\alpha$ or, equivalently,

$$\nabla_0 Z_k = \frac{1}{\alpha} \widetilde{\nabla} X_k - Z_k \otimes d\alpha = \sum_{i=1}^{m} X_i \otimes \omega_{k,i} + \sum_{j=m+1}^{n} X_j \otimes \nu_{k,j} - Z_k \otimes d\alpha.$$

Using that Fil_j corresponds to polynomials of degree $\leq j$ in $X_{m+1}, \ldots, X_n$ and Leibniz's rule, the result follows. □

2.2 Application to modular curves

Fix a prime $p > 3$ and an integer $N \geq 4$ such that $p \nmid N$. Let $X = X_1(N)$ be the compactified modular curve of level $\Gamma_1(N)$ over $\mathbb{Z}_p$. Let $\mathfrak{X}$ denote the formal completion of X along its special fibre and let $\mathcal{X}$ be its (adic) analytic generic fibre. Consider the universal semiabelian scheme $\pi \colon E^{\mathrm{univ}} \to X$ and its de Rham cohomology $\mathscr{H} = \mathrm{H}^1_{\mathrm{dR}}(E^{\mathrm{univ}}/X) = \mathbb{R}^1\pi_* \Omega^\bullet_{E^{\mathrm{univ}}/X}$, that is naturally endowed with the Hodge filtration

$$0 \longrightarrow \underline{\omega} \longrightarrow \mathscr{H} \longrightarrow \underline{\omega}^{-1} \longrightarrow 0$$

(where $\underline{\omega} = \pi_* \Omega^1_{E^{\mathrm{univ}}/X}$). We define Hdg to be the ideal of $\mathscr{O}_{\mathfrak{X}}$ that is locally generated by p and a fixed lift of the Hasse invariant (that is, the Eisenstein series E_{p-1}).

2.2.1 The weight space

Consider the Iwasawa algebra $\Lambda = \mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!] \cong \mathbb{Z}_p\big[(\mathbb{Z}/p\mathbb{Z})^\times\big][\![T]\!]$ and its subalgebra $\Lambda^0 = \mathbb{Z}_p[\![1+p\mathbb{Z}_p]\!] \cong \mathbb{Z}_p[\![T]\!]$ (the isomorphisms sending $[\exp(p)]$ to $1+T$). We consider the weight spaces

$$\mathfrak{W} = \mathrm{Spf}(\Lambda), \quad \mathcal{W} = \mathrm{Spa}(\Lambda, \Lambda)^{\mathrm{an}},$$
$$\mathfrak{W}^0 = \mathrm{Spf}(\Lambda^0), \quad \mathcal{W}^0 = \mathrm{Spa}(\Lambda^0, \Lambda^0)^{\mathrm{an}}.$$

(Here, the superscript an means that we take only the analytic points, which are those whose supports are not open.)

For any interval of the form $I = [p^a, p^b] \subset [0, \infty]$ with $a \in \mathbb{Z}_{\geq 0} \cup \{-\infty\}$ and $b \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, we define

$$\mathcal{W}_I = \{\, x \in \mathcal{W} : |p|_x \leq |T^{p^a}|_x \neq 0 \text{ and } |T^{p^b}|_x \leq |p|_x \neq 0 \,\}.$$

We will consider two main cases:

(1) If $I = [0, p^b]$ with $b \neq \infty$, then

$$\mathcal{W}_I = \mathrm{Spa}\left(\Lambda\Big\langle \frac{T^{p^b}}{p} \Big\rangle \Big[\frac{1}{p}\Big], \Lambda\Big\langle \frac{T^{p^b}}{p} \Big\rangle\right).$$

In this case, we set

$$\Lambda_I = \Lambda\Big\langle \frac{T^{p^b}}{p} \Big\rangle$$

(and similarly for Λ_I^0) and $\alpha = p$.

(2) If $I = [p^a, p^b]$ with $a \neq -\infty$, then

$$\mathcal{W}_I = \mathrm{Spa}\left(\Lambda\Big\langle \frac{p}{T^{p^a}}, \frac{T^{p^b}}{p} \Big\rangle \Big[\frac{1}{T}\Big], \Lambda\Big\langle \frac{p}{T^{p^a}}, \frac{T^{p^b}}{p} \Big\rangle\right).$$

In this case, we set

$$\Lambda_I = \Lambda\Big\langle \frac{p}{T^{p^a}}, \frac{T^{p^b}}{p} \Big\rangle$$

(and similarly for Λ_I^0) and $\alpha = T$.

For an interval I as above, let $\kappa_I \colon \mathbb{Z}_p^\times \to \Lambda_I^\times$ be the universal weight on $\mathcal{W}_I$

(i.e., the character given by $t \mapsto [t]$). We decompose it into the restriction $\kappa_{I,\mathrm{f}}$ to the finite part $(\mathbb{Z}/p\mathbb{Z})^\times$ and $\kappa_I^0 = \kappa_I \kappa_{I,\mathrm{f}}^{-1} \colon \mathbb{Z}_p^\times \to (\Lambda_I^0)^\times$. Define $\mathfrak{X}_0 = \mathfrak{X} \times_{\mathbb{Z}_p} \mathfrak{W}^0$ and $\mathfrak{X}_I = \mathfrak{X} \times_{\mathbb{Z}_p} \mathrm{Spf}(\Lambda_I)$.

Assumption 8. We restrict to one of the following two cases:

(1) $I = [0,1]$ and we choose $r \geq 2$ and $1 \leq n \leq r$.

(2) $I = [p^a, p^b]$ and we choose $r \geq 1$ such that $r + a \geq b + 2$ and $1 \leq n \leq a + r$.

Under this assumption, there exists a unique

$$u_{\kappa_I} = \frac{\log(\kappa_I(t))}{\log(t)} \in p^{1-n}\Lambda_I^0$$

such that $\kappa_I(t) = \exp(u_{\kappa_I} \log(t))$ for all $t \in 1 + p^n\mathbb{Z}_p$.

2.2.2 The Igusa tower

Define $\mathfrak{X}_{r,I}$ to be the formal scheme over $\mathfrak{X}_I$ representing the functor that sends an α–adically complete Λ_I^0–algebra R to the set of equivalence classes of pairs (f, η), where $f \colon \mathrm{Spf}(R) \to \mathfrak{X}_I$ and $\eta \in \mathrm{H}^0(\mathrm{Spf}(R), f^*\underline{\omega}^{(1-p)p^{r+1}})$ satisfies that $\eta \cdot f^* E_{p-1}^{p^{r+1}} \equiv \alpha \bmod p^2$. (We omit the description of the equivalence relation.)

By assumption 8, the universal semiabelian formal scheme $\mathfrak{E}_{r,I}$ over $\mathfrak{X}_{r,I}$ admits a canonical subgroup H_n generically of order p^n. Thus, we will work instead over the Igusa curve $\mathcal{IG}_{n,r,I} = \mathrm{Isom}_{\mathcal{X}_{r,I}}\big(\underline{(\mathbb{Z}/p^n\mathbb{Z})^\times}, (H_n)^\vee\big) \to \mathcal{X}_{r,I}$ and over the normalization $\mathfrak{IG}_{n,r,I}$ of $\mathfrak{X}_{r,I}$ in $\mathcal{IG}_{n,r,I}$. The universal trivialization over $\mathfrak{IG}_{n,r,I}$ yields a section P_n^{univ} of $(H_n)^\vee$ ("the image of 1" under $\underline{(\mathbb{Z}/p^n\mathbb{Z})^\times} \cong (H_n)^\vee$). From it we will obtain the desired *marked section*.

2.2.3 The modular sheaves

Recall that we work in the situation of the diagram

$$\begin{tikzcd} H_n \arrow[r, hook] \arrow[dr, "\pi_{H_n}"'] & \mathfrak{E}_{n,r,I} \arrow[d, "\pi"] \\ & \mathfrak{IG}_{n,r,I} \end{tikzcd}$$

and we define the sheaves $\underline{\omega} = \pi_*\Omega^1_{\mathfrak{E}_{n,r,I}/\mathfrak{IG}_{n,r,I}}$ and $\underline{\omega}_{H_n} = (\pi_{H_n})_*\Omega^1_{H_n/\mathfrak{IG}_{n,r,I}}$. It turns out that the canonical map $\underline{\omega} \to \underline{\omega}_{H_n}$ has kernel $\underline{\beta}'_n\underline{\omega}$, where

$$\underline{\beta}'_n = p^n \,\mathrm{Hdg}^{-\frac{p^n-1}{p-1}},$$

and we get an isomorphism $\underline{\omega}/\underline{\beta}'_n\underline{\omega} \cong \underline{\omega}_{H_n}$.

All in all, we have a commutative diagram

$$\begin{tikzcd}
 & \underline{\omega} \arrow[d] \arrow[dr, two heads] & \\
\mathrm{dlog}\colon (H_n)^\vee \arrow[r] & \underline{\omega}_{H_n} \arrow[r, "\cong"] & \underline{\omega}/\underline{\beta}'_n\underline{\omega} \\
(H_n \xrightarrow{\varphi} \mathbb{G}_\mathrm{m}) \arrow[r, mapsto] & \varphi^*\left(\frac{dT}{T}\right) &
\end{tikzcd}$$

and we want to use $s = \mathrm{dlog}(P_n^{\mathrm{univ}})$ (where P_n^{univ} is the universal point of $(H_n)^\vee$ over $\mathfrak{IG}_{n,r,I}$) as a marked section. The problem is that s does not generate $\underline{\omega}/\underline{\beta}'_n\omega$. In fact, it generates $\underline{\delta\omega}/\underline{\beta}'_n\underline{\omega}$, where $\underline{\delta} = \mathrm{Hdg}^{1/(p-1)}$. Therefore, we define the modified sheaf

$$\underline{\omega}^\sharp = \underline{\delta\omega}$$

and regard s as a section of $\underline{\omega}^\sharp/\underline{\beta}_n\underline{\omega}^\sharp$, where $\underline{\beta}_n = \underline{\delta}^{-1}\underline{\beta}'_n$. Another problem that we encounter is that $\underline{\omega}^\sharp$ is not a locally direct summand of $\mathscr{H}$, which we also modify in the following way: we define

$$\mathscr{H}^\sharp = \underline{\omega}^\sharp + \underline{\delta}^p\mathscr{H} \subset \mathscr{H}.$$

Equivalently, $\mathscr{H}^\sharp$ is the push-out

$$\begin{tikzcd}
\underline{\delta}^p\underline{\omega} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near end] & \underline{\delta}^p\mathscr{H} \arrow[d] \\
\underline{\delta\omega} \arrow[r] & \mathscr{H}^\sharp
\end{tikzcd}$$

and fits in a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \underline{\delta}^p\underline{\omega} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near end] & \underline{\delta}^p\mathscr{H} \arrow[r] \arrow[d, hook] & \underline{\delta}^p\underline{\omega}^{-1} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \underline{\omega}^\sharp \arrow[r] \arrow[d, hook] & \mathscr{H}^\sharp \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", very near start] & \underline{\delta}^p\underline{\omega}^{-1} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \underline{\omega} \arrow[r] & \mathscr{H} \arrow[r] & \underline{\omega}^{-1} \arrow[r] & 0
\end{tikzcd}$$

with exact rows (the first and the third ones coming from the Hodge filtration). In particular, $\mathscr{H}^\sharp$ is locally free of rank 2.

All in all, the data given by $(\underline{\omega}^\sharp, s) \subset (\mathscr{H}^\sharp, s)$ as defined above satisfies the hypotheses of the first part and we can apply the theory of vector bundles with marked sections. (In the notation of section 2.1, we have the locally free sheaves $\mathscr{F} = \underline{\omega}^\sharp$ and $\mathscr{E} = \mathscr{H}^\sharp$ on the formal scheme $\mathfrak{S} = \mathfrak{IG}_{n,r,I}$ with ideal of definition $\mathscr{I} = \underline{\beta}_n = p^n \operatorname{Hdg}^{-p^n/(p-1)}$ and the marked section s of $\bar{\mathscr{F}} \subset \bar{\mathscr{E}}$.) In this way, we obtain

$$\begin{tikzcd}
\mathbb{V}_0(\underline{\omega}^\sharp, s) \arrow[dr, "f_n"] \arrow[ddr, bend right, "f_0"'] & & \mathbb{V}_0(\mathscr{H}^\sharp, s) \arrow[ll] \arrow[dl, "g_n"'] \arrow[ddl, bend left, "g_0"] \\
& \mathfrak{IG}_{n,r,I} \arrow[d, "h_n"] & \\
& \mathfrak{X}_{r,I} &
\end{tikzcd}$$

and a filtration on $g_{0,*}\mathscr{O}_{\mathbb{V}_0(\mathscr{H}^\sharp, s)}$.

Let $\mathfrak{T}$ be the formal group scheme over $\mathfrak{IG}_{n,r,I}$ whose points are given by

$$\mathfrak{T}(\psi\colon \mathfrak{S} \to \mathfrak{IG}_{n,r,I}) = \big(1 + (\psi^* \underline{\beta}_n)\mathscr{O}_{\mathfrak{S}}(\mathfrak{S})\big)^\times$$

where $\underline{\beta}_n = p^n \operatorname{Hdg}^{-p^n/(p-1)}$. (By abuse of notation, we write $\underline{\beta}_n$ both for the ideal sheaf over $\mathfrak{X}_{r,I}$ and its pull-back over $\mathfrak{IG}_{n,r,I}$.) We extend it to $\mathfrak{X}_{r,I}$ by considering the formal group scheme over $\mathfrak{X}_{r,I}$ whose points are given by

$$\mathfrak{T}^{\mathrm{ext}}(\varphi\colon \mathfrak{S} \to \mathfrak{X}_{r,I}) = \mathbb{Z}_p^\times \big(1 + (\varphi^* \underline{\beta}_n)\mathscr{O}_{\mathfrak{S}}(\mathfrak{S})\big)^\times.$$

These formal tori act on $\mathbb{V}_0(\underline{\omega}^\sharp, s)$.

(1) The action of $\mathfrak{T}$ on $\mathbb{V}_0(\underline{\omega}^\sharp, s) \to \mathfrak{IG}_{n,r,I}$ can be described on points as follows: given $\psi\colon \mathfrak{S} \to \mathfrak{IG}_{n,r,I}$, $v \in \mathbb{V}_0(\underline{\omega}^\sharp, s)(\mathfrak{S})$ (i.e., a morphism $v\colon \psi^*\underline{\omega}^\sharp \to \mathscr{O}_{\mathfrak{S}}$ such that $(v \bmod \psi^*\underline{\beta}_n)(\psi^*(s)) = 1$) and $t \in \mathfrak{T}(\mathfrak{S}) = 1 + (\psi^*\underline{\beta}_n)\mathscr{O}_{\mathfrak{S}}(\mathfrak{S})$, then $t * v = tv$.

(2) The action of $\mathfrak{T}^{\mathrm{ext}}$ on $\mathbb{V}_0(\underline{\omega}^\sharp, s) \to \mathfrak{X}_{r,I}$ is the extension of the previous case as follows: given $\varphi\colon \mathfrak{S} \to \mathfrak{X}_{r,I}$, a point of $\mathbb{V}_0(\underline{\omega}^\sharp, s)(\mathfrak{S})$ consists of a pair (ψ, v) as in the previous case. For every $\lambda \in \mathbb{Z}_p^\times$, let $\bar{\lambda}$ denote its image in $(\mathbb{Z}/p^n\mathbb{Z})^\times \cong \operatorname{Gal}(\mathcal{IG}_{n,r,I}/\mathcal{X}_{r,I})$. Then $\bar{\lambda}$ gives an automorphism of $\mathfrak{IG}_{n,r,I}$ over $\mathfrak{X}_{r,I}$ (the identification of the Galois group with $\mathbb{Z}/p^n\mathbb{Z}$ is the one characterized by $\bar{\lambda}^*(\operatorname{dlog}(P_n^{\mathrm{univ}})) = \bar{\lambda}^{-1} \cdot \operatorname{dlog}(P_n^{\mathrm{univ}})$). We also have an isomorphism $\gamma_\lambda\colon \bar{\lambda}^*\underline{\omega}^\sharp \to \underline{\omega}^\sharp$ (multiplication by $\lambda \in \mathbb{Z}_p^\times$) which, after pull-back to $\mathfrak{S}$, induces $\psi^*(\gamma_\lambda)\colon (\bar{\lambda} \circ \psi)^*\underline{\omega}^\sharp = \psi^*(\bar{\lambda}^*\underline{\omega}^\sharp) \to \psi^*\underline{\omega}^\sharp$. Then $\lambda * (\psi, v) = \big(\bar{\lambda} \circ \psi, v \circ \psi^*(\gamma_\lambda)\big)$.

There are analogous actions of $\mathfrak{T}$ and $\mathfrak{T}^{\mathrm{ext}}$ on $\mathbb{V}_0(\mathscr{H}^\sharp, s)$ over $\mathfrak{IG}_{n,r,I}$ and over $\mathfrak{X}_{r,I}$, respectively.

The actions of $\mathfrak{T}$ and $\mathfrak{T}^{\mathrm{ext}}$ defined above can be used to define actions on the (push-forwards of the) structure sheaves $\mathscr{O}_{\mathbb{V}_0(\underline{\omega}^\sharp,s)}$ and $\mathscr{O}_{\mathbb{V}_0(\mathscr{H}^\sharp,s)}$ as follows.

(1) Regard both $\mathfrak{T}$ and $f_{n,*}\mathscr{O}_{\mathbb{V}_0(\underline{\omega}^\sharp,s)}$ as sheaves on $\mathfrak{IG}_{n,r,I}$ (i.e., we view $\mathfrak{T}$ only as a sheaf on the Zariski site of $\mathfrak{IG}_{n,r,I}$ and forget that it is a formal scheme). Given an open $\mathfrak{S} \hookrightarrow \mathfrak{IG}_{n,r,I}$, we observe that $\mathbb{V}_0(\underline{\omega}^\sharp, s)(\mathfrak{S})$ is (a subset of the) dual to $f_{n,*}\mathscr{O}_{\mathbb{V}_0(\underline{\omega}^\sharp,s)}$. For $t \in \mathfrak{T}(\mathfrak{S})$ and $x \in f_{n,*}\mathscr{O}_{\mathbb{V}_0(\underline{\omega}^\sharp,s)}(\mathfrak{S})$, we define $t * x$ to be the element of $f_{n,*}\mathscr{O}_{\mathbb{V}_0(\underline{\omega}^\sharp,s)}(\mathfrak{S})$ satisfying that

$$v(t*x) = (t*v)(x) \quad \text{for all } v \in \mathbb{V}_0(\underline{\omega}^\sharp, s)(\mathfrak{S}).$$

(2) Similarly, regard both $\mathfrak{T}^{\mathrm{ext}}$ and $f_{0,*}\mathscr{O}_{\mathbb{V}_0(\underline{\omega}^\sharp,s)}$ as sheaves on $\mathfrak{X}_{r,I}$. Given an open $\mathfrak{S} \hookrightarrow \mathfrak{X}_{r,I}$, we can consider the pull-back $\mathfrak{S} \times_{\mathfrak{X}_{r,I}} \mathfrak{IG}_{n,r,I} \hookrightarrow \mathfrak{IG}_{n,r,I}$ and we observe that $\mathbb{V}_0(\underline{\omega}^\sharp, s)(\mathfrak{S} \times_{\mathfrak{X}_{r,I}} \mathfrak{IG}_{n,r,I})$ is (a subset of the) dual to $f_{0,*}\mathscr{O}_{\mathbb{V}_0(\underline{\omega}^\sharp,s)}(\mathfrak{S}) = f_{n,*}\mathscr{O}_{\mathbb{V}_0(\underline{\omega}^\sharp,s)}(\mathfrak{S} \times_{\mathfrak{X}_{r,I}} \mathfrak{IG}_{n,r,I})$. For $t \in \mathfrak{T}^{\mathrm{ext}}(\mathfrak{S})$ and $x \in f_{0,*}\mathscr{O}_{\mathbb{V}_0(\underline{\omega}^\sharp,s)}(\mathfrak{S})$, we define $t * x$ to be the element of $f_{0,*}\mathscr{O}_{\mathbb{V}_0(\underline{\omega}^\sharp,s)}(\mathfrak{S})$ satistying that

$$v(t*x) = (t*v)(x) \quad \text{for all } v \in \mathbb{V}_0(\underline{\omega}^\sharp, s)(\mathfrak{S} \times_{\mathfrak{X}_{r,I}} \mathfrak{IG}_{n,r,I}).$$

There are analogous actions of $\mathfrak{T}$ and $\mathfrak{T}^{\mathrm{ext}}$ on $g_{n,*}\mathscr{O}_{\mathbb{V}_0(\mathscr{H}^\sharp,s)}$ and $g_{0,*}\mathscr{O}_{\mathbb{V}_0(\mathscr{H}^\sharp,s)}$, respectively. The definition of the actions will become much clearer in the next talk with the local calculations.

Recall that, by assumption 8, the universal character $\kappa_I \colon \mathbb{Z}_p^\times \to \Lambda_I^\times$ admits an analytic expansion on elements with which it makes sense to evaluate it at sections of $\mathfrak{T}^{\mathrm{ext}}$.

Definition 9. We define the following *modular sheaves*:

(1) $\underline{\omega}^{\kappa_I,\mathrm{f}} = (h_{1,*}\mathscr{O}_{\mathfrak{IG}_{1,r,I}} \hat{\otimes}_{\Lambda^0} \Lambda)[\kappa_{I,\mathrm{f}}^{-1}]$ (i.e., the subsheaf on which $(\mathbb{Z}/p\mathbb{Z})^\times$ acts via $\kappa_{I,\mathrm{f}}^{-1}$);

(2) $\underline{\omega}^{\kappa_I,0} = f_{0,*}\mathscr{O}_{\mathbb{V}_0(\underline{\omega}^\sharp,s)}[\kappa_I^0]$ (i.e., the subsheaf on which $\mathfrak{T}^{\mathrm{ext}}$ acts via κ_I^0);

(3) $\underline{\omega}^{\kappa_I} = \underline{\omega}^{\kappa_I,0} \hat{\otimes}_{\mathscr{O}_{\mathfrak{X}_{r,I}}} \underline{\omega}^{\kappa_I,\mathrm{f}}$;

(4) $\mathbb{W}^0_{\kappa_I} = \begin{cases} g_{0,*}\mathscr{O}_{\mathbb{V}_0(\mathscr{H}^\sharp,s)}[\kappa_I^0] & \text{if } I = [p^a, p^b] \text{ with } b \neq \infty, \\ \varprojlim_{n\to\infty} \mathbb{W}^0_{\kappa_{[p,p^n]}} & \text{if } I = [p, \infty], \end{cases}$ and

(5) $\mathbb{W}_{\kappa_I} = \mathbb{W}^0_{\kappa_I} \hat{\otimes}_{\mathscr{O}_{\mathfrak{X}_{r,I}}} \underline{\omega}^{\kappa_I,\mathrm{f}}$.

Remark. In the definition of $\underline{\omega}^{\kappa_{I,\mathrm{f}}}$, the action of the character $\kappa_{I,\mathrm{f}}$ appears with an inverse because the actions of the torus on the Igusa curve and on the vector bundle with marked section are "kind of dual". Namely, the Igusa curve parametrizes P_n^{univ}, whereas the vector bundle parametrizes functions that (when reduced modulo some ideal of definition) send $\mathrm{dlog}(P_n^{\mathrm{univ}})$ to 1. Thus, if we multiply P_n^{univ} by a number, then we have to multiply those functions by its inverse to preserve the property that $\mathrm{dlog}(P_n^{\mathrm{univ}}) \mapsto 1$. For a more precise statement, see lemma 3.7 in the "Triple product…" article [2].[1]

[1] I'm not totally convinced about this. It seems to me that Andreatta–Iovita mix the two possible identifications $\mathrm{Gal}(\mathcal{IG}_{n,r,I}/\mathcal{X}_{r,I}) \cong (\mathbb{Z}/p^n\mathbb{Z})^\times$. Namely, in lemma 3.7 they use the "inverse" identification of that used in the definition of the action of the extended torus. That would account for the "duality" twice, which is wrong.

3 The Gauss–Manin connection

3.1 Local description of the sheaves

We are going to describe the sheaves constructed in the previous talk affine locally. Fix r, n, I as in assumption 8. Recall that we constructed vector bundles with marked sections using $\mathscr{H} = \mathrm{H}^1_{\mathrm{dR}}(\mathfrak{E}_{n,r,I}/\mathfrak{IG}_{n,r,I})$ and $\underline{\omega} = \pi_*\Omega^1_{\mathfrak{E}_{n,r,I}/\mathfrak{IG}_{n,r,I}}$. Choose affine opens

$$\begin{tikzcd} \mathrm{Spf}(R) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathfrak{IG}_{n,r,I} \arrow[d, "h_n"] \\ \mathrm{Spf}(R_0) \arrow[r, hook] & \mathfrak{X}_{r,I} \end{tikzcd}$$

such that the rings R_0 and R are α–torsion-free (admissibility) and the sheaves $\mathscr{H}|_R$ and $\underline{\omega}|_R$ are free (of ranks 2 and 1, respectively). In fact, the sheaves $\mathscr{H}$ and $\underline{\omega}$ are obtained by pull-back from the corresponding sheaves on $\mathfrak{X}_{r,I}$, that we write with the same symbols by abuse of notation. We choose an R_0–basis ω, η of $\mathscr{H}|_{R_0}$ adapted to the Hodge filtration (i.e., ω is an R_0–basis of $\underline{\omega}|_{R_0}$). We assume also that Hdg is free and generated by $E_{p-1}(\mathfrak{E}_{r,I}/R_0, \omega)$ and set

$$\delta = E_{p-1}(\mathfrak{E}_{r,I}/R_0, \omega)^{\frac{1}{p-1}} \quad \text{and} \quad \beta_n = p^n \cdot E_{p-1}(\mathfrak{E}_{r,I}/R_0, \omega)^{-\frac{p^n}{p-1}}$$

(generators of the ideal sheaves $\underline{\delta}|_R$ and $\underline{\beta}_n|_R$, respectively)

Fix a basis (f, e) of $\mathscr{H}^\sharp|_R$ with the property that $f \equiv s = \mathrm{dlog}(P_n^{\mathrm{univ}}) \bmod \beta_n R$. It is easy to see that we can identify

$$\mathbb{V}_0(\underline{\omega}^\sharp, s)(R) = \{\, v \in \mathrm{Hom}_R(\underline{\omega}^\sharp|_R, R) : (v \mod \beta_n)(s) = 1 \,\} = (1 + \beta_n R) f^\vee$$

and

$$\begin{aligned} \mathbb{V}_0(\mathscr{H}^\sharp, s)(R) &= \{\, v \in \mathrm{Hom}_R(\mathscr{H}^\sharp|_R, R) : (v \mod \beta_n)(s) = 1 \,\} \\ &= (1 + \beta_n R) f^\vee + R e^\vee, \end{aligned}$$

where $f^\vee, e^\vee$ is the dual basis of f, e. In fact, by the construction in the proof of theorem 4, $g_{n,*}\mathscr{O}_{\mathbb{V}_0(\mathscr{H}^\sharp, s)|_R}$ is (the coherent sheaf associated with) $R\langle Z, Y\rangle$ and $f_{n,*}\mathscr{O}_{\mathbb{V}_0(\underline{\omega}^\sharp, s)|_R}$ is (the coherent sheaf associated with) $R\langle Z\rangle$, where the formal variable $Z = \frac{X-1}{\beta_n}$ corresponds to $\frac{f-1}{\beta_n}$ and the variable Y corresponds to e. Therefore, we identify $(1 + \beta_n a) f^\vee + b e^\vee$ with the morphism of R–algebras $R\langle Z, Y\rangle \to R$

defined by

$$Z \mapsto \frac{(1+\beta_n a)-1}{\beta_n} = a \quad \text{and} \quad Y \mapsto b.$$

Recall that we defined the action of $\mathfrak{T}(R) = 1+\beta_n R$ on $\mathbb{V}_0(\underline{\omega}^\sharp, s)(R)$ by

$$t * (1+\beta_n a)f^\vee = t(1+\beta_n a)f^\vee = \big(1+\beta_n(s+a+\beta_n sa)\big)f^\vee \quad \text{if } t = 1+\beta_n s.$$

We define an action of $\mathfrak{T}(R)$ on $f_{n,*}\mathscr{O}_{\mathbb{V}_0(\underline{\omega}^\sharp,s)|_R}$ by imposing the condition

$$v(t*Z) = (t*v)(Z) \quad \text{for all } v \in (1+\beta_n R)f^\vee.$$

A direct computation shows that

$$t*Z = \frac{t-1}{\beta_n} + tZ.$$

Lemma 10. *The sheaf $f_{n,*}\mathscr{O}_{\mathbb{V}_0(\underline{\omega}^\sharp,s)}[\kappa_I]$ on $\mathfrak{IG}_{n,r,I}$ is locally given by*

$$R\langle Z\rangle[\kappa_I] = R\cdot\kappa_I(1+\beta_n Z).$$

Proof. We have to prove that $R\langle Z\rangle[\kappa_I] \subseteq R\cdot\kappa_I(1+\beta_n Z)$ (the other inclusion is straight-forward).

We first claim that $R\langle Z\rangle^{\mathfrak{T}(R)=1} = R$. Indeed, take $g \in R\langle Z\rangle^{\mathfrak{T}(R)=1}$ and express it as

$$g = \sum_{m=0}^{\infty} a_m Z^m.$$

For every $t \in \mathfrak{T}(R)$,

$$g(Z) = t*g(Z) = \sum_{m=0}^{\infty} a_m\Big(\frac{t-1}{\beta_n} + tZ\Big)^m$$

and, evaluating at $Z=0$, we deduce that

$$\sum_{m=1}^{\infty} a_m u^m = 0 \quad \text{for all } u = \frac{t-1}{\beta_n} \in R.$$

By Weierstrass's preparation theorem, this is only possible if $a_m = 0$ for all $m \geq 1$. This completes the proof of the claim.

Now, for any $g \in R\langle Z\rangle[\kappa_I]$, we have that

$$\frac{g}{\kappa_I(1+\beta_n Z)} \in R\langle Z\rangle^{\mathfrak{T}(R)=1} = R. \qquad \square$$

We can describe $\underline{\omega}^{\kappa_I,0}$ over $\mathfrak{X}_{r,I}$ locally as well. Given $\lambda \in \mathbb{Z}_p^\times$, we write $\bar{\lambda}$ for its image in $(\mathbb{Z}/p^n\mathbb{Z})^\times \cong \mathrm{Gal}(\mathcal{IG}_{n,r,I}/\mathcal{X}_{r,I})$. The identification with the Galois group (unique up to sign) is determined by $\bar{\lambda} * P_n^{\mathrm{univ}} = \bar{\lambda}^{-1} \cdot P_n^{\mathrm{univ}}$ (i.e., $\bar{\lambda} * P_n^{\mathrm{univ}}$ is the image of $\bar{\lambda}^{-1}$ under the universal trivialization $\mathbb{Z}/p^n\mathbb{Z} \to (H_n)^\vee$). Up to changing ω, we may assume that $f = \delta\omega$ and still $f \equiv \mathrm{dlog}(P_n^{\mathrm{univ}}) \bmod \beta_n R$. Then $\underline{\omega}^\sharp|_R = fR$ and $\bar{\lambda}^*\underline{\omega}^\sharp|_R = \lambda^{-1} fR$ and we recall that the action of $\mathbb{Z}_p^\times$ on $\mathbb{V}_0(\underline{\omega}^\sharp, s)(R_0)$ was given by composition with

$$\begin{aligned} \gamma_\lambda\colon \bar{\lambda}^*\underline{\omega}^\sharp|_R &\longrightarrow \underline{\omega}^\sharp|_R \\ \lambda^{-1} f &\longmapsto f \end{aligned}$$

(i.e., multiplication by λ). More precisely, consider a point $(\psi, v) \in \mathbb{V}_0(\underline{\omega}^\sharp, s)(R_0)$, where $\psi\colon \mathrm{Spf}(R_0) \to \mathrm{Spf}(R)$ is a section of the structure morphism h_n and then $v = (1+\beta^n a) f^\vee \colon \underline{\omega}^\sharp|_R \to R_0$ is a morphism of R–modules (note that R_0 is an R–algebra via ψ) such that $(v \bmod \beta_n)(s) = 1$. Then $\lambda * (\psi, v) = (\bar{\lambda} \circ \psi, v')$, where

$$v' = (1+\beta_n a)(\lambda^{-1} f)^\vee \colon \bar{\lambda}^*\underline{\omega}^\sharp|_R \xrightarrow{\psi^*(\gamma_\lambda)} \underline{\omega}^\sharp|_R \xrightarrow{\ v\ } R_0$$

is a morphism of R–modules (but now the R–algebra structure on R_0 comes from $\bar{\lambda} \circ \psi$ instead!). In fact, by the universal property of fibre products we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Spf}(R_0) \arrow[dr, "v\circ\psi^*(\gamma_\lambda)"] \arrow[drrr, bend left, "v'"] \arrow[ddr, bend right, "\psi"'] & & & \\
& \mathbb{V}_0(\bar{\lambda}^*\underline{\omega}^\sharp, \bar{\lambda}^* s)|_R \arrow[rr] \arrow[d, "\widetilde{f}_n"] \arrow[drr, phantom, "\ulcorner"] & & \mathbb{V}_0(\underline{\omega}^\sharp, s)|_R \arrow[d, "f_n"] \\
& \mathrm{Spf}(R) \arrow[rr, "\bar{\lambda}"] \arrow[dr] & & \mathrm{Spf}(R) \arrow[dl] \\
& & \mathrm{Spf}(R_0) &
\end{tikzcd}$$

that we can use to describe v'. Recall that $f_{n,*}\mathscr{O}_{\mathbb{V}_0(\underline{\omega}^\sharp, s)|_R}$ is (the coherent sheaf associated with) $R\langle Z \rangle$, where the formal variable Z corresponds to $\frac{f-1}{\beta_n}$, and $\widetilde{f}_{n,*}\mathscr{O}_{\mathbb{V}_0(\bar{\lambda}^*\underline{\omega}^\sharp, \bar{\lambda}^* s)|_R}$ is (the coherent sheaf associated with) $R\langle \widetilde{Z} \rangle$, where the formal variable $\widetilde{Z}$ corresponds to $\frac{\lambda^{-1}f-1}{\beta_n}$. From the relation

$$\frac{f-1}{\beta_n} = \lambda\Big(\frac{\lambda^{-1}f-1}{\beta_n}\Big) + \frac{\lambda-1}{\beta_n},$$

we see that the top horizontal arrow is given by

$$\begin{array}{ccc} R\langle\widetilde{Z}\rangle & \longleftarrow & R\langle Z\rangle \\ \lambda\widetilde{Z} + \dfrac{\lambda-1}{\beta_n} & \longleftarrow\!\!\shortmid & Z \end{array}$$

(this is a morphism of R_0–algebras, but not of R–algebras!). We can identify v with the morphism of R–algebras (the structure given by ψ) defined by

$$Z \mapsto \frac{(1+\beta_n a)-1}{\beta_n} = a$$

and similarly $v \circ \psi^*(\gamma_\lambda)$ with the morphism of R–algebras (the structure given by ψ again) defined by

$$\widetilde{Z} \mapsto \frac{(1+\beta_n a)-1}{\beta_n} = a.$$

Therefore, v' is identified with

$$\begin{array}{ccccc} R\langle Z\rangle & \longrightarrow & R\langle\widetilde{Z}\rangle & \longrightarrow & R_0 \\ Z & \longmapsto & \lambda\widetilde{Z} + \dfrac{\lambda-1}{\beta_n} & \longmapsto & \lambda a + \dfrac{\lambda-1}{\beta_n} \end{array}$$

(morphism of R_0–algebras, but not of R–algebras!). Since the action of $\mathfrak{T}^{\mathrm{ext}}(R_0)$ on $f_{0,*}\mathscr{O}_{\mathbb{V}_0(\underline{\omega}^\sharp,s)|_R}$ is defined by imposing the condition

$$v(\lambda * Z) = (\lambda * v)(Z) \quad \text{for all } v \in (1+\beta_n R_0)f^\vee,$$

we conclude that

$$\lambda * Z = \frac{\lambda-1}{\beta_n} + \lambda Z.$$

Therefore,

$$\lambda * \kappa_I^0(1+\beta_n Z) = \kappa_I^0(\lambda)\kappa_I^0(1+\beta_n Z)$$

and $\underline{\omega}^{\kappa_I,0}$ is locally given[2] by

$$R\langle Z\rangle[\kappa_I^0] = R_0 \cdot \kappa_I^0(1+\beta_n Z).$$

For $\mathbb{V}_0(\mathscr{H}^\sharp, s)$, we have similarly defined actions. In particular,

$$t * Z = \frac{t-1}{\beta_n} + tZ \quad \text{and } t * Y = tY.$$

Lemma 11. *The sheaf $g_{n,*}\mathscr{O}_{\mathbb{V}_0(\mathscr{H}^\sharp,s)}[\kappa_I]$ on $\mathfrak{IG}_{n,r,I}$ is locally given by*

$$R\langle Z, Y\rangle[\kappa_I] = \left\{ \sum_{m=0}^{\infty} a_m \kappa_I(1+\beta_n Z) \frac{Y^m}{(1+\beta_n Z)^m} : a_m \to 0 \text{ as } m \to \infty \right\}.$$

Proof. The proof is completely analogous to that of lemma 10. We first prove that $R\langle Z, Y\rangle^{\mathfrak{T}(R)=1} = R\langle V\rangle$, where

$$V = \frac{Y}{1+\beta_n Z}.$$

If $g \in R\langle Z, Y\rangle$ is expressed as

$$g = \sum_{m,i} a_{m,i} Z^m Y^i = \sum_{m,i} b_{m,i} Z^m V^i,$$

we see that $g \in R\langle Z, Y\rangle^{\mathfrak{T}(R)=1}$ implies that

$$g = t * g = \sum_{m,i} b_{m,i} \Big(\frac{t-1}{\beta_n} + tZ\Big)^m V^i.$$

Comparing the coefficients of V^i for a fixed $i \geq 0$, we get

$$\sum_{m\geq 0} b_{m,i} \Big(\frac{t-1}{\beta_n}\Big)^m = \sum_{m\geq 0} b_{m,i} Z^m.$$

The same trick as before, evaluating at $Z = 0$, shows that

$$\sum_{m\geq 1} b_{m,i} u^m = 0 \quad \text{for all } u = \frac{t-1}{\beta_n} \in R$$

and so the result follows by Weierstrass's preparation theorem.

[2] This part in subsection 3.2.1 of the Andreatta–Iovita's paper "Triple product…" [2] is not very detailed and has several mistakes. Ju-Feng and I have had many discussions about these things (and how the finite part of the character, $\kappa_{I,\mathrm{f}}$, should come into play if we want to compute $\underline{\omega}^{\kappa_I}$ instead). I'm not very satisfied with this part, as we should write better what happens with the extension R/R_0. In loc. cit., Andreatta–Iovita wrote that the explicit formula should be valid only for $n = 1$. Presumably because in that case R is (essentially?) $R_0 \otimes_{\Lambda_I^0} \Lambda^I$ and we have a basis given by the Teichmüller lifts of $(\mathbb{Z}/p\mathbb{Z})^\times$. We should ask Adrian how this works at some point.

Finally, for any $g \in R\langle Z, Y\rangle[\kappa_I]$,

$$\frac{g}{\kappa_I(1+\beta_n Z)} \in R\langle Z, Y\rangle^{\mathfrak{T}(R)=1} = R\langle V\rangle. \qquad \square$$

We are going to use the following important theorem.

Theorem 12. *(This is theorem 3.11 of Andreatta–Iovita's paper "Triple product…" [2].)*

(1) *The action of the torus* $\mathfrak{T}^{\mathrm{ext}}$ *on the sheaf* $f_{0,*}\mathscr{O}_{\mathbb{V}_0(\mathscr{H}^\sharp,s)}$ *(over* $\mathfrak{X}_{r,I}$*) preserves the filtration. Write* $\mathrm{Fil}_j\,\mathbb{W}^0_{\kappa_I} = \mathrm{Fil}_j\, f_{0,*}\mathscr{O}_{\mathbb{V}_0(\mathscr{H}^\sharp,s)}[\kappa_I^0]$*. Then*
   - (i) $\mathrm{Fil}_j\,\mathbb{W}^0_{\kappa_I}$ *is a locally free* $\mathscr{O}_{\mathfrak{X}_{r,I}}$*–module,*
   - (ii) $\mathbb{W}^0_{\kappa_I}$ *is the* α*–adic completion of* $\varinjlim_j \mathrm{Fil}_j\,\mathbb{W}^0_{\kappa_I}$ *and*
   - (iii) $\mathrm{Fil}_0\,\mathbb{W}^0_{\kappa_I} \cong \underline{\omega}^{\kappa_I,0}$ *and* $\mathrm{Gr}_j\,\mathbb{W}^0_{\kappa_I} \cong \underline{\omega}^{\kappa_I,0}\,\hat{\otimes}_{\mathscr{O}_{\mathfrak{X}_{r,I}}}\,\mathrm{Hdg}^j\,\underline{\omega}^{-2j}$.

(2) *Defining* $\mathrm{Fil}_j\,\mathbb{W}_{\kappa_I} = \mathrm{Fil}_j\,\mathbb{W}^0_{\kappa_I}\,\hat{\otimes}_{\mathscr{O}_{\mathfrak{X}_{r,I}}}\,\underline{\omega}^{\kappa_I,\mathrm{f}}$*, the filtered* $\mathscr{O}_{\mathfrak{X}_{r,I}}$*–module* $\mathbb{W}_{\kappa_I}$ *satisfies properties analogous to (i), (ii) and (iii) above (replacing* $\underline{\omega}^{\kappa_I,0}$ *with* $\underline{\omega}^{\kappa_I}$*).*

(3) *Specializing* κ_I *at a classical weight* $k \in \mathbb{Z}_{>0}$*, there is a canonical identification*

$$\mathrm{Sym}^k(\mathscr{H})\left[\frac{1}{\alpha}\right] = \mathrm{Fil}_k\,\mathbb{W}_k\left[\frac{1}{\alpha}\right]$$

(as sheaves on $\mathcal{X}_{r,I}$*) that is compatible with the filtrations (recall that on* $\mathrm{Sym}^k(\mathscr{H})$ *we have the Hodge filtration).*

3.2 The Gauss–Manin connection

From now on, we assume further that $I \subset [0,\infty)$. There is a Gauss–Manin connection ∇ on $\mathscr{H}$ and we would like to "restrict" it to a connection on $\mathscr{H}^\sharp \subset \mathscr{H}$. However, this is not possible over $\mathfrak{IG}_{n,r,I}$ and we have to modify it.

Let $\mathcal{IG}'_{n,r,I} \to \mathcal{IG}_{n,r,I}$ be the $\mathcal{IG}_{n,r,I}$–adic space parametrizing trivializations $\mathcal{E}[p^n] \cong (\mathbb{Z}/p^n\mathbb{Z})^2$ compatible with the trivializations $(H_n)^\vee \cong \mathbb{Z}/p^n\mathbb{Z}$. Define $\mathfrak{IG}'_{n,r,I}$ to be the normalization of $\mathfrak{IG}_{n,r,I}$ in $\mathcal{IG}'_{n,r,I}$.

Fact. The Gauss–Manin connection on $\mathscr{H} = \mathrm{H}^1_{\mathrm{dR}}(\mathfrak{E}_{n,r,I}/\mathfrak{IG}'_{n,r,I})$ provides an integrable connection $\nabla^\sharp$ on $\mathscr{H}^\sharp$ over $\mathfrak{IG}'_{n,r,I}$.

Remark. Again by abuse of notation we use the same symbols for the sheaves on $\mathfrak{X}_{r,I}$ and for their pull-backs to $\mathfrak{IG}_{n,r,I}$ or $\mathfrak{IG}'_{n,r,I}$. If it is not clear from the context, we will specify the base space.

Fix affine opens

$$\begin{tikzcd}
\mathrm{Spf}(R') \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathfrak{IG}'_{n,r,I} \arrow[d] \arrow[dd, bend left, "h'_n"] \\
\mathrm{Spf}(R) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathfrak{IG}_{n,r,I} \arrow[d, "h_n"] \\
\mathrm{Spf}(R_0) \arrow[r, hook] & \mathfrak{X}_{r,I}
\end{tikzcd}$$

as in the previous section, with a basis ω, η of $\mathscr{H}|_{R_0}$. In this case, we consider δ and β_n in R'. We obtain a basis $f = \delta\omega, e = \delta^p \eta$ of $\mathscr{H}^\sharp|_{R'}$ (i.e., locally over $\mathfrak{IG}'_{n,r,I}$).

Write $\mathfrak{P}^{(1)}_{R'/\Lambda_I^0}$ for the first infinitesimal neighbourhood of $\mathrm{Spf}(R')$ with respect to the diagonal morphism $\Delta\colon \mathrm{Spf}(R') \to \mathrm{Spf}(R' \hat{\otimes}_{\Lambda_I^0} R')$. Let $I(\Delta)$ be the ideal of definition of $\Delta(\mathrm{Spf}(R'))$ (we may assume that Δ is a closed immersion over $\mathrm{Spf}(R')$), so that $\mathfrak{P}^{(1)}_{R'/\Lambda_I^0} = \mathrm{Spf}\big((R' \hat{\otimes}_{\Lambda_I^0} R')/I(\Delta)^2\big)$. Let $j_1, j_2\colon \mathfrak{P}^{(1)}_{R'/\Lambda_I^0} \to \mathrm{Spf}(R')$ be the morphisms induced by the two natural projections.

By Grothendieck's formalism, we have an isomorphism

$$\varepsilon^\sharp\colon j_2^*\mathscr{H}^\sharp|_{R'} \to j_1^*\mathscr{H}^\sharp|_{R'}$$

(with some extra conditions) corresponding to the Gauss–Manin connection $\nabla^\sharp$. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2\big(\mathfrak{P}^{(1)}_{R'/\Lambda_I^0}\big)$$

be the matrix of $\varepsilon^\sharp$ with respect to the bases $1 \otimes f, 1 \otimes e$ of $j_2^*\mathscr{H}^\sharp|_{R'}$ and $f \otimes 1, e \otimes 1$ of $j_1^*\mathscr{H}^\sharp|_{R'}$, so that

$$\varepsilon^\sharp(1 \otimes f \quad 1 \otimes e) = (f \otimes 1 \quad e \otimes 1)\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Lemma 13.

(1) *We can express* $a = 1 + a_0$ *and* $d = 1 + d_0$ *with* $a_0, b, c, d_0 \in I(\Delta)$. *In particular,* $a_0^2 = b^2 = c^2 = d_0^2 = 0$ *in* $\mathfrak{P}^{(1)}_{R'/\Lambda_I^0}$.

(2) *Regarding* $a_0, b, c, d_0 \in I(\Delta)/I(\Delta)^2 \cong \Omega^1_{R'/\Lambda_I^0}$,

$$a_0, b, c, d_0 \in \frac{1}{\mathrm{Hdg}}\big((h'_n)^*\Omega^1_{\mathfrak{X}_{r,I}/\Lambda_I^0}\big)\Big|_{R'} = \frac{1}{\mathrm{Hdg}}(h'_n)^*\Omega^1_{R_0/\Lambda_I^0}.$$

Moreover, $\mathrm{Hdg} \cdot c$ *is (the pull-back of) the Kodaira–Spencer differential* $\mathrm{KS}(\omega, \eta)$.

Proof. The first part is just restating that $\varepsilon^\sharp \equiv \mathrm{id} \bmod I(\Delta)$.

We can express $\delta^{p-1} = E_{p-1}(\mathfrak{E}_{n,r,I}/R', \omega) = u|_{R'}$ for some $u \in R_0$ (a generator of Hdg), so

$$du|_{R'} = d\delta^{p-1} = (p-1)\delta^{p-2}\, d\delta = (p-1)u|_{R'}\, \mathrm{dlog}(\delta).$$

Therefore,

$$\mathrm{dlog}(\delta) = \frac{1}{p-1}\frac{du|_{R'}}{u|_{R'}} \in \frac{1}{\mathrm{Hdg}}\Omega^1_{R_0/\Lambda_I^0}|_{R'}.$$

On the other hand, the Kodaira–Spencer isomorphism

$$\mathrm{KS}\colon \underline{\omega} \hookrightarrow \mathscr{H} \xrightarrow{\nabla} \mathscr{H}\,\hat{\otimes}_{\mathscr{O}_{\mathfrak{X}_{r,I}}}\Omega^1_{\mathfrak{X}_{r,I}/\Lambda_I^0} \twoheadrightarrow \underline{\omega}^{-1}\,\hat{\otimes}_{\mathscr{O}_{\mathfrak{X}_{r,I}}}\Omega^1_{\mathfrak{X}_{r,I}/\Lambda_I^0}$$

provides a basis $\Theta = \mathrm{KS}(\omega,\eta)$ of $\Omega^1_{R_0/\Lambda_I^0}$ defined by $\mathrm{KS}(\omega) = \bar{\eta}\otimes \mathrm{KS}(\omega,\eta)$ (where $\bar{\eta}$ is the image of η in $\omega^{-1}|_{R_0}$). Thus, we can write

$$\begin{aligned} \nabla(\omega) &= x\omega\otimes\Theta + \eta\otimes\Theta \quad \text{and} \\ \nabla(\eta) &= y\omega\otimes\Theta + z\eta\otimes\Theta \end{aligned}$$

for some $x,y,z\in R_0$.

Now, over R' (omitting the pull-back from the notation), if we write $du = t_u\Theta$, we can express

$$\begin{aligned} \nabla^\sharp(f) =&\ \nabla(\delta\omega) = \delta\big(x\omega\otimes\Theta + \eta\otimes\Theta\big) + \delta\omega\otimes\mathrm{dlog}(\delta) \\ &= xf\otimes\Theta + \frac{e}{\delta^{p-1}}\otimes\Theta + f\otimes\frac{du}{(p-1)u} \\ &= \Big(x + \frac{t_u}{(p-1)u}\Big)(f\otimes\Theta) + \frac{1}{\delta^{p-1}}(e\otimes\Theta) \end{aligned}$$

and similarly

$$\begin{aligned} \nabla^\sharp(e) =&\ \nabla(\delta^p\omega) = \delta^p\big(y\omega\otimes\Theta + z\eta\otimes\Theta\big) + p\delta^p\eta\otimes\mathrm{dlog}(\delta) \\ &= y\delta^{p-1}f\otimes\Theta + ze\otimes\Theta + pe\otimes\frac{du}{(p-1)u} \\ &= y\delta^{p-1}(f\otimes\Theta) + \Big(z + \frac{pt_u}{(p-1)u}\Big)(e\otimes\Theta). \end{aligned}$$

Recall that $\varepsilon^\sharp(1\otimes g) = g\otimes 1 + \nabla^\sharp(g)$. That is,

$$\nabla^\sharp(f) = a_0(f\otimes 1) + c(e\otimes 1) \quad \text{and} \quad \nabla^\sharp(e) = b(f\otimes 1) + d_0(e\otimes 1).$$

Comparing coefficients, we see that

$$\begin{aligned}
a_0 &= \Big(x + \frac{t_u}{(p-1)u}\Big)\Theta,\\
b &= y\delta^{p-1}\Theta,\\
c &= \frac{1}{\delta^{p-1}}\Theta,\\
d_0 &= \Big(z + \frac{pt_u}{(p-1)u}\Big)\Theta.
\end{aligned}$$

This completes the proof, as $u|_{R'} = \delta^{p-1}$ generates Hdg. $\square$

We next use these computations to define a Gauss–Manin connection ∇_{κ_I} on $\mathbb{W}_{\kappa_I}$ over the generic fibre $\mathcal{X}_{r,I}$ of $\mathfrak{X}_{r,I}$.

Consider the commutative diagram

$$\begin{tikzcd}
\mathbb{V}_0(\mathscr{H}^\sharp, s)|_{R'} \arrow[r, hook] \arrow[d] & \mathbb{V}_0(\mathscr{H}^\sharp, s) \arrow[d, "g'_n"] \\
\mathrm{Spf}(R') \arrow[r, hook] & \mathfrak{I}\mathfrak{G}'_{n,r,I}
\end{tikzcd}$$

(where the vertical arrows are the structure morphisms). A computation analogous to that of lemma 11 (but over $\mathfrak{I}\mathfrak{G}'_{n,r,I}$ instead of $\mathfrak{I}\mathfrak{G}_{n,r,I}$) shows that,

$$g'_{n,*}\mathscr{O}_{\mathbb{V}_0(\mathscr{H}^\sharp,s)}[\kappa_I]\big|_{R'} = R'\langle V\rangle \cdot \kappa_I(1+\beta_n Z), \quad \text{where } V = \frac{Y}{1+\beta_n Z},$$

and so, for $i \in \{1, 2\}$,

$$j_i^*\big(g'_{n,*}\mathscr{O}_{\mathbb{V}_0(\mathscr{H}^\sharp,s)}[\kappa_I]\big|_{R'}\big) = \big((R' \hat{\otimes}_{\Lambda_I^0} R')/I(\Delta)^2\big)\langle V\rangle \cdot \kappa_I(1+\beta_n Z).$$

Define $\varepsilon_{\kappa_I} : j_2^*\big(g'_{n,*}\mathscr{O}_{\mathbb{V}_0(\mathscr{H}^\sharp,s)}[\kappa_I]\big|_{R'}\big) \to j_1^*\big(g'_{n,*}\mathscr{O}_{\mathbb{V}_0(\mathscr{H}^\sharp,s)}[\kappa_I]\big|_{R'}\big)$ to be the isomorphism induced by $\varepsilon^\sharp$ (i.e., determined by the matrix A). Namely,

$$\begin{aligned}
\varepsilon_{\kappa_I}(1+\beta_n Z) &= \varepsilon^\sharp(1\otimes f) = a(f\otimes 1) + c(e\otimes 1)\\
&= a(1+\beta_n Z) + cY = (1+\beta_n Z)(a + cV)
\end{aligned}$$

and

$$\begin{aligned}
\varepsilon_{\kappa_I}(Y) &= \varepsilon^\sharp(1\otimes e) = b(f\otimes 1) + d(e\otimes 1)\\
&= b(1+\beta_n Z) + dY = (1+\beta_n Z)(b + dV),
\end{aligned}$$

whence

$$\begin{aligned}
\varepsilon_{\kappa_I}\big(\kappa_I(1+\beta_n Z)V^m\big) &= \varepsilon_{\kappa_I}\left(\kappa_I(1+\beta_n Z)\Big(\frac{Y}{1+\beta_n Z}\Big)^m\right) \\
&= \kappa_I\big((1+\beta_n Z)(a+cV)\big)\left(\frac{(1+\beta_n Z)(b+dV)}{(1+\beta_n Z)(a+cV)}\right)^m \\
&= \kappa_I(1+\beta_n Z)\cdot(\kappa_I - m)(a+cV)\cdot(b+dV)^m.
\end{aligned}$$

But we can express

$$(\kappa_I - m)(a+cV) = \exp\big((u_{\kappa_I} - m)\log(1+a_0+cV)\big) = 1 + (u_{\kappa_I} - m)(a_0 + cV)$$

(using the power series expansions of exp and log and that $a_0^2 = c^2 = a_0 c = 0$ in $I(\Delta)/I(\Delta)^2$) and

$$(b+dV)^m = \big(b + (1+d_0)V\big)^m = (1+md_0)V^m + mbV^{m-1}$$

(using the binomial expansions and that $b^2 = d_0^2 = bd_0 = 0$ in $I(\Delta)/I(\Delta)^2$). Therefore, a straight-forward computation shows that $\varepsilon_{\kappa_I}\big(\kappa_I(1+\beta_n Z)V^m\big)$ is given by

$$\kappa_I(1+\beta_n Z)\Big[(u_{\kappa_I} - m)cV^{m+1} + \big(1 + md_0 + (u_{\kappa_I} - m)a_0\big)V^m + mbV^{m-1}\Big].$$

and so

$$\begin{aligned}
\nabla_{\kappa_I}\big(\kappa_I(1+\beta_n Z)V^m\big) &= \varepsilon_{\kappa_I}\big(\kappa_I(1+\beta_n Z)V^m\big) - \kappa_I(1+\beta_n Z)V^m \\
&= \kappa_I(1+\beta_n Z)\big[(u_{\kappa_I} - m)V^{m+1}\otimes c + mV^m \otimes d_0 + \\
&\quad + (u_{\kappa_I} - m)V^m \otimes a_0 + mV^{m-1}\otimes b\big],
\end{aligned}$$

which is an element of $p^{1-n}\kappa_I(1+\beta_n Z)R'\langle V\rangle \hat{\otimes}_{R'} \Omega^1_{R'/\Lambda^0_I}$.

All in all, we obtain a local formula for the Gauss–Manin connection ∇_{κ_I} over $\mathfrak{IG}'_{n,r,I}$ which clearly satisfies Griffiths's transversality. Combining the formula with lemma 13, we obtain the main result of this section.

Theorem 14. *(This is theorem 3.18 of Andreatta–Iovita's paper "Triple product…" [2].)*

(1) *The Gauss–Manin connection ∇_{κ_I} on $g'_{n,*}\mathscr{O}_{\mathbb{V}_0(\mathscr{H}^\sharp,s)}[\kappa_I]$ (over $\mathfrak{IG}'_{n,r,I}$) descends to an integrable connection on $g_{n,*}\mathscr{O}_{\mathbb{V}_0(\mathscr{H}^\sharp,s)}[\kappa_I]$ (over $\mathfrak{IG}_{n,r,I}$) that also satisfies Griffiths's transversality if we invert α.*

(2) *Again after inverting α, the Gauss–Manin connection descends further to an integ-*

rable connection

$$\nabla_{\kappa_I}\colon \mathbb{W}^0_{\kappa_I} \to \mathbb{W}^0_{\kappa_I} \hat{\otimes}_{\mathscr{O}_{\mathfrak{X}_{r,I}}} \Omega^1_{\mathfrak{X}_{r,I}/\Lambda^0_I}[\alpha^{-1}]$$

satisfying Griffiths's transversality and that induces an $\mathscr{O}_{\mathfrak{X}_{r,I}}$–linear morphism

$$\mathrm{Gr}_j \nabla_{\kappa_I}\colon\ \mathrm{Gr}_j \mathbb{W}_{\kappa_I} \to \mathrm{Gr}_{j+1} \mathbb{W}_{\kappa_I} \hat{\otimes}_{\mathscr{O}_{\mathfrak{X}_{r,I}}} \Omega^1_{\mathfrak{X}_{r,I}/\Lambda^0_I}[\alpha^{-1}]$$

on the j–th graded piece that is an isomorphism multiplied by $u_{\kappa_I} - j$.

(3) *Similarly, there is an induced Gauss–Manin connection*

$$\nabla_{\kappa_I}\colon \mathbb{W}_{\kappa_I} \to \mathbb{W}_{\kappa_I} \hat{\otimes}_{\mathscr{O}_{\mathfrak{X}_{r,I}}} \Omega^1_{\mathfrak{X}_{r,I}/\Lambda^0_I}[\alpha^{-1}]$$

with analogous properties.

(4) *If we specialize κ_I at $k \in \mathbb{Z}_{>0}$, the identification $\mathrm{Sym}^k \mathscr{H}[\alpha^{-1}] = \mathrm{Fil}_k \mathbb{W}_k[\alpha^{-1}]$ is compatible with the Gauss–Manin connections on both sides.*

3.3 Nearly overconvergent modular forms

Let $\mathfrak{IG}^{\mathrm{ord}}_{n,r,I}$ denote the open formal subscheme of $\mathfrak{IG}_{n,r,I}$ defined by the inverse image of the ordinary locus of $\mathfrak{X}_I$.

Fact. Over $\mathfrak{IG}^{\mathrm{ord}}_{n,r,I}$, we have $\mathscr{H}^\sharp = \mathscr{H} = \underline{\omega} \oplus \underline{\omega}^{-1}$. Indeed, the first equality is a consequence of Hdg being invertible and the splitting comes from identifying $\underline{\omega}^{-1}$ with the submodule where a fixed lift of the Frobenius acts invertibly.

By the functoriality of the vector bundles with marked sections, we obtain maps

$$\mathbb{V}_0(\mathscr{H}^\sharp, s)^{\mathrm{ord}} = \mathbb{V}_0\big(\mathscr{H}^\sharp|_{\mathfrak{IG}^{\mathrm{ord}}_{n,r,I}}, s\big) \to \mathbb{V}\big(\mathscr{H}^\sharp|_{\mathfrak{IG}^{\mathrm{ord}}_{n,r,I}}\big) \to \mathbb{V}(\underline{\omega}^{-1}),$$

whence we get an isomorphism

$$\mathbb{V}_0(\mathscr{H}^\sharp, s)^{\mathrm{ord}} \to \mathbb{V}_0(\underline{\omega}^\sharp, s)^{\mathrm{ord}} \times_{\mathfrak{IG}^{\mathrm{ord}}_{n,r,I}} \mathbb{V}(\underline{\omega}^{-1}).$$

Therefore, we obtain an isomorphism

$$\mathbb{W}^{\mathrm{ord},0}_{\kappa_I} = \mathbb{W}^0_{\kappa_I}|_{\mathfrak{X}^{\mathrm{ord}}_I} \cong \underline{\omega}^{\kappa^0_I}_{\mathrm{ord}} \hat{\otimes}_{\mathscr{O}_{\mathfrak{X}^{\mathrm{ord}}_I}} \mathrm{Sym}\,\underline{\omega}^{-2}$$

and similarly for $\mathbb{W}^{\mathrm{ord}}_{\kappa_I}$. (This isomorphism follows because, locally, $\mathbb{W}^0_{\kappa_I}$ consists of elements of the form

$$\sum_{m=0}^{\infty} a_m \kappa_I(1+\beta_n Z) \frac{Y^m}{(1+\beta_n Z)^m}$$

and both Y and $1+\beta_n Z$ generate $\underline{\omega}^{-1}$.)

Fact. The space of Katz's p–adic modular forms of weight κ_I^0 (resp. κ_I) is identified with $\mathrm{H}^0(\mathfrak{X}_I^{\mathrm{ord}}, \underline{\omega}_{\mathrm{ord}}^{\kappa_I^0})$ (resp. $\mathrm{H}^0(\mathfrak{X}_I^{\mathrm{ord}}, \underline{\omega}_{\mathrm{ord}}^{\kappa_I^0})$).

We can define the q–expansion map for the sheaf $\mathbb{W}_{\kappa_I}^0$ as the composition

$$\mathrm{H}^0(\mathfrak{X}_{r,I}, \mathbb{W}_{\kappa_I}^0) \xrightarrow{\mathrm{Res}} \mathrm{H}^0(\mathfrak{X}_I^{\mathrm{ord}}, \mathbb{W}_{\kappa_I}^{\mathrm{ord},0}) \longrightarrow \mathrm{H}^0(\mathfrak{X}_I^{\mathrm{ord}}, \underline{\omega}_{\mathrm{ord}}^{\kappa_I^0}) \xrightarrow{q\text{–exp}} \Lambda_I^0[\![q]\!]$$

(and similarly for $\mathbb{W}_{\kappa_I}$).

Definition 15. A p–adic modular form g (in the sense of Katz) of weight κ_I is called a *nearly overconvergent modular form* if there exist r and I as above such that the q–expansion of g in $\Lambda_I^0[\![q]\!]$ lies in the image of the q–expansion map for $\mathbb{W}_{\kappa_I}$.

4 Operations on the sheaves and q–expansions

We keep the notation of the previous talks. In particular, we fix r, n and I as in assumption 8 and use them to construct the sheaves $\underline{\omega}^{\kappa_I,0}$, $\underline{\omega}^{\kappa_I}$, $\mathbb{W}^0_{\kappa_I}$ and $\mathbb{W}_{\kappa_I}$ on the formal scheme $\mathfrak{X}_{r,I}$.

4.1 The U and V operators

Motivation. Let R_0 be a p–adically complete ring. For every $r_0 \in R_0$, let $M_0(N; r_0)$ denote the algebra of Katz's p–adic modular functions of level N and "growth condition" r_0 (i.e., the locus of overconvergence of these functions is defined by the condition $v_p(E_{p-1}^{r_0}) \leq 1$) that are holomorphic at infinity. In his article "p–adic properties of modular schemes and modular forms" [3], Katz used the theory of canonical subgroups to define a Frobenius[3]

$$\varphi\colon M_0(N; r_0^p) \longmapsto M_0(N; r_0) \quad \text{whenever } v_p(r_0) < \frac{1}{p+1}$$

defined as follows. Given a triple (E, ψ, ι), where E is an elliptic curve over some R_0–algebra, $\psi\colon \mu_N \to E$ is a level $\Gamma_1(N)$–structure and $\iota\colon \widehat{E} \to \mathbb{G}_{\mathrm{m}}$ is a trivialization of the formal group, we can form a similar triple (E', ψ', ι') using the quotient $\pi\colon E \to E/H_1 = E'$ by the canonical subgroup H_1 of E and its dual $\check{\pi}$. Namely, there are commutative diagrams

$$\begin{tikzcd} & E \\ \mu_N \arrow[ur, "\psi"] \arrow[dr, "\psi'"'] & \\ & E' \arrow[uu, "\check{\pi}"'] \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \widehat{E} \arrow[dr, "\iota"] & \\ & \mathbb{G}_{\mathrm{m}} \\ \widehat{E}' \arrow[uu, "\check{\pi}"] \arrow[ur, "\iota'"'] & \end{tikzcd}$$

characterizing ψ' and ι', respectively. Then

$$\varphi(f)(E, \psi, \iota) = f(E', \psi', \iota').$$

A computation with Tate curves shows that φ corresponds to the V operator given on q–expansions by

$$\mathrm{V}\Big(\sum_{m \geq 0} a_m q^m\Big) = \sum_{m \geq 0} a_m q^{pm}.$$

[3] Recall that the canonical subgroup is a lift from characteristic p of the kernel of the relative Frobenius morphism of an elliptic curve. Hence the name Frobenius for φ.

Moreover, if $r_0 = 1$ or after inverting p, the morphism φ is finite and flat (hence locally free by noetherianness) and admits a well-defined trace. Again, a computation with Tate curves shows that $\frac{1}{p}\operatorname{tr}(\varphi)$ corresponds to the U operator given on q–expansions by

$$\mathrm{U}\Big(\sum_{m\geq 0} a_m q^m\Big) = \sum_{m\geq 0} a_{pm} q^m.$$

In the setting of Andreatta–Iovita, we have to define a morphism at the level of modular curves[4] that we can use to define the operators U and V on the modular sheaves. We consider the morphism $\Phi\colon \mathfrak{X}_{r+1,I} \to \mathfrak{X}_{r,I}$ defined in terms of moduli by $\mathcal{E} \mapsto \mathcal{E}' = \mathcal{E}/H_1$. We consider as well $i\colon \mathfrak{X}_{r+1,I} \to \mathfrak{X}_{r,I}$ defined in terms of moduli by $\mathcal{E} \mapsto \mathcal{E}$. (More precisely, we can define such morphisms in terms of moduli on the generic fibres and then take normalizations to obtain the morphisms of formal schemes.) We would like to define

$$\mathrm{V}\colon\ \mathrm{H}^0(\mathfrak{X}_{r,I}, \underline{\omega}^{\kappa_I}) \xrightarrow{\Phi^*} \mathrm{H}^0(\mathfrak{X}_{r+1,I}, \Phi^*\underline{\omega}^{\kappa_I}) \xrightarrow{?} \mathrm{H}^0(\mathfrak{X}_{r+1,I}, i^*\underline{\omega}^{\kappa_I})$$

and

$$\mathrm{U}\colon\ \mathrm{H}^0(\mathfrak{X}_{r+1,I}, i^*\underline{\omega}^{\kappa_I}) \xrightarrow{??} \mathrm{H}^0(\mathfrak{X}_{r+1,I}, \Phi^*\underline{\omega}^{\kappa_I}) \xrightarrow{\frac{1}{p}\operatorname{tr}(\Phi)} \mathrm{H}^0(\mathfrak{X}_{r,I}, \underline{\omega}^{\kappa_I})\Big[\frac{1}{p}\Big].$$

Fact. The morphism Φ is finite and flat (of degree p). Therefore, it admits a well-defined trace $\operatorname{tr}(\Phi)\colon \mathscr{O}_{\mathfrak{X}_{r,I}} \to \Phi^*\mathscr{O}_{\mathfrak{X}_{r+1,I}}$.

It turns out that we can define natural maps filling the gaps ? and ?? above via the formalism of vector bundles with marked sections. We use lemma 6.4 and corollary 6.5 of Andreatta–Iovita's article "Triple product…" [2] (technical results):

Lemma 16. *Let $\mathfrak{S} = \mathrm{Spf}(R)$ be an affine open of $\mathfrak{IG}_{\tilde{n},r,I}$ or of $\mathfrak{IG}'_{\tilde{n},r,I}$ for some $\tilde{n} \geq n$. Let $\mathfrak{E}/\mathfrak{S}$ and $\widetilde{\mathfrak{E}}/\mathfrak{S}$ be elliptic curves coming from $\mathfrak{IG}_{n,r,I}$ (in the sense that they are endowed with canonical subgroups H_n and $\widetilde{H}_n$ of order p^n and trivializations of $(H_n)^\vee$ and $(\widetilde{H}_n)^\vee$, respectively) or from $\mathfrak{IG}'_{n,r,I}$. Assume that $\mathrm{Hdg}(\widetilde{\mathfrak{E}}) \subset \mathrm{Hdg}(\mathfrak{E})$ and work with the ideal of definition $I = p^n\,\mathrm{Hdg}(\widetilde{\mathfrak{E}})^{-p^n/(p-1)}$. Let $\lambda\colon \widetilde{\mathfrak{E}} \to \mathfrak{E}$ be an isogeny of degree p^d for some $d \geq 1$. If λ induces a morphism $\widetilde{H}_n \to H_n$ that is generically an*

[4] Notice that the ring of modular functions is the ring of global sections of the structure sheaf on the modular curve. Thus φ essentially defines the desired map of modular curves.

isomorphism, then $\lambda^*\colon \mathrm{H}^1_{\mathrm{dR}}(\mathfrak{E}/\mathfrak{S}) \to \mathrm{H}^1_{\mathrm{dR}}(\widetilde{\mathfrak{E}}/\mathfrak{S})$ *induces a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & \underline{\omega}^\sharp \arrow[r] \arrow[d, "\lambda^*", "\wr"'] & \mathscr{H}^\sharp \arrow[r] \arrow[d, "\lambda^\sharp"] & \mathrm{Hdg}(\mathfrak{E})^{\frac{p}{p-1}}\underline{\omega}^\vee \arrow[r] \arrow[d, "((\lambda^\vee)^*)^\vee"] & 0 \\
0 \arrow[r] & \underline{\widetilde{\omega}}^\sharp \arrow[r] & \widetilde{\mathscr{H}}^\sharp \arrow[r] & \mathrm{Hdg}(\widetilde{\mathfrak{E}})^{\frac{p}{p-1}}\underline{\widetilde{\omega}}^\vee \arrow[r] & 0
\end{tikzcd}$$

in which the first vertical arrow is an isomorphism and the last vertical arrow is injective with image $\tau_\lambda \,\mathrm{Hdg}(\widetilde{\mathfrak{E}})^{p/(p-1)}$*, where*

$$\tau_\lambda = p^d \cdot \mathrm{Hdg}(\widetilde{\mathfrak{E}})^{\frac{(p+1)(p^d-1)}{p^d(p-1)}}.$$

Corollary 17. *In the situation of lemma 16, assume further that* $d = 1$ *and consider the commutative diagram*

$$\begin{tikzcd}
\mathbb{V}_0(\underline{\widetilde{\omega}}^\sharp, \widetilde{s}) \arrow[d] \arrow[ddr, bend right=60, "\widetilde{f}"'] & & \mathbb{V}_0(\widetilde{\mathscr{H}}^\sharp, \widetilde{s}) \arrow[ll] \arrow[d, "\lambda^\sharp"] \arrow[ddl, bend left=60, "\widetilde{g}"] \\
\mathbb{V}_0(\underline{\omega}^\sharp, s) \arrow[dr, "f"] & & \mathbb{V}_0(\mathscr{H}^\sharp, s) \arrow[ll] \arrow[dl, "g"'] \\
& \mathfrak{S} &
\end{tikzcd}$$

induced by the functoriality of vector bundles with marked sections. The morphism $g_*\mathscr{O}_{\mathbb{V}_0(\mathscr{H}^\sharp, s)} \to \widetilde{g}_*\mathscr{O}_{\mathbb{V}_0(\widetilde{\mathscr{H}}^\sharp, s')}$ *preserves the filtrations and induces on the m–th graded pieces (via the identifications in proposition 5) the morphism*

$$f_*\mathscr{O}_{\mathbb{V}_0(\underline{\omega}^\sharp, s)} \hat{\otimes}_{\mathscr{O}_\mathfrak{S}} \mathrm{Sym}^m\big(\mathrm{Hdg}(\mathfrak{E})^{\frac{p}{p-1}}\underline{\omega}^\vee\big) \to \widetilde{f}_*\mathscr{O}_{\mathbb{V}_0(\underline{\widetilde{\omega}}^\sharp, \widetilde{s})} \hat{\otimes}_{\mathscr{O}_\mathfrak{S}} \mathrm{Sym}^m\big(\mathrm{Hdg}(\widetilde{\mathfrak{E}})^{\frac{p}{p-1}}\underline{\widetilde{\omega}}^\sharp\big)$$

defined by the isomorphism $f_*\mathscr{O}_{\mathbb{V}_0(\underline{\omega}^\sharp, s)} \to \widetilde{f}_*\mathscr{O}_{\mathbb{V}_0(\underline{\widetilde{\omega}}^\sharp, \widetilde{s})}$ *and* $\big((\lambda^\vee)^*\big)^\vee$*. If, furthermore,* $\mathfrak{E}/\mathfrak{S}$ *and* $\widetilde{\mathfrak{E}}/\mathfrak{S}$ *come from* $\mathfrak{I}\mathfrak{G}'_{n,r,I}$ *(in the sense that they are endowed with trivializations of* $\mathfrak{E}[p^n]$ *and* $\widetilde{\mathfrak{E}}[p^n]$*, respectively), then* $\lambda^\sharp$ *is compatible with the Gauss–Manin connections.*

To simplify the notation, let $\mathfrak{E}$ denote the universal (generalized) elliptic curve $\mathfrak{E}_{n,r,I}/\mathfrak{I}\mathfrak{G}_{n,r,I}$ or $\mathfrak{E}_{n+1,r+1,I}/\mathfrak{I}\mathfrak{G}_{n+1,r+1,I}$ (depending on the context; this abuse of notation should not create confusion). Since we want to apply lemma 16 and corollary 17, we lift $i, \Phi\colon \mathfrak{X}_{r+1,I} \to \mathfrak{X}_{r,I}$ to $i, \Phi\colon \mathfrak{I}\mathfrak{G}_{n+1,r+1,I} \to \mathfrak{I}\mathfrak{G}_{n,r,I}$ in the obvious way (we use the same symbols again in a harmless abuse of notation). In particular, to extend the definition of Φ in terms of moduli we observe that $H'_n = H_{n+1}/H_1$

and $(H_n')^\vee = (H_{n+1})^\vee[p^n]$, so for the trivializations we use $P_{n+1} \mapsto P_n' = p \cdot P_{n+1}$. We obtain cartesian diagrams

$$\begin{array}{ccc} \mathfrak{E} & \xrightarrow{\quad\quad\quad} & \mathfrak{E} \\ \downarrow & \ulcorner & \downarrow \\ \mathfrak{IG}_{n+1,r+1,I} & \xrightarrow{i} & \mathfrak{IG}_{n,r,I} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathfrak{E}' = \mathfrak{E}/H_1 & \xrightarrow{\quad\quad} & \mathfrak{E} \\ \downarrow & \ulcorner & \downarrow \\ \mathfrak{IG}_{n+1,r+1,I} & \xrightarrow{\Phi} & \mathfrak{IG}_{n,r,I} \end{array}$$

from which we see that $i^*(\mathscr{H}^\sharp, s) = (\mathscr{H}^\sharp, s)$[5] and $\Phi^*(\mathscr{H}^\sharp, s) = \big((\mathscr{H}')^\sharp, s'\big)$ (note that we use the ideal of definition $\underline{\beta}_n$ even to construct the sheaves and marked sections over $\mathfrak{IG}_{n+1,r+1,I}$, as we only get a canonical subgroup of order p^n by pull-back from $\mathfrak{IG}_{n,r,I}$).

Let $\lambda\colon \mathfrak{E}' \to \mathfrak{E}$ be the isogeny between elliptic curves over $\mathfrak{IG}_{n+1,r+1,I}$ dual to the quotient $\mathfrak{E} \to \mathfrak{E}/H_1 = \mathfrak{E}'$. We are in the situation of lemma 16, so we obtain

$$\begin{array}{ccc} i^*(\mathscr{H}^\sharp, s) & \xrightarrow{\lambda^\sharp} & \Phi^*(\mathscr{H}^\sharp, s) \\ \cup & & \cup \\ i^*(\underline{\omega}^\sharp, s) & \xrightarrow{\cong} & \Phi^*(\underline{\omega}^\sharp, s) \end{array}$$

inducing the desired morphisms filling the gaps ? and ?? (the latter works even for $\mathbb{W}_{\kappa_I}$, while the former only for $\underline{\omega}^{\kappa_I}$ because we need to invert the direction of the arrows).

Definition 18. The *operators* U and V are defined to be

$$\mathrm{U}\colon\ \mathrm{H}^0(\mathfrak{X}_{r+1,I}, i^*\underline{\omega}^{\kappa_I}) \longrightarrow \mathrm{H}^0(\mathfrak{X}_{r+1,I}, \Phi^*\underline{\omega}^{\kappa_I}) \xrightarrow{\frac{1}{p}\operatorname{tr}(\Phi)} \mathrm{H}^0(\mathfrak{X}_{r,I}, \underline{\omega}^{\kappa_I})\Big[\frac{1}{p}\Big]$$

and

$$\mathrm{V}\colon\ \mathrm{H}^0(\mathfrak{X}_{r,I}, \underline{\omega}^{\kappa_I}) \xrightarrow{\Phi^*} \mathrm{H}^0(\mathfrak{X}_{r+1,I}, \Phi^*\underline{\omega}^{\kappa_I}) \longrightarrow \mathrm{H}^0(\mathfrak{X}_{r+1,I}, i^*\underline{\omega}^{\kappa_I})$$

(where the unlabelled arrows are the maps induced by $\lambda^\sharp$ and by the inverse of its restriction λ^* as above; cf. lemma 16).

Proposition 19. *For every $h \in \mathbb{Q}_{\geq 0}$, the groups $\mathrm{H}^0(\mathcal{X}_{r,I}, \mathbb{W}_{\kappa_I})$ and $\mathrm{H}^0(\mathcal{X}_{r,I}, \mathrm{Fil}_m \mathbb{W}_{\kappa_I})$ admit (locally on the weight space, in the sense that we might have to shrink I) h–slope*

[5] Again, this is an abuse of notation. In the left-hand side, $(\mathscr{H}^\sharp, s)$ denotes the sheaf obtained from $\mathrm{H}^1_{\mathrm{dR}}(\mathfrak{E}_{n,r,I}/\mathfrak{IG}_{n,r,I})$ (with the marked section with respect to the ideal of definition $\underline{\beta}_n$). In the right-hand side, however, $(\mathscr{H}^\sharp, s)$ denotes the sheaf obtained from $\mathrm{H}^1_{\mathrm{dR}}(\mathfrak{E}_{n+1,r+1,I}/\mathfrak{IG}_{n+1,r+1,I})$ (with the marked section with respect to the ideal of definition $\underline{\beta}_n$, not $\underline{\beta}_{n+1}$ as one might initially expect!).

decompositions for U*:*

$$\mathrm{H}^0(\mathcal{X}_{r,I}, \mathbb{W}_{\kappa_I}) = \mathrm{H}^0(\mathcal{X}_{r,I}, \mathbb{W}_{\kappa_I})^{\leq h} \oplus \mathrm{H}^0(\mathcal{X}_{r,I}, \mathbb{W}_{\kappa_I})^{>h}$$

(and similarly for $\mathrm{Fil}_m \mathbb{W}_{\kappa_I}$*). Furthermore,*

$$\mathrm{H}^0(\mathcal{X}_{r,I}, \mathrm{Fil}_m \mathbb{W}_{\kappa_I})^{\leq h} = \mathrm{H}^0(\mathcal{X}_{r,I}, \mathbb{W}_{\kappa_I})^{\leq h} \quad \textit{if } m \gg 0.$$

Idea of the proof. Locally (i.e., shrinking I if necessary),

$$\mathrm{H}^0(\mathcal{X}_{r,I}, \mathrm{Fil}_m \mathbb{W}_{\kappa_I}) = \mathrm{H}^0(\mathcal{X}_{r,I}, \mathrm{Fil}_m \mathbb{W}_{\kappa_I})^{\leq h} \oplus \mathrm{H}^0(\mathcal{X}_{r,I}, \mathrm{Fil}_m \mathbb{W}_{\kappa_I})^{>h}$$

because $\mathrm{Fil}_m \mathbb{W}_{\kappa_I}$ is coherent and U is a compact[6] operator. By lemma 16 and corollary 17, $\lambda^\sharp$ induces a map on Gr_{m+1} that is $0 \bmod \tau_\lambda^{m+1} \subset \left(\alpha^{\lfloor (m+1)/p \rfloor}\right)$. In particular, the operator U acting on $\mathrm{H}^0(\mathcal{X}_{r,I}, \mathbb{W}_{\kappa_I} / \mathrm{Fil}_m \mathbb{W}_{\kappa_I})$ is divisible by p^{h+1} if $m \gg 0$ and so this part admits a trivial decomposition.

Now the result for $\mathbb{W}_{\kappa_I}$ follows[7] from the short exact sequence

$$0 \longrightarrow \mathrm{H}^0(\mathcal{X}_{r,I}, \mathrm{Fil}_m \mathbb{W}_{\kappa_I}) \longrightarrow \mathrm{H}^0(\mathcal{X}_{r,I}, \mathbb{W}_{\kappa_I}) \longrightarrow \mathrm{H}^0(\mathcal{X}_{r,I}, \mathbb{W}_{\kappa_I} / \mathrm{Fil}_m \mathbb{W}_{\kappa_I}) \longrightarrow 0$$

(where the exactness on the right comes from the fact that $\mathrm{Fil}_m \mathbb{W}_{\kappa_I}$ is coherent on the affinoid $\mathcal{X}_{r,I}$, hence acyclic). □

4.2 Twists by finite characters

Motivation. Let $n \in \mathbb{Z}_{\geq 1}$ with the usual assumptions with respect to r and I (cf. assumption 8) and fix a primitive p^n–th root of unity $\zeta \in \overline{\mathbb{Q}}_p$. Given a primitive character $\chi\colon (\mathbb{Z}/p^n\mathbb{Z})^\times \to \Lambda_I[\zeta]^\times$, there is a twist-by-$\chi$ operator θ^χ defined easily on q–expansions by

$$\theta^\chi\Big(\sum_{i \geq 0} a_i q^i\Big) = \sum_{i \geq 0} \chi(i) a_i q^i.$$

But we can express (factoring out the common values of χ and rearranging the sum)

$$\sum_{i \geq 0} \chi(i) a_i q^i = \sum_{l} \chi(l) \sum_{i \geq 0} \Big(\frac{1}{\varphi(p^n)} \cdot \sum_j \zeta^{(l-i)j}\Big) a_i q^i$$

[6]Giovanni said this isn't necessary: coherence means that we work in finite dimensions and in that case we always have decompositions.

[7]Giovanni doesn't think this is so simple. Apparently one needs a technical condition (PR) to obtain the splitting.

$$
\begin{aligned}
&= \frac{1}{\varphi(p^n)} \sum_j \chi^{-1}(j) \sum_l \chi(lj)\zeta^{lj} \sum_{i\geq 0} a_i \zeta^{-ij} q^i \\
&= \frac{\mathfrak{g}_\chi}{\varphi(p^n)} \sum_j \chi^{-1}(j) \sum_i a_i (\zeta^{-j} q)^i
\end{aligned}
$$

where the summation indices j and l run over $(\mathbb{Z}/p^n\mathbb{Z})^\times$ and

$$
\mathfrak{g}_\chi = \sum_l \chi(l)\zeta^l
$$

is the Gauss sum of χ. Thus, we need to define for every $j \in (\mathbb{Z}/p^n\mathbb{Z})^\times$ an operation on the modular curves (i.e., in terms of moduli) that corresponds to the transformation of Tate curves $\mathrm{Tate}(q^N) \mapsto \mathrm{Tate}(q^{Np^n}) \mapsto \mathrm{Tate}(\zeta^{-j} q^N)$. But such transformations are well-known: they occur in the definition of T_{p^n} already!

The arguments are similar to those of section 4.1. Consider the morphism $t\colon \mathfrak{X}_{r+n,I} \to \mathfrak{X}_{r,I}$ defined in terms of moduli by $\mathcal{E} \mapsto \mathcal{E}' = \mathcal{E}/H_n$. (More precisely, we can define a morphism on the generic fibres using the moduli interpretations and then take normalizations to obtain the morphism of formal schemes.) In fact, the same arguments as in section 4.1 show that we obtain from this construction a cartesian diagram

$$
\begin{tikzcd}
\mathfrak{E}' = \mathfrak{E}/H_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathfrak{E} \arrow[d] \\
\mathfrak{IG}_{2n,r+n,I} \arrow[r, "t"] & \mathfrak{IG}_{n,r,I}
\end{tikzcd}
$$

(where, again by abuse of notation, $\mathfrak{E}$ denotes two universal generalized elliptic curves: either $\mathfrak{E}_{n,r,I}/\mathfrak{IG}_{n,r,I}$ or $\mathfrak{E}_{2n,r+n,I}/\mathfrak{IG}_{2n,r+n,I}$ depending on the context). The isogeny $\lambda\colon \mathfrak{E}' \to \mathfrak{E}$ between elliptic curves over $\mathfrak{IG}_{2n,r+n,I}$ dual to the quotient $\mathfrak{E} \to \mathfrak{E}/H_n = \mathfrak{E}'$ satisfies the hypotheses of lemma 16.

We want to define similarly $t_j\colon \mathfrak{IG}_{2n,r+n,I} \to \mathfrak{IG}_{n,r,I}$, first on moduli (i.e., on the generic fibres). Given a triple $(\mathcal{E}, H_{2n}, P_{2n})$ consisting of an elliptic curve $\mathcal{E}$ with a canonical subgroup H_{2n} of order p^{2n} and a trivialization P_{2n} of $(H_{2n})^\vee$, we can form $(\mathcal{E}', H_n', P_n')$ taking the quotient by H_n as above. Let $\check{H}_n'$ be the kernel of the isogeny $\mathcal{E}' \to \mathcal{E}$ dual to the quotient $\mathcal{E} \to \mathcal{E}/H_n = \mathcal{E}'$. Then $\mathcal{E}'[p^n] = H_n' \times \check{H}_n'$ and the Weil pairing induces a duality isomorphism $\check{H}_n' \cong (H_n')^\vee$. Therefore, P_n' induces isomorphisms

$$
\sigma\colon \underline{\mathbb{Z}/p^n\mathbb{Z}} \to \check{H}_n' \quad \text{and} \quad \sigma^\vee\colon H_n' \to \mu_{p^n}.
$$

Assume that we work over some base ring containing ζ. There is an obvious identification

$$\begin{aligned} \operatorname{Hom}\big(\underline{\mathbb{Z}/p^n\mathbb{Z}}, \mu_{p^n}\big) &\cong \mathbb{Z}/p^n\mathbb{Z} \\ (1 \mapsto \zeta^j) &\longleftarrow\!\shortmid\; j \end{aligned}$$

(depending on the fixed ζ). Furthermore, there is a bijection

$$\begin{aligned} \eta\colon \operatorname{Hom}(\check{H}'_n, H'_n) &\longrightarrow \operatorname{Hom}\big(\underline{\mathbb{Z}/p^n\mathbb{Z}}, \mu_{p^n}\big) \\ \rho &\longmapsto \sigma^\vee \circ \rho \circ \sigma \end{aligned}$$

and so we obtain a morphism $\rho_j\colon \check{H}'_n \to H'_n$ for every $j \in \mathbb{Z}/p^n\mathbb{Z}$. We define the subgroup $H'_{\rho_j} = (\rho_j \times \mathrm{id}_{\check{H}'_n})(\check{H}'_n) \subset H'_n \times \check{H}'_n = \mathcal{E}'[p^n]$. Equivalently, the cartesian diagram

$$\begin{tikzcd} H'_{\rho_j} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{E}'[p^n] \arrow[d, "\sigma^\vee \times \sigma^{-1}", "\wr" swap] \\ \underline{\mathbb{Z}/p^n\mathbb{Z}} \arrow[r, hook] & \mu_{p^n} \times \underline{\mathbb{Z}/p^n\mathbb{Z}} \\ 1 \arrow[r, mapsto] & (\zeta^j, 1) \end{tikzcd}$$

characterizes H'_{ρ_j}. (Observe that, as j varies, we obtain all subgroups of order p^n that have trivial intersection with H'_n, which corresponds to $\mu_{p^n} \times \{\, 0 \,\}$ in the bottom row.) We define $\mathcal{E}'_j = \mathcal{E}'/H'_{\rho_j}$. The canonical isogeny $\mathcal{E}' \to \mathcal{E}'_j$ "translates" the additional structure (i.e., maps H'_n isomorphically onto the canonical subgroup $H'_{n,j}$ of order p^n of $\mathcal{E}'_j$). Therefore, we can define t_j in terms of moduli by $(\mathcal{E}, H_{2n}, P_{2n}) \mapsto (\mathcal{E}'_j, H'_{n,j}, P'_{n,j})$. We obtain a cartesian diagram

$$\begin{tikzcd} \mathfrak{E}'_j = \mathfrak{E}'/H'_{\rho_j} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathfrak{E} \arrow[d] \\ \mathfrak{IG}_{2n,r+n,I} \arrow[r, "t_j"] & \mathfrak{IG}_{n,r,I} \end{tikzcd}$$

analogous to the one given by t. Also, the quotient isogeny $\lambda_j\colon \mathfrak{E}' \to \mathfrak{E}'_j$ over $\mathfrak{IG}_{2n,r+n,I}$ satisfies the hypotheses of lemma 16.

Remark. Observe that, for $\mathrm{Tate}(q^N)$ regarded as $\mathbb{G}_\mathrm{m}/q^{N\mathbb{Z}}$, the canonical subgroup is simply $H_n = \mu_{p^n}$ and $\mathrm{Tate}(q^N)/H_n \cong \mathrm{Tate}(q^{Np^n})$, as desired. Then again the canonical subgroup of $\mathrm{Tate}(q^{Np^n})$ is $H'_n = \mu_{p^n}$, whereas $\check{H}'_n = \langle q^N \rangle$. The Weil pairing identifies[8] q^N with $(\zeta \mapsto \zeta^{-1})$, so that $\sigma^\vee(\zeta) = \zeta$ and $\sigma(1) = q^{-N}$. Therefore, $H'_{\rho_j} = \langle \zeta^{-j} q^N \rangle$ and $\mathrm{Tate}(q^{Np^n})/H'_{\rho_j} \cong \mathrm{Tate}(\zeta^{-j} q^N)$, also as desired.

By lemma 16, we have injective morphisms

$$\mathscr{H}^\sharp \xhookrightarrow{\lambda^\sharp} (\mathscr{H}')^\sharp \xhookleftarrow{\lambda_j^\sharp} (\mathscr{H}_j')^\sharp = t_j^* \mathscr{H}^\sharp$$

(of sheaves on $\mathfrak{IG}_{2n,r+n,I}$) with the same image. Therefore, we obtain isomorphisms $\theta_j \colon t_j^* \mathbb{W}_{\kappa_I} \to \mathbb{W}_{\kappa_I}$ for all $j \in \mathbb{Z}/p^n\mathbb{Z}$.

Definition 20. The *twist-by-χ operator θ^χ* is defined[9] to be

$$\theta^\chi = \frac{\mathfrak{g}_\chi}{\varphi(p^n)} \sum_{j \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \chi(j)^{-1} (\theta_j \circ t_j^*) \colon \mathrm{H}^0(\mathfrak{X}_{r,I}, \mathbb{W}_{\kappa_I}) \longrightarrow \mathrm{H}^0(\mathfrak{X}_{r+n,I}, \mathbb{W}_{\kappa_I + 2\chi}) \left[\frac{1}{p}\right]$$

(where $\mathfrak{g}_\chi$ is the Gauss sum of χ with respect to the fixed ζ).

Remark. The fact that θ^χ transforms the weights as stated in definition 20 follows from a rather straight-forward computation, which is the content of lemma 3.31 of Andreatta–Iovita's article "Triple product. . . " [2].

4.3 The overconvergent projection

We defined the Gauss–Manin connection $\nabla_{\kappa_I} \colon \mathbb{W}_{\kappa_I} \to \mathbb{W}_{\kappa_I} \hat{\otimes}_{\mathscr{O}_{\mathcal{X}_{r,I}}} \Omega^1_{\mathcal{X}_{r,I}/\mathcal{W}_I^0}$ over $\mathcal{X}_{r,I}$ by descent from $\mathcal{IG}_{n,r,I}$ using an explicit local formula. (Observe that we work over the generic fibres, as we had to invert α; cf. theorem 14). By Kodaira–Spencer's isomorphism $\Omega^1_{\mathcal{X}_{r,I}/\mathcal{W}_I^0} \cong \underline{\omega}^2$ and regarding $\underline{\omega}^2 = \mathrm{Fil}_0 \mathbb{W}_2 \subset \mathbb{W}_2$, we can interpret the Gauss–Manin connection as a morphism $\nabla_{\kappa_I} \colon \mathbb{W}_{\kappa_I} \to \mathbb{W}_{\kappa_I+2}$. In fact, for every $m \geq 0$, we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Fil}_m \mathbb{W}_{\kappa_I} \arrow[r] \arrow[d, "\nabla_{\kappa_I}"] & \mathbb{W}_{\kappa_I} \arrow[r] \arrow[d, "\nabla_{\kappa_I}"] & \mathbb{W}_{\kappa_I} / \mathrm{Fil}_m \mathbb{W}_{\kappa_I} \arrow[r] \arrow[d, "\nabla_{\kappa_I}"] & 0 \\
0 \arrow[r] & \mathrm{Fil}_{m+1} \mathbb{W}_{\kappa_I+2} \arrow[r] & \mathbb{W}_{\kappa_I+2} \arrow[r] & \mathbb{W}_{\kappa_I+2} / \mathrm{Fil}_{m+1} \mathbb{W}_{\kappa_I+2} \arrow[r] & 0
\end{tikzcd}$$

[8] Since the Weil pairing is alternating, there are two possible identifications of $\check{H}_n'$ with the dual of H_n', depending on which factor is "the first" and which is "the second". This computation isn't done in the article "Triple product. . . " [2], so I'm not sure if that's how Andreatta–Iovita thought about it. This is however the sign that makes their formula work, I believe. Ju-Feng thinks the "standard identification" is the other one. Another concern is that there are several possibilities for σ, I simply chose the "most natural" one for this computation, but shouldn't all intervene?

[9] Lennart and Giovanni don't think such a definition can be integral (as it seems to be in proposition 3.29 of Andreatta–Iovita's article "Triple product. . . " [2]) when $n \geq 1$. I agree and so added the $\frac{1}{p}$ at the end.

with exact rows. We view the columns of this diagram as de Rham complexes $(\mathrm{Fil}_m \mathbb{W}_{\kappa_I})^\bullet$, $\mathbb{W}_{\kappa_I}^\bullet$ and $(\mathbb{W}_{\kappa_I} / \mathrm{Fil}_m \mathbb{W}_{\kappa_I})^\bullet$ and define $\mathrm{H}^i_{\mathrm{dR}}(\mathcal{X}_{r,I}, \mathbb{W}_{\kappa_I}^\bullet)$ to be the i–th hypercohomology group of $\mathbb{W}_{\kappa_I}^\bullet$ (and similarly for the other complexes). Since both $\mathrm{Fil}_m \mathbb{W}_{\kappa_I}$ and $\mathrm{Fil}_{m+1} \mathbb{W}_{\kappa_I+2}$ are coherent sheaves and $\mathcal{X}_{r,I}$ is affinoid, we can actually compute

$$\mathrm{H}^i_{\mathrm{dR}}(\mathcal{X}_{r,I}, (\mathrm{Fil}_m \mathbb{W}_{\kappa_I})^\bullet) = \mathrm{H}^i\Big(\mathrm{H}^0(\mathcal{X}_{r,I}, \mathrm{Fil}_m \mathbb{W}_{\kappa_I}) \xrightarrow{\nabla_{\kappa_I}} \mathrm{H}^0(\mathcal{X}_{r,I}, \mathrm{Fil}_{m+1} \mathbb{W}_{\kappa_I+2})\Big)$$

(i.e., the hypercohomology is simply the cohomology of the complex of global sections in this situation). In particular, the only non-trivial de Rham cohomology groups of $(\mathrm{Fil}_m \mathbb{W}_{\kappa_I})^\bullet$ appear in degrees $i = 0$ and 1.

Lemma 21. *For every $m \geq 0$ there is a* U*–equivariant*[10] *short exact sequence*

$$0 \longrightarrow \mathrm{H}^0(\mathcal{X}_{r,I}, \underline{\omega}^{\kappa_I+2}) \longrightarrow \mathrm{H}^1_{\mathrm{dR}}(\mathcal{X}_{r,I}, (\mathrm{Fil}_m \mathbb{W}_{\kappa_I})^\bullet) \longrightarrow \bigoplus_{i=0}^{m} \mathrm{H}^0(\mathcal{X}_{r,I}, j_{i,*}\underline{\omega}^{-i}) \longrightarrow 0,$$

where the first arrow is induced by the inclusion $\underline{\omega}^{\kappa_I+2} = \mathrm{Fil}_0 \mathbb{W}_{\kappa_I+2} \subset \mathrm{Fil}_{m+1} \mathbb{W}_{\kappa_I+2}$ and $j_i \colon \mathcal{X}_{r,I} \times_{\mathcal{W}_I} \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p) \hookrightarrow \mathcal{X}_{r,I}$ is given by specialization at the classical weight i (viewed as a $\mathbb{Q}_p$–valued point of $\mathcal{W}_I$).

Sketch of the proof. Over $\mathcal{X}_{r,I}$, the Gauss–Manin connection ∇_{κ_I} induces on graded pieces a morphism

$$\underline{\omega}^{\kappa_I-2m} \cong \mathrm{Gr}_m \mathbb{W}_{\kappa_I} \xrightarrow{\nabla_{\kappa_I}} \mathrm{Gr}_{m+1} \mathbb{W}_{\kappa_I+2} \cong \underline{\omega}^{\kappa_I-2m}$$

that is (an isomorphism times) multiplication by $u_{\kappa_I} - m$ (cf. theorem 14). In particular, its kernel is trivial and its cokernel is $\underline{\omega}^{\kappa_I-2m} / (u_{\kappa_I} - m)\underline{\omega}^{\kappa_I-2m} \cong \underline{\omega}^{-m}$.

[10]This lemma is (part of) lemma 3.32 of Andreatta–Iovita's article "Triple product…" [2]. In loc. cit. they write that the action on the last term has to be modified, but I think it is a mistake and that the modification should appear in the second part of their lemma, when they deal with the torsion-free part. That makes more sense to me seeing the end of their proof (in particular the relation between U and Coleman's operator θ^{i+1}).

We argue by induction on m. We have a commutative diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{Fil}_{m-1}\mathbb{W}_{\kappa_I} \arrow[r] \arrow[d, "\nabla_{\kappa_I}"] & \mathrm{Fil}_m \mathbb{W}_{\kappa_I} \arrow[r] \arrow[d, "\nabla_{\kappa_I}"] & \mathrm{Gr}_m \mathbb{W}_{\kappa_I} \arrow[r] \arrow[d, "\nabla_{\kappa_I}"] & 0 \\
0 \arrow[r] & \mathrm{Fil}_m \mathbb{W}_{\kappa_I+2} \arrow[r] & \mathrm{Fil}_{m+1} \mathbb{W}_{\kappa_I+2} \arrow[r] & \mathrm{Gr}_{m+1} \mathbb{W}_{\kappa_I+2} \arrow[r] \arrow[d] & 0 \\
 & & & \underline{\omega}^{-m} \arrow[d] & \\
 & & & 0 &
\end{tikzcd}$$

with exact rows and column. By the observation before, if we apply $\mathrm{H}^0(\mathcal{X}_{r,I}, -)$, the resulting rows will still be exact (coherent sheaves on affinoids are acyclic) and then the snake lemma yields a short exact sequence

$$0 \to \mathrm{H}^1_{\mathrm{dR}}\big(\mathcal{X}_{r,I}, (\mathrm{Fil}_{m-1}\mathbb{W}_{\kappa_I})^\bullet\big) \to \mathrm{H}^1_{\mathrm{dR}}\big(\mathcal{X}_{r,I}, (\mathrm{Fil}_m \mathbb{W}_{\kappa_I})^\bullet\big) \to \mathrm{H}^0(\mathcal{X}_{r,I}, j_{m,*}\underline{\omega}^{-m}) \to 0$$

from which the result follows. $\square$

Remark. Since $j_{i,*}\,\underline{\omega}^{-i} \cong \mathrm{Gr}_i \mathbb{W}_{\kappa_I}/(u_{\kappa_I} - i)\,\mathrm{Gr}_i \mathbb{W}_{\kappa_I}$, if we invert (i.e., localize at) $u_{\kappa_I} - i$, then this part of the sequence goes away.

Definition 22. For each $m \geq 0$, the *overconvergent projection* is defined to be the isomorphism

$$\begin{aligned} H^\dagger_m \colon\ \mathrm{H}^1_{\mathrm{dR}}\big(\mathcal{X}_{r,I}, (\mathrm{Fil}_m \mathbb{W}_{\kappa_I})^\bullet\big) \otimes_{\Lambda_I} \Lambda_I\Big[\prod_{i=0}^m (u_{\kappa_I} - i)^{-1}\Big] \cong \\ \cong \mathrm{H}^0(\mathcal{X}_{r,I}, \underline{\omega}^{\kappa_I+2}) \otimes_{\Lambda_I} \Lambda_I\Big[\prod_{i=0}^m (u_{\kappa_I} - i)^{-1}\Big] \end{aligned}$$

induced by the inclusion $\mathrm{H}^0(\mathcal{X}_{r,I}, \underline{\omega}^{\kappa_I+2}) \hookrightarrow \mathrm{H}^1_{\mathrm{dR}}\big(\mathcal{X}_{r,I}, (\mathrm{Fil}_m \mathbb{W}_{\kappa_I})^\bullet\big)$ (cf. lemma 21).

Given $h \in \mathbb{Q}_{\geq 0}$, we can find locally on the weight space h–slope decompositions for U of the de Rham cohomology groups above[11] and choose $m = m(h)$ large enough so that $\mathrm{H}^1_{\mathrm{dR}}\big(\mathcal{X}_{r,I}, (\mathbb{W}_{\kappa_I}/\,\mathrm{Fil}_m \mathbb{W}_{\kappa_I})^\bullet\big)^{\leq h} = 0$. In this situation, we can define the *overconvergent projection*

$$H^\dagger \colon\ \mathrm{H}^1_{\mathrm{dR}}\big(\mathcal{X}_{r,I}, \mathbb{W}^\bullet_{\kappa_I}\big)^{\leq h} \otimes_{\Lambda_I} \Lambda_I\Big[\prod_{i=0}^m (u_{\kappa_I} - i)^{-1}\Big] \cong$$

$$\cong \mathrm{H}^0\big(\mathcal{X}_{r,I}, \underline{\omega}^{\kappa_I+2}\big)^{\leq h} \otimes_{\Lambda_I} \Lambda_I\left[\prod_{i=0}^{m}(u_{\kappa_I} - i)^{-1}\right]$$

similarly.

4.4 The Gauss–Manin connection on q–expansions

Consider the Tate curve $\mathrm{Tate}(q^N)$ over $R = \Lambda_I^0((q))$ with its canonical differential ω_{can}. Let $\mathscr{H}(q) = \mathrm{H}^1_{\mathrm{dR}}(\mathrm{Tate}(q^N)/R)$. Let $\nabla\colon \mathscr{H}(q) \to \mathscr{H}(q) \hat{\otimes} \Omega^1_{R/\Lambda_I^0}$ be the Gauss–Manin connection and let $\mathrm{KS}\colon \underline{\omega}(q)^2 \cong \Omega^1_{R/\Lambda_I^0}$ be the Kodaira–Spencer isomorphism. It is well-known that

$$\mathrm{KS}(\omega_{\mathrm{can}}^2) = \frac{dq}{q}.$$

Its dual derivation is

$$\partial = q\frac{d}{dq}.$$

We use the basis $\omega_{\mathrm{can}}, \eta_{\mathrm{can}} = \nabla(\partial)(\omega_{\mathrm{can}})$ of $\mathscr{H}(q)$. Explicit formulae for the Gauss–Manin connection on Tate curves are also well-known (cf. appendix A1 of Katz's "p–adic properties of modular schemes and modular forms" [3], for example):

$$\nabla\big(\omega_{\mathrm{can}} \quad \eta_{\mathrm{can}}\big) = \big(\omega_{\mathrm{can}} \quad \eta_{\mathrm{can}}\big)\begin{pmatrix} 0 & 0 \\ \frac{dq}{q} & 0 \end{pmatrix}.$$

In this section, let $\kappa\colon \mathbb{Z}_p^\times \to \Lambda_I$ be any locally analytic character (often the universal character κ_I), so that for $c \gg 0$ there exists $u_\kappa \in p^{1-c}\Lambda_I^0$ with the property that $\kappa(t) = \exp\big(u_\kappa \log(t)\big)$ for all $t \in 1 + p^c\mathbb{Z}_p$. As usual, we decompose κ into its finite part $\kappa_{\mathrm{f}}\colon (\mathbb{Z}/p\mathbb{Z})^\times \to \Lambda_I$ and $\kappa^0 = \kappa \cdot \kappa_{\mathrm{f}}^{-1}\colon \mathbb{Z}_p^\times \to \Lambda_I^0$.

Let $\mathbb{W}_\kappa^0(q)$ (resp. $\mathbb{W}_\kappa(q)$, $\underline{\omega}^{\kappa,0}(q)$ or $\underline{\omega}^\kappa(q)$) be the pull-back of $\mathbb{W}_\kappa^0$ (resp. $\mathbb{W}_\kappa$, $\underline{\omega}^{\kappa,0}$ or $\underline{\omega}^\kappa$) by the morphism $\mathrm{Spf}(R) \to \mathfrak{X}_{r,I}$ corresponding to $\mathrm{Tate}(q^N)$ (in the moduli interpretation of $\mathfrak{X}_{r,I}$). For simplicity, we use $n = 1$ in the construction of these sheaves and p instead of $\beta_1 = p \cdot E_{p-1}(q)^{-p/(p-1)}$ as a generator of $\underline{\beta}_1$ (this makes sense because $E_{p-1}(q)$ is invertible in R). Now, using the formal variables $X = 1 + pZ$ and Y corresponding to the basis $f = \omega_{\mathrm{can}}$ and $e = \eta_{\mathrm{can}}$ of

[11] This is not obvious; it is a proposition (here not written as such for conciseness). It is proved in the same way as proposition 19, using the result in loc. cit. and that the de Rham cohomology groups of $(\mathrm{Fil}_m \mathbb{W}_{\kappa_I})^\bullet$ can be computed from the global sections.

$\mathscr{H}^{\sharp}(q) = \mathscr{H}(q)$, the local computations of section 3.1 show that

$$\underline{\omega}^{\kappa,0}(q) = R \cdot \kappa(1 + pZ)$$

and

$$\mathbb{W}_{\kappa}^{0}(q) = R\langle V \rangle \cdot \kappa(1 + pZ), \quad \text{where } V = \frac{Y}{1 + pZ}.$$

Define (in general, for any two locally analytic characters κ and s)

$$V_{\kappa,s} = s(V) \cdot \kappa(1 + pZ) = s(Y) \cdot (\kappa - s)(1 + pZ).$$

With this notation,

$$\mathrm{Fil}_m \mathbb{W}_{\kappa}^{0}(q) = \sum_{i=0}^{m} R \cdot V_{\kappa,i} \quad \text{for every } m \geq 0$$

and the q–expansion map is given by

$$\begin{aligned} \mathbb{W}_{\kappa}^{0}(q) &\longrightarrow R \\ \sum_{i \geq 0} \lambda_i \cdot V_{\kappa,i} &\longmapsto \lambda_0 \end{aligned}$$

(projection onto the piece with no powers of Y).

By the explicit formula[12] obtained right before theorem 14, we can write

$$\begin{aligned} \nabla_{\kappa} \colon \mathbb{W}_{\kappa}^{0}(q) &\longrightarrow \mathbb{W}_{\kappa}^{0}(q) \hat{\otimes}_R \Omega^1_{R/\Lambda_I^0} \\ \lambda \cdot V_{\kappa,i} &\longmapsto \partial(\lambda) V_{\kappa,i} \otimes \frac{dq}{q} + \lambda(u_{\kappa} - i) V_{\kappa,i+1} \otimes \frac{dq}{q} \end{aligned}$$

or, equivalently if we regard $\nabla_{\kappa} \colon \mathbb{W}_{\kappa}^{0}(q) \to \mathbb{W}_{\kappa+2}^{0}(q)$ (with $\frac{dq}{q}$ corresponding to $(1 + pZ)^2$ by Kodaira–Spencer's isomorphism),

$$\nabla_{\kappa}(\lambda V_{\kappa,i}) = \partial(\lambda) V_{\kappa+2,i} + \lambda(u_{\kappa} - i) V_{\kappa+2,i+1}.$$

Proposition 23. *Consider a class* $[\gamma] \in \mathrm{H}^1_{\mathrm{dR}}(\mathcal{X}_{r,I}, (\mathrm{Fil}_m \mathbb{W}_{\kappa}^{0})^{\bullet})$ *represented by some* $\gamma \in \mathrm{H}^0(\mathcal{X}_{r,I}, \mathrm{Fil}_{m+1} \mathbb{W}_{\kappa+2}^{0})$ *and express the evaluation of* γ *at* $\mathrm{Tate}(q^N)/R$ *as*

$$\gamma(q) = \sum_{i=0}^{m+1} \gamma_i(q) \cdot V_{\kappa+2,i}.$$

[12] In that notation, $a_0 = b = d_0 = 0$ and $c = \frac{dq}{q}$, so most of the terms simply cancel.

The q–expansion of $H_m^\dagger([\gamma])$ is

$$\sum_{i=0}^{m+1} \frac{\partial^i(\gamma_i(q))}{(u_\kappa - i+1)(u_\kappa - i+2)\cdots u_\kappa}$$

Proof. The idea is to "get rid of" the powers V^i using that the class $[\gamma]$ is defined in $\mathrm{Coker}\big(\nabla_\kappa\colon\ \mathrm{H}^0(\mathcal{X}_{r,I}, \mathrm{Fil}_m\,\mathbb{W}_\kappa^0) \to \mathrm{H}^0(\mathcal{X}_{r,I}, \mathrm{Fil}_{m+1}\,\mathbb{W}_{\kappa+2}^0)\big)$ (i.e., modulo the image of ∇_κ). The formula for ∇_κ above implies that

$$\nabla_\kappa\big(\gamma_i(q)V_{\kappa,i-1}\big) = \partial\big(\gamma_i(q)\big)V_{\kappa+2,i-1} + \gamma_i(q)(u_\kappa - i+1)V_{\kappa+2,i}$$

for $0 \le i \le m+1$. We deduce that

$$\begin{aligned}\gamma_i(q)V_{\kappa+2,i} &\equiv \frac{\partial\big(\gamma_i(q)\big)}{u_\kappa - i+1}V_{\kappa+2,i-1} \equiv \frac{\partial^2\big(\gamma_i(q)\big)}{(u_\kappa - i+1)(u_\kappa - i+2)}V_{\kappa+2,i-2} \equiv \cdots \\ &\equiv \frac{\partial^i\big(\gamma_i(q)\big)}{(u_\kappa - i+1)(u_\kappa - i+2)\cdots u_\kappa}V_{\kappa+2,0} \mod \nabla_\kappa\big(\mathrm{H}^0(\mathcal{X}_{r,I}, \mathrm{Fil}_m\,\mathbb{W}_\kappa^0)\big).\end{aligned}$$

The last term is in $\underline{\omega}^{\kappa+2,0}(q)$, so it is precisely $H_m^\dagger\big([\gamma(q)]\big)$ multiplied by the basis element $V_{\kappa+2,0}$. $\square$

In the remainder of this talk, we try to give some formulae for iterates of the Gauss–Manin connection in terms of q–expansions. Such formulae will be useful when we define p–adic iterations later.

Lemma 24. *Let $\lambda = \lambda(q) \in R$ and let $s \in \mathbb{Z}_{\ge 1}$. We can write*

$$\nabla_\kappa^s(\lambda V_{\kappa,i}) = \sum_{j=0}^{s} c_{s,j}\cdot \partial^{s-j}(\lambda)V_{\kappa+2s,i+j},$$

where

$$c_{s,j} = \binom{s}{j}\prod_{l=1}^{j}(u_\kappa - i + s - l) = \binom{s}{j}(u_\kappa - i+s-1)\cdots(u_\kappa - i + s - j).$$

Proof. We prove it by induction on s using that

$$\nabla_\kappa(\lambda V_{\kappa,l}) = \partial(\lambda)V_{\kappa+2,l} + \lambda(u_\kappa - l)V_{\kappa+2,l+1}.$$

The base case $s=1$ is clear from this. Moreover, the formula implies that

$$c_{s+1,j} = c_{s,j} + (u_{\kappa+2s} - i - j + 1)c_{s,j-1}$$

$$= (u_\kappa - i + s - 1) \cdots (u_\kappa - i + s + 1 - j) \cdot$$
$$\cdot \left[\binom{s}{j} (u_\kappa - i + s - j) + \binom{s}{j-1} (u_\kappa + 2s - i - j + 1) \right]$$

and this last part inside the square brackets can be expressed as

$$\frac{s!}{j!(s+1-j)!} \left[(s+1-j)(u_\kappa - i + s - j) + j(u_\kappa - i + s - j + s + 1) \right] =$$
$$= \frac{s!}{j!(s+1-j)!} \left[(s+1-j+j)(u\kappa - i + s - j) + j(s+1) \right]$$
$$= \frac{s!}{j!(s+1-j)!} (s+1)(u_\kappa - i + s) = \binom{s+1}{j} (u_\kappa - i + s).$$

All in all,

$$c_{s+1,j} = \binom{s+1}{j} (u_\kappa - i + s) \cdots (u_\kappa - i + s + 1 - j)$$

as claimed. □

Remark. This lemma suggests that, for a more general (locally analytic) weight $s \colon \mathbb{Z}_p^\times \to \Lambda_I^\times$, we should have

$$\nabla_\kappa^s (\lambda V_{\kappa,i}) = \sum_{j \geq 0} \binom{u_s}{j} \left[\prod_{l=1}^{j} (u_\kappa + u_s - i - l) \right] \partial^{s-j}(\lambda) V_{\kappa+2s,i+j}.$$

For such a formula to make sense, we will need some "divisibility by powers of p" condition on u_κ and u_s.

From now on, we sometimes omit the weight in the subscript of the Gauss–Manin connection (as we will mix several weights at the same time).

Proposition 25. *Let $\lambda = \lambda(q) \in R^{\mathrm{U}=0}$ and let $c \in \mathbb{Z}_{\geq 1}$ such that $p^{c-1} u_\kappa \in \Lambda_I^0$. For every $M \in \mathbb{Z}_{\geq 1}$, we can write*

$$\left(\nabla_\kappa^{p-1} - 1 \right)^{Mp} (\lambda V_{\kappa,0}) = \sum_{i=0}^{(p-1)Mp} \sum_{h \geq 0} p^{2M-(c+1)i-2h} \left[(1+pZ)^{2(p-1)} - 1 \right]^{hp} g_{i,h} V_{\kappa,i}$$

(where the 1 on the left-hand side means $\mathrm{id}_{\mathbb{W}_\kappa^0(q)}$) with $g_{i,h} \in R^{\mathrm{U}=0}[1+pZ]$ (i.e., these are polynomials with coefficients in $R^{\mathrm{U}=0}$ evaluated at $1+pZ$). If, moreover, $u_\kappa \in p\Lambda_I^0$ (in particular we can take $c = 1$), then $p^{M-2i-h} g_{i,h} \in R^{\mathrm{U}=0}[1+pZ]$ too.

Idea of the proof. Using lemma 24, we can compute for $H \in \mathbb{Z}_{\geq 1}$

$$
\begin{aligned}
\left(\nabla^{p-1}-1\right)^{H}(\lambda V_{\kappa,l}) &= \sum_{s=0}^{H}(-1)^{H-s}\,\nabla^{(p-1)s}(\lambda V_{\kappa,l}) \\
&= \sum_{s=0}^{H}\sum_{j=0}^{(p-1)s}\binom{H}{s}(-1)^{H-s}c_{(p-1)s,j}\partial^{(p-1)s-j}(\lambda)V_{\kappa+2(p-1)s,l+j} \\
&= \sum_{s=1}^{H}\sum_{j=1}^{(p-1)s}\binom{H}{s}(-1)^{H-s}c_{(p-1)s,j}\partial^{(p-1)s-j}(\lambda)(1+pZ)^{2(p-1)s}V_{\kappa,l+j}+ \\
&\quad+\sum_{s=0}^{H}\binom{H}{s}(-1)^{H-s}\partial^{(p-1)s-j}(\lambda)(1+pZ)^{2(p-1)s}V_{\kappa,l} \\
&= \sum_{s=1}^{H}\sum_{j=1}^{(p-1)s}\binom{H}{s}(-1)^{H-s}c_{(p-1)s,j}\partial^{(p-1)s-j}(\lambda)(1+pZ)^{2(p-1)s}V_{\kappa,l+j}+ \\
&\quad+\lambda\left[(1+pZ)^{2(p-1)}-1\right]^{H}V_{\kappa,l}+ \\
&\quad+\sum_{s=1}^{H}\binom{H}{s}(-1)^{H-s}\left[\partial^{(p-1)s}(\lambda)-\lambda\right](1+pZ)^{2(p-1)s}V_{\kappa,l}
\end{aligned}
$$

(in the third equality we separated $j \geq 1$ from $j = 0$ and in the last equality we added and subtracted $\lambda\left[(1+pZ)^{2(p-1)}-1\right]^{H}V_{\kappa,l}$).

Now the proposition can be proved by induction on M.

- For the base case $M = 1$, we set $H = p$ and $l = 0$ in the previous computation and make the following observations.
  - Writing

    $$\lambda(q) = \sum_{i\geq 0} a_i q^i \quad \text{and} \quad \partial^{(p-1)s}\big(\lambda(q)\big) = \sum_{i\geq 0} i^{(p-1)s}a_i q^i,$$

    we see coefficient by coefficient that

    $$\partial^{(p-1)s}(\lambda) \equiv \lambda \mod p \text{ if } 1 \leq s < p \ \text{ (resp. } \bmod p^2 \text{ if } s = p).$$

    Therefore,

    $$\binom{p}{s}\left[\partial^{(p-1)s}(\lambda)-\lambda\right] \in p^2 R^{\mathrm{U}=0} \ \text{ for } 1 \leq s \leq p.$$

    Thus, the third term of the computation gives the piece $(i,h) = (0,0)$ in the statement of the proposition.
  - The term $\lambda\left[(1+pZ)^{2(p-1)-1}\right]^{p}V_{\kappa,0}$ gives the piece $(i,h) = (0,1)$ in the statement of the proposition.

– By lemma 24, $c_{(p-1)s,j}$ is a polynomial in u_κ with integer coefficients and degree $\leq j$, whence $p^{j(c-1)}c_{(p-1)s,j} \in \Lambda_I^0$. Thus, the first term of the previous computation gives the pieces $(i,h) = (j,0)$ for $j \geq 1$ in the statement of the proposition.

- For the inductive step, it suffices to prove the following claim: for every $g \in R^{\mathrm{U}=0}[1+pZ]$, we can write

$$\left(\nabla^{p-1}-1\right)^p \left(p^{2M-(c+1)i-2h}\left[(1+pZ)^{2(p-1)}-1\right]^{ph} g V_{\kappa,i}\right) = \\ = \sum_{j=0}^{(p-1)p} \sum_{v \geq 0} p^{2M+2-(c+1)(i+j)-2v}\left[(1+pZ)^{2(p-1)}-1\right]^{pv} f_{j,v} V_{\kappa,i+j}$$

with $f_{j,v} \in R^{\mathrm{U}=0}$; if, moreover, $u_\kappa \in p\Lambda_I^0$ and $p^{M-2i-h}g \in R^{\mathrm{U}=0}[1+pZ]$, then $p^{M+1-2(i+j)-v}f_{j,v} \in R^{\mathrm{U}=0}[1+pZ]$ too. This claim can be proved with a similar analysis separating

$$\begin{aligned} &\left(\nabla^{p-1}-1\right)^p \left(p^{2M-(c+1)i-2h}\left[(1+pZ)^{2(p-1)}-1\right]^{ph} g V_{\kappa,i}\right) = \\ &= \sum_{s=0}^{p} \binom{p}{s} (-1)^{p-s} p^{2M-(c+1)i-2h} \sum_{t=0}^{(p-1)s} \binom{(p-1)s}{t} \cdot \\ &\qquad \cdot \nabla^t\left(\left[(1+pZ)^{2(p-1)}-1\right]^{ph}\right) \nabla^{(p-1)s-t}(gV_{\kappa,i}) \\ &= \sum_{s=1}^{p} \binom{p}{s} (-1)^{p-s} p^{2M-(c+1)i-2h} \sum_{t=1}^{(p-1)s} \binom{(p-1)s}{t} \cdot \\ &\qquad \cdot \nabla^t\left(\left[(1+pZ)^{2(p-1)}-1\right]^{ph}\right) \nabla^{(p-1)s-t}(gV_{\kappa,i}) + \\ &\quad + p^{2M-(c+1)i-2h}\left[(1+pZ)^{2(p-1)}-1\right]^{ph} \left(\nabla^{p-1}-1\right)^p (gV_{\kappa,i}) \end{aligned}$$

and studying each term using the computation in the beginning of the proof and the next lemma. □

Lemma 26. *For every $t, h \in \mathbb{Z}_{\geq 1}$, we can express*

$$\nabla^t\left(\left[(1+pZ)^{2(p-1)}-1\right]^{ph}\right) = \sum_{j \geq \max\{h-t,0\}}^{h-1} p^{h-j}\left[(1+pZ)^{2(p-1)}-1\right]^{pj} \cdot P_j \cdot V_{0,t}$$

with $P_j \in \mathbb{Z}[1+pZ]$.[13]

[13] In lemma 3.40 of Andreatta–Iovita's article "Triple product. . . " [2], this lemma has a second claim which is false. However, that part is unnecessary in the proof of proposition 25 because the inequality where they use it isn't tight.

Corollary 27. *Let $g(q) \in \underline{\omega}^{\kappa}(q)$ with $\mathrm{U}\big(g(q)\big) = 0$ and let $c \in \mathbb{Z}_{\geq 1}$ such that $p^{c-1}u_\kappa \in \Lambda_I^0$. For every $M \in \mathbb{Z}_{\geq 1}$,*

$$\big(\nabla_\kappa^{p-1} - 1\big)^{Mp}\big(g(q)\big) \in \sum_{i=0}^{(p-1)Mp} p^{2M-(c+1)i} \cdot \underline{\omega}^{\kappa,\mathrm{f}}(q)[Z] \cdot V_{\kappa,i}.$$

If, moreover, $u_\kappa \in p\Lambda_I^0$, then

$$\big(\nabla_\kappa^{p-1} - 1\big)^{Mp}\big(g(q)\big) \in p^M \cdot \sum_{i=0}^{(p-1)Mp} \underline{\omega}^{\kappa,\mathrm{f}}(q)[Z] \cdot V_{\kappa,i}.$$

References

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[2] Andreatta, F. and Iovita, A. *Triple product p–adic L–functions associated to finite slope p–adic families of modular forms*. Version 2 (preprint). 2019. arXiv: `1708.02785v2 [math.NT]`.

[3] Katz, N. M. 'p-adic properties of modular schemes and modular forms'. In: *Modular functions of one variable III*. Ed. by Serre, J.-P. and Kuijk, W. Lecture notes in mathematics 350. Berlin, Germany: Springer-Verlag, 1973, pp. 69–190.