Iwasawa theory and p–adic Hodge theory for relative Lubin–Tate extensions

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Abstract

These are notes on some aspects of Iwasawa theory and p–adic Hodge theory that I worked on in 2020. My goal back then was to compare (big) Iwasawa cohomology classes attached to a Hida family with other p–adic L–functions. But the tower of extensions that I had to deal with arised from the p–power torsion of an elliptic curve with CM, which made it a relative Lubin–Tate tower. As far as I know, there has been a lot of recent work trying to adapt the more classical cyclotomic p–adic Hodge theory to Lubin–Tate extensions. However, nobody published results for the relative Lubin–Tate case (which is only slightly more complex but sometimes appears more naturally in applications). These notes contain slight generalizations of other authors' works with proofs to convince oneself that the theory works essentially in the same way. My main source of inspiration was the work in progress of Peter Schneider and Otmar Venjakob.

WARNING: this document has never been revised by anyone else and I will most likely leave it in this unfinished state. As I wrote it for personal use, it is very different from an article meant for publication. I share it only so that others can avoid redoing the tedious work of adapting proofs to relative Lubin–Tate towers.

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1 Introduction

TODO!!!

Part I

The classical (cyclotomic) theory

2 Some p–adic Hodge theory

Throughout this section, let K be a finite extension of $\mathbb{Q}_p$ and let k denote its residue field. Let $F = W(k)[p^{-1}]$ be the maximal unramified subextension of K. Fix an algebraic closure $\overline{K}$ of K and let $G_K = \mathrm{Gal}(\overline{K}/K)$. Let $\mathbb{C}_p$ be the completion of $\overline{K}$ with the natural action of G_K obtained by requiring continuity.

For every $n \in \mathbb{Z}_{\geq 1}$, let μ_{p^n} denote the subgroup of p^n–th roots of unity in $\overline{K}$ and set $K_n = K(\mu_{p^n})$. Define also

$$K_\infty = \bigcup_{n \geq 1} K_n$$

and let $H_K = G_{K_\infty} = \mathrm{Gal}(\overline{K}/K_\infty)$ and $\Gamma_K = G_K/H_K \cong \mathrm{Gal}(K_\infty/K)$. We will use analogous notation for F and other field extensions of $\mathbb{Q}_p$. Write $e_K = [K_\infty : F_\infty]$ and let F' denote the maximal unramified extension of F inside F_∞.

Fix once and for all a compatible system of primitive p^n–th roots of unity $\varepsilon^{(n)}$ in $\overline{K}$ for all $n \geq 0$. That is, $\varepsilon^{(1)} \neq 1$ and $\left(\varepsilon^{(n+1)}\right)^p = \varepsilon^{(n)}$ for all $n \geq 0$. Equivalently, we have fixed a $\mathbb{Z}_p$–basis of $\mathbb{Z}_p(1) = \varprojlim \mu_{p^n}$ and the cyclotomic character $\chi_{\mathrm{cyc}} \colon G_K \to \mathbb{Z}_p^\times$ is now defined by the condition

$$\sigma\left(\varepsilon^{(n)}\right) = \left(\varepsilon^{(n)}\right)^{\chi_{\mathrm{cyc}}(\sigma)} \quad \text{for all } n \geq 0 \text{ and } \sigma \in G_K.$$

Observe that χ_{cyc} induces an isomorphism from Γ_K to an open subgroup of $\mathbb{Z}_p^\times$.

Definition 1. A $\mathbb{Q}_p$*–representation of* G_K is a finite-dimensional $\mathbb{Q}_p$–vector space V endowed with a continuous and linear action of the group G_K. Equivalently, the $\mathbb{Q}_p$–representation is given by a continuous morphism $\rho \colon G_K \to \mathrm{Aut}_{\mathbb{Q}_p}(V)$. Let $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ denote the category of such representations.

Definition 2. A $\mathbb{Z}_p$*–representation of* G_K is a finite free $\mathbb{Z}_p$–module T endowed with a continuous and linear action of the group G_K. Let $\mathrm{Rep}_{\mathbb{Z}_p}(G_K)$ denote the category of such representations.

Remark. Sometimes the notation $\mathrm{Rep}_{\mathbb{Z}_p}(G_K)$ is used to refer to a more general notion of $\mathbb{Z}_p$–representations in which T is allowed to be any finitely generated

$\mathbb{Z}_p$–module (i.e., allowing the possibility of having torsion). I am only interested in free $\mathbb{Z}_p$–representations, that arise as G_K–stable lattices of $\mathbb{Q}_p$–representations.

Fontaine started a theory of rings of periods which allows us to study Galois representations in terms of objects and structures resembling those of linear algebra. In the remainder of this section, we recall the construction of all the rings that might be useful and the corresponding Dieudonné modules. The reference that I found the most useful is Rebecca Bellovin's thesis [1] (the results of which are also published in the form of an article [2]). Cherbonnier–Colmez's article [13] has a very clear summary of the rings too and the last appendix of Berger's article [5] contains a helpful diagram outlining their relations.

2.1 Perfect rings of characteristic p

Consider the ring

$$\widetilde{\mathbf{E}}^{+} = \varprojlim_{x \mapsto x^p} \mathscr{O}_{\mathbb{C}_p} = \left\{ (x^{(0)}, x^{(1)}, \ldots) \in \prod_{n \geq 0} \mathscr{O}_{\mathbb{C}_p} : \left(x^{(n+1)}\right)^p = x^{(n)} \text{ for all } n \geq 0 \right\}$$

with the following operations: for $x = \left(x^{(n)}\right)_{n \geq 0}$ and $y = \left(y^{(n)}\right)_{n \geq 0}$ in $\widetilde{\mathbf{E}}^{+}$, the elements $x + y$ and xy of $\widetilde{\mathbf{E}}^{+}$ are defined by

$$(x+y)^{(n)} = \lim_{m \to \infty} \left(x^{(n+m)} + y^{(n+m)}\right)^{p^m} \quad \text{and} \quad (xy)^{(n)} = x^{(n)} y^{(n)}.$$

These operations arise from the natural coordinate-wise addition and multiplication of the perfection of $\mathscr{O}_{\mathbb{C}_p} / p\mathscr{O}_{\mathbb{C}_p}$ via the bijection

$$\begin{aligned} \varprojlim_{x \mapsto x^p} \left(\mathscr{O}_{\mathbb{C}_p} / p\mathscr{O}_{\mathbb{C}_p}\right) \cong \varprojlim_{x \mapsto x^p} \mathscr{O}_{\mathbb{C}_p} &= \widetilde{\mathbf{E}}^{+} \\ (\bar{x}_n)_{n \geq 0} \mapsto \left(x^{(n)} = \lim_{m \to \infty} x_{n+m}^{p^m}\right)_{n \geq 0} \end{aligned}$$

(also known as tilting). Recall that we have fixed $\varepsilon = \left(\varepsilon^{(0)}, \varepsilon^{(1)}, \ldots\right) \in \widetilde{\mathbf{E}}^{+}$.

There is a valuation $\mathrm{v}_{\widetilde{\mathbf{E}}}$ on $\widetilde{\mathbf{E}}^{+}$ defined by

$$\mathrm{v}_{\widetilde{\mathbf{E}}}(x) = \mathrm{v}_p\left(x^{(0)}\right)$$

and in fact $\widetilde{\mathbf{E}}^{+}$ is a valuation ring with fraction field

$$\widetilde{\mathbf{E}} = \varprojlim_{x \mapsto x^p} \mathbb{C}_p = \left\{ (x^{(0)}, x^{(1)}, \ldots) \in \prod_{n \geq 0} \mathbb{C}_p : \left(x^{(n+1)}\right)^p = x^{(n)} \text{ for all } n \geq 0 \right\}.$$

One can prove that $\widetilde{\mathbf{E}}$ is an algebraically closed field of characteristic p, perfect and complete with respect to $\mathrm{v}_{\widetilde{\mathbf{E}}}$. We write φ for the Frobenius endomorphism $x \mapsto x^p$ on $\widetilde{\mathbf{E}}$ as well as for its restriction to $\widetilde{\mathbf{E}}^+$.

2.2 Perfect rings of characteristic 0

Consider the rings of Witt vectors $\widetilde{\mathbf{A}}^+ = \mathrm{W}(\widetilde{\mathbf{E}}^+)$ and $\widetilde{\mathbf{A}} = \mathrm{W}(\widetilde{\mathbf{E}})$, which naturally admit lifts of the Frobenius morphisms that we also call φ. Define $\widetilde{\mathbf{B}}^+ = \widetilde{\mathbf{A}}^+[p^{-1}]$ and $\widetilde{\mathbf{B}} = \widetilde{\mathbf{A}}[p^{-1}]$. Then $\widetilde{\mathbf{A}}$ is a complete discrete valuation ring with field of fractions $\widetilde{\mathbf{B}}$ and residue field $\widetilde{\mathbf{E}}$.

Every element $x \in \widetilde{\mathbf{A}}$ can be expressed as

$$x = \sum_{k \geq 0} p^k [x_k] \quad \text{with } x_k \in \widetilde{\mathbf{E}} \text{ for all } k$$

in a unique way. Similarly, every element $x \in \widetilde{\mathbf{B}}$ is of the form

$$x = \sum_{k \gg -\infty} p^k [x_k] \quad \text{with } x_k \in \widetilde{\mathbf{E}} \text{ for all } k.$$

Set $\pi = [\varepsilon] - 1 \in \widetilde{\mathbf{A}}^+$, where $[\,\cdot\,]$ denotes the Teichmüller lift.

We mostly work with the weak topology. On $\widetilde{\mathbf{A}}$ one can define the weak topology via the basis of open neighbourhoods of 0

$$p^k \widetilde{\mathbf{A}} + \pi^n \widetilde{\mathbf{A}}^+ \quad \text{for } k, n \geq 0$$

(or alternatively as the product topology coming from the valuation $\mathrm{v}_{\widetilde{\mathbf{E}}}$ on $\widetilde{\mathbf{E}}$). Then the weak topologies on $\widetilde{\mathbf{A}}^+$, $\widetilde{\mathbf{B}}^+$ and $\widetilde{\mathbf{B}}$ are the induced ones from the weak topology on $\widetilde{\mathbf{A}}$ regarding

$$\widetilde{\mathbf{A}}^+ \subset \widetilde{\mathbf{A}}, \qquad \widetilde{\mathbf{B}} = \bigcup_{n \geq 0} p^{-n} \widetilde{\mathbf{A}}, \qquad \widetilde{\mathbf{B}}^+ \subset \widetilde{\mathbf{B}}.$$

All these rings are equipped with continuous actions of φ and of G_K (even of $G_{\mathbb{Q}_p}$) given by

$$\varphi\Big(\sum_k p^k [x_k]\Big) = \sum_k p^k [x_k^p] \qquad \text{and} \qquad \sigma\Big(\sum_k p^k [x_k]\Big) = \sum_k p^k [\sigma(x_k)]$$

(where the Galois action on $\widetilde{\mathbf{E}}$ is naturally induced by that on $\mathbb{C}_p$). The two actions commute.

2.2.1 Overconvergent elements

We will also want to use overconvergent versions of these rings. Take $s \in \mathbb{R}_{>0}$. We define

$$\widetilde{\mathbf{A}}^{\dagger,s} = \Big\{ \sum_{k\geq 0} p^k[x_k] \in \widetilde{\mathbf{A}} : \mathrm{v}_{\widetilde{\mathbf{E}}}(x_k) + \frac{psk}{p-1} \geq 0 \text{ for all } k \geq 0 \\ \text{and } \lim_{k\to\infty} \mathrm{v}_{\widetilde{\mathbf{E}}}(x_k) + \frac{psk}{p-1} = \infty \Big\}$$

and $\widetilde{\mathbf{B}}^{\dagger,s} = \widetilde{\mathbf{A}}^{\dagger,s}[p^{-1}]$. It turns out that these rings are complete with respect to the topology given by the valuation

$$\mathrm{w}_s(x) = \inf_{k\geq 0}\Big(\mathrm{v}_{\widetilde{\mathbf{E}}}(x_k) + \frac{psk}{p-1}\Big)$$

and thus become Banach spaces. (On $\widetilde{\mathbf{B}}^{\dagger,s}$ we consider the inductive limit after expressing

$$\widetilde{\mathbf{B}}^{\dagger,s} = \bigcup_{n\geq 0} p^{-n}\widetilde{\mathbf{A}}^{\dagger,s}$$

and using the topology given by w_s on each piece.) In general, we are interested in these kinds of rings only for $s \gg 0$. Therefore, we define

$$\widetilde{\mathbf{B}}^{\dagger} = \bigcup_{s>0} \widetilde{\mathbf{B}}^{\dagger,s} \quad \text{and} \quad \widetilde{\mathbf{A}}^{\dagger} = \widetilde{\mathbf{B}}^{\dagger} \cap \widetilde{\mathbf{A}}$$

with the induced topologies as inductive limit and subspace. (Note that $\widetilde{\mathbf{A}}^{\dagger}$ is strictly larger than the union of the $\widetilde{\mathbf{A}}^{\dagger,s}$.) Set $\widetilde{\mathbf{A}}_K^{\dagger,s} = (\widetilde{\mathbf{A}}^{\dagger,s})^{H_K}$, $\widetilde{\mathbf{B}}_K^{\dagger,s} = (\widetilde{\mathbf{B}}^{\dagger,s})^{H_K}$, $\widetilde{\mathbf{A}}_K^{\dagger} = (\widetilde{\mathbf{A}}^{\dagger})^{H_K}$ and $\widetilde{\mathbf{B}}_K^{\dagger} = (\widetilde{\mathbf{B}}^{\dagger})^{H_K}$ (and similarly for other fields).

Remark. Let $s_0 = (p-1)/p$ and $s_n = p^n s_0 = p^{n-1}(p-1)$ for every $n \geq 1$. Then $\widetilde{\mathbf{B}}^{\dagger,s_n}$ consists of the elements x of $\widetilde{\mathbf{B}}$ for which the series $\varphi^{-n}(x)$ converges in the ring of periods $\mathbf{B}_{\mathrm{dR}}^{+}$ that we will see in section 2.5.1. The appearance of the "strange" factor is due to the fact that, for $\bar{\pi} = \varepsilon - 1$,

$$\mathrm{v}_{\widetilde{\mathbf{E}}}(\bar{\pi}) = \frac{p}{p-1}.$$

More generally, for $0 \leq s \leq s' \leq \infty$ with $s, s' \in \mathbb{Z}[p^{-1}] \cup \{\infty\}$, we could define

$$\widetilde{\mathbf{A}}_{[s,s']} = \widetilde{\mathbf{A}}^{+}\Big\langle \frac{p}{[\bar{\pi}]^s}, \frac{[\bar{\pi}]^{s'}}{p} \Big\rangle$$

(the convergence for the Tate algebra being with respect to the p–adic topology), where by convention

$$\frac{p}{[\bar{\pi}]^\infty} = \frac{1}{[\bar{\pi}]} \quad \text{and} \quad \frac{[\bar{\pi}]^\infty}{p} = 0,$$

and $\widetilde{\mathbf{B}}_{[s,s']} = \widetilde{\mathbf{A}}_{[s,s']}[p^{-1}]$, which is naturally a Banach algebra. From the structure on $\widetilde{\mathbf{A}}^+$, we have continuous actions of G_K on $\widetilde{\mathbf{A}}_{[s,s']}$ and on $\widetilde{\mathbf{B}}_{[s,s']}$ and continuous Frobenius morphisms $\widetilde{\mathbf{A}}_{[s,s']} \to \widetilde{\mathbf{A}}_{[ps,ps']}$ and $\widetilde{\mathbf{B}}_{[s,s']} \to \widetilde{\mathbf{B}}_{[ps,ps']}$. In particular,

$$\widetilde{\mathbf{B}}_{[s,\infty]} = \widetilde{\mathbf{B}}^{\dagger,s}, \qquad \widetilde{\mathbf{A}}_{[0,\infty]} = \widetilde{\mathbf{A}}^+ \quad \text{and} \quad \widetilde{\mathbf{A}}_{[\infty,\infty]} = \widetilde{\mathbf{A}}^+$$

(cf. section 2.1 of Berger's article [4]).

2.3 Imperfect rings of characteristic p

There is an embedding of $k((T))$ into $\widetilde{\mathbf{E}}$ given by $T \mapsto \bar{\pi} = \varepsilon - 1$. One can prove that its image is independent of the choice of ε. Let $\mathbf{E}_F = \mathrm{Im}\big(k((T)) \hookrightarrow \widetilde{\mathbf{E}}\big)$. Let $\mathbf{E}$ denote the separable closure of $\mathbf{E}_F$ inside $\widetilde{\mathbf{E}}$ and let $\mathbf{E}^+$ be its ring of integers. One can prove that $\mathbf{E}$ is a dense subfield of $\widetilde{\mathbf{E}}$ and Fontaine–Wintenberger's theory of fields of norms yields an identification $\mathrm{Gal}(\mathbf{E}/\mathbf{E}_F) \cong H_F$. We define $\mathbf{E}_K = \mathbf{E}^{H_K}$, which is a finite field extension of $\mathbf{E}_F$ of degree $e_K = [K_\infty : F_\infty] = |H_F/H_K|$. Set $\mathbf{E}_K^+ = (\mathbf{E}^+)^{H_K}$, which is the ring of integers of $\mathbf{E}_K$.

2.4 Imperfect rings of characteristic 0

We want to define imperfect versions of all the rings with tildes and we start by lifting the construction of the previous subsection to characteristic 0.

There is an embedding of $\mathscr{O}_F((T))$ into $\widetilde{\mathbf{A}}$ defined by $T \mapsto \pi = [\varepsilon] - 1$. Let $\mathbf{A}_F$ denote the p–adic completion of $\mathrm{Im}\big(\mathscr{O}_F((T)) \hookrightarrow \widetilde{\mathbf{A}}\big)$ (recall that $\widetilde{\mathbf{A}}$ is complete with the weak topology and so with the p–adic topology too). We can express

$$\mathbf{A}_F = \left\{ \sum_{k\in\mathbb{Z}} a_k \pi^k \in \widetilde{\mathbf{A}} : a_k \in \mathscr{O}_F \text{ for all } k \in \mathbb{Z} \text{ and } \lim_{k\to-\infty} \mathrm{v}_p(a_k) = \infty \right\},$$

one can prove that $\mathbf{A}_F$ is a complete discrete valuation ring with residue field $\mathbf{E}_F$. Let $\mathbf{B}_F = \mathbf{A}_F[p^{-1}]$ be its field of fractions. Since the actions of φ and G_F are given by

$$\varphi(\pi) = (1+\pi)^p - 1 \quad \text{and} \quad \sigma(\pi) = (1+\pi)^{\chi_{\mathrm{cyc}}(\sigma)} - 1,$$

we see that both $\mathbf{A}_F$ and $\mathbf{B}_F$ are stable under φ and G_F.

Let $\mathbf{B}$ be the p–adic completion of the maximal unramified extension of $\mathbf{B}_F$ inside $\widetilde{\mathbf{B}}$ and put $\mathbf{A} = \mathbf{B} \cap \widetilde{\mathbf{A}}$, so that $\mathbf{B} = \mathbf{A}[p^{-1}]$. Then $\mathbf{A}$ is a complete discrete valuation ring with field of fractions $\mathbf{B}$ and residue field $\mathbf{E}$. Similarly, define $\mathbf{A}^+ = \mathbf{B} \cap \widetilde{\mathbf{A}}^+$ and $\mathbf{B}^+ = \mathbf{B} \cap \widetilde{\mathbf{B}}^+$. Since extensions of $\mathbf{E}_F$ correspond to unramified extensions of $\mathbf{B}_F$, the rings $\mathbf{A}^+$, $\mathbf{A}$, $\mathbf{B}^+$ and $\mathbf{B}$ are all stable under φ and G_K.

Set $\mathbf{A}_K = \mathbf{A}^{H_K}$ and $\mathbf{B}_K = \mathbf{B}^{H_K}$ (if $K = F$, we indeed recover the rings $\mathbf{A}_F$ and $\mathbf{B}_F$ defined before). Then $\mathbf{A}_K$ is a complete discrete valuation ring with field of fractions $\mathbf{B}_K = \mathbf{A}_K[p^{-1}]$ and residue field $\mathbf{E}_K$. The theory of fields of norms provides us with a uniformizer $\bar{\pi}_K$ of $\mathbf{E}_K$ and we can take its Teichmüller lift $\pi_K = [\bar{\pi}_K] \in \mathbf{A}_K$. Then we can identify

$$\mathbf{B}_K = \Big\{ \sum_{k\in\mathbb{Z}} a_k \pi_K^k \in \widetilde{\mathbf{B}} : a_k \in F' \text{ for all } k \in \mathbb{Z}, \\ \inf_{k\in\mathbb{Z}} \mathrm{v}_p(a_k) > -\infty \text{ and } \lim_{k\to-\infty} \mathrm{v}_p(a_k) = \infty \Big\}.$$

Remark. The rings with tilde are perfect and so φ is bijective. However, for the rings without tilde introduced in this subsection, φ is no longer surjective. On the other hand, one can prove that the ring $\widetilde{\mathbf{A}}_K$ (resp. the field $\widetilde{\mathbf{B}}_K$) contains $\varphi^{-\infty}(\mathbf{A}_K)$ as a dense subring (resp. $\varphi^{-\infty}(\mathbf{B}_K)$ as a dense subfield).

2.4.1 The operator ψ

The field $\mathbf{B}$ is a totally ramified extension of degree p of $\varphi(\mathbf{B})$. Therefore, the Frobenius morphism $\varphi\colon \mathbf{B} \to \mathbf{B}$ is injective but not surjective. We can at least define a left inverse ψ of φ as follows:

$$\psi(x) = \frac{1}{p}\varphi^{-1}\big(\mathrm{Tr}_{\mathbf{B}/\varphi(\mathbf{B})}(x)\big).$$

By definition, it is clear that $\psi\big(\varphi(x)\big) = x$. The operator ψ commutes with the action of G_K because so does φ.

More explicitly, one can check that $1, [\varepsilon], \ldots, [\varepsilon]^{p-1}$ is a basis of $\mathbf{B}$ over $\varphi(\mathbf{B})$. Then every element $x \in \mathbf{B}$ is of the form $x_0 + x_1[\varepsilon] + \cdots + x_{p-1}[\varepsilon]^{p-1}$ for some $x_0, x_1, \ldots, x_{p-1} \in \varphi(\mathbf{B})$ and

$$\psi(x) = \psi\big(x_0 + x_1[\varepsilon] + \cdots + x_{p-1}[\varepsilon]^{p-1}\big) = x_0.$$

In particular, $\psi(\mathbf{A}) \subset \mathbf{A}$.

2.4.2 Overconvergent elements

We define overconvergent rings analogously. For $s > 0$, let $\mathbf{B}^{\dagger,s} = \mathbf{B} \cap \widetilde{\mathbf{B}}^{\dagger,s}$ and $\mathbf{A}^{\dagger,s} = \mathbf{A} \cap \widetilde{\mathbf{A}}^{\dagger,s}$, each endowed with the subspace topology of its version with tilde (i.e., the topologies are given by the valuation w_s). In general, we are interested in these kinds of rings only for $s \gg 0$. Therefore, we define

$$\mathbf{B}^{\dagger} = \bigcup_{s>0} \mathbf{B}^{\dagger,s} = \mathbf{B} \cap \widetilde{\mathbf{B}}^{\dagger} \quad \text{and} \quad \mathbf{A}^{\dagger} = \mathbf{B}^{\dagger} \cap \mathbf{A} = \mathbf{B} \cap \widetilde{\mathbf{A}}^{\dagger}$$

with the induced topologies as inductive limit and subspace. (Note that, as is the case with the versions with tildes, $\mathbf{A}^{\dagger}$ is strictly larger than the union of the $\mathbf{A}^{\dagger,s}$.)

For $s > 0$, we set $\mathbf{B}_K^{\dagger,s} = (\mathbf{B}^{\dagger,s})^{H_K} = \mathbf{B}_K \cap \widetilde{\mathbf{B}}^{\dagger,s}$ and $\mathbf{A}_K^{\dagger,s} = (\mathbf{A}^{\dagger,s})^{H_K} = \mathbf{A}_K \cap \widetilde{\mathbf{A}}^{\dagger,s}$ with the induced topologies as subspaces of $\widetilde{\mathbf{B}}^{\dagger,s}$. Similarly, we set

$$\mathbf{B}_K^{\dagger} = (\mathbf{B}^{\dagger})^{H_K} = \bigcup_{s>0} \mathbf{B}_K^{\dagger,s} \quad \text{and} \quad \mathbf{A}_K^{\dagger} = (\mathbf{A}^{\dagger})^{H_K} = \mathbf{B}_K^{\dagger} \cap \mathbf{A}_K.$$

Since $\mathbf{A}_K$ is not perfect, for every $n \in \mathbb{Z}_{\geq 0}$ we define $\mathbf{A}_{K,n}^{\dagger,s} = \varphi^{-n}\big(\mathbf{A}_K^{\dagger,p^n s}\big)$. Finally, for $s \gg 0$, we can identify

$$\mathbf{B}_K^{\dagger,s} = \Big\{ \sum_{k\in\mathbb{Z}} a_k \pi_K^k \in \mathbf{B} : a_k \in F' \text{ for all } k \in \mathbb{Z}, \\ \inf_{k\in\mathbb{Z}} \mathrm{v}_p(a_k) > -\infty \text{ and } \lim_{k\to-\infty} \mathrm{v}_p(a_k) + \frac{k}{e_k s} = \infty \Big\}$$

(i.e., these are unbounded Laurent series with bounded coefficients in F' and convergent on the half-open annulus $0 < \mathrm{v}_p(T) \leq 1/(e_K s)$ evaluated at $T = \pi_K$; cf. proposition I.3 of Berger's article [5] and corollary II.2.4 of Cherbonnier–Colmez's article [13]).

2.5 Rings of periods

The rings above can be used to define Fontaine's rings of periods. There is a general formalism that uses these rings of periods to study p–adic Galois representations of G_K that are "nice" in some sense in terms of semilinear algebra structures. One could say that this theory simplifies the structures at the expense of making the coefficients much more complicated.

These rings of periods allow us to define different notions of admissibility on the algebraic side that should correspond to geometric properties of the Galois representations, at least conjecturally. For instance, what are known as de Rham (or

$\mathbf{B}_{\mathrm{dR}}$–admissible) representations are those that come from geometry and crystalline (or $\mathbf{B}_{\mathrm{crys}}$–admissible) representations are those that have good reduction at p.

2.5.1 The ring $\mathbf{B}_{\mathrm{dR}}$

There is a Galois-equivariant morphism of rings $\theta\colon \widetilde{\mathbf{B}}^+ \to \mathbb{C}_p$ defined by

$$\theta\Big(\sum_{k\gg-\infty} p^k[x_k]\Big) = \sum_{k\gg-\infty} p^k x_k^{(0)}$$

that is continuous with respect to the weak topology on $\widetilde{\mathbf{B}}^+$ and the p–adic topology on $\mathbb{C}_p$. One can show that θ is surjective and that $\mathrm{Ker}(\theta)$ is a principal ideal generated by

$$\omega = \frac{\pi}{\varphi^{-1}(\pi)} = \frac{[\varepsilon]-1}{[\varepsilon^{1/p}]-1}.$$

Then we define $\mathbf{B}_{\mathrm{dR}}^+$ to be the $\mathrm{Ker}(\theta)$–adic completion of $\widetilde{\mathbf{B}}^+$. There is an induced action of G_K on $\mathbf{B}_{\mathrm{dR}}^+$ but there is no natural way to define φ because $\mathrm{Ker}(\theta)$ is not stable under φ. It turns out that $\mathbf{B}_{\mathrm{dR}}^+$ is a discrete valuation ring with field of fractions $\mathbf{B}_{\mathrm{dR}}$ and residue field $\mathbb{C}_p$ and any generator of $\mathrm{Ker}(\theta)$ is a uniformizer. Since $\theta(\pi) = 0$, we can define a distinguished element

$$t = \log([\varepsilon]) = \log\big(1 + ([\varepsilon]-1)\big) = \sum_{k\geq 1} (-1)^{k-1}\frac{\pi^k}{k} \in \mathbf{B}_{\mathrm{dR}}^+$$

(i.e., the series converges in $\mathbf{B}_{\mathrm{dR}}^+$) which can be regarded as a p–adic analogue of the complex period $2\pi i$. One can prove that t is a uniformizer of $\mathbf{B}_{\mathrm{dR}}^+$ and that G_K acts on t through the cyclotomic character:

$$\sigma(t) = \log\big([\sigma(\varepsilon)]\big) = \log\big([\varepsilon]^{\chi_{\mathrm{cyc}}(\sigma)}\big) = \chi_{\mathrm{cyc}}(\sigma)t \quad \text{for all } \sigma \in G_K.$$

In particular, $\mathbf{B}_{\mathrm{dR}} = \mathbf{B}_{\mathrm{dR}}^+[t^{-1}]$ and the powers of t define a filtration on $\mathbf{B}_{\mathrm{dR}}$. One can also prove that $(\mathbf{B}_{\mathrm{dR}})^{G_K} = K$.

As a matter of fact, the valuation topology on $\mathbf{B}_{\mathrm{dR}}^+$ is not good enough and the action of G_K is not continuous. That is why we need to define a more complicated (and finer) topology. To do so, we consider on $\widetilde{\mathbf{A}}^+/\big(\mathrm{Ker}(\theta)^h \cap \widetilde{\mathbf{A}}^+\big)$ the quotient topology induced by the weak topology on $\widetilde{\mathbf{A}}^+$, which turns out to coincide with the one obtained from the p–adic topology on $\widetilde{\mathbf{A}}^+$. Then

$$\widetilde{\mathbf{B}}^+/\,\mathrm{Ker}(\theta)^h = \bigcup_{n\geq 0} \widetilde{\mathbf{A}}^+/\big(\mathrm{Ker}(\theta)^h \cap \widetilde{\mathbf{A}}^+\big)$$

can be made into a p–adic Banach space and so $\mathbf{B}_{\mathrm{dR}}^+$ becomes a Fréchet space.

Definition 3. Let V be a p–adic representation of G_K. We define the de Rham Dieudonné module

$$\mathbf{D}_{\mathrm{dR},K}(V) = (\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K},$$

which is a filtered K–vector space of dimension $\leq \dim_{\mathbb{Q}_p} V$. We say that V is a *de Rham representation* if $\dim_K \mathbf{D}_{\mathrm{dR},K}(V) = \dim_{\mathbb{Q}_p} V$.

2.5.2 The rings $\mathbf{B}_{\mathrm{crys}}$ and $\mathbf{B}_{\max}$

The ring $\mathbf{B}_{\mathrm{dR}}$ has the defect that we cannot define a Frobenius action on it. To remedy this one needs to focus on smaller subrings that are stable under φ. Thinking in these terms leads to the construction of a ring of periods $\mathbf{B}_{\mathrm{crys}}$ and the notion of *crystalline representations*. The problem is that the topology of $\mathbf{B}_{\mathrm{crys}}$ has some undesirable properties (cf. the second paragraph of section III.2 of Colmez's article [15]). Therefore, we actually work with a slightly larger ring $\mathbf{B}_{\max}$ that has a "nicer" topology. The two rings are interchangeable in the study of representations because $\varphi(\mathbf{B}_{\max}) \subset \mathbf{B}_{\mathrm{crys}} \subset \mathbf{B}_{\max}$ and the periods of crystalline representations actually lie in

$$\bigcap_{n\geq 0} \varphi^n(\mathbf{B}_{\mathrm{crys}}) = \bigcap_{n\geq 0} \varphi^n(\mathbf{B}_{\max}).$$

We can define $\mathbf{B}_{\max}^+ = \widetilde{\mathbf{B}}_{[0,s_0]}$ (see the end of section 2.2 for the meaning of this notation). By definition, $\mathbf{B}_{\max}^+$ is a Banach space and has continuous actions of both φ and G_K arising naturally from those on $\widetilde{\mathbf{A}}^+$. Furthermore, one can prove that there is a natural Galois-equivariant inclusion $\mathbf{B}_{\max}^+ \hookrightarrow \mathbf{B}_{\mathrm{dR}}^+$ and that $\mathbf{B}_{\max}^+$ contains the distinguished period t. In particular, $\varphi(t) = pt$. As a subring of $\mathbf{B}_{\mathrm{dR}}^+$, the ring $\mathbf{B}_{\max}^+$ can be identified with

$$\left\{ \sum_{k\geq 0} a_k \frac{\omega^k}{p^k} : a_k \in \widetilde{\mathbf{B}}^+ \text{ for all } k \geq 0 \text{ and } \lim_{k\to\infty} a_k = 0 \text{ for the } p\text{–adic topology} \right\}.$$

Define $\mathbf{B}_{\max} = \mathbf{B}_{\max}^+[t^{-1}]$ with the induced actions of φ and G_K. One can prove that $\mathbf{B}_{\max}^{G_K} = F$ and there is a G_K–equivariant embedding $\mathbf{B}_{\max} \otimes_F K \hookrightarrow \mathbf{B}_{\mathrm{dR}}$, whence $\mathbf{B}_{\max} \otimes_F K$ can be endowed with the subspace filtration.

Proposition 4. *There is a short exact sequence*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Q}_p & \longrightarrow & \mathbf{B}_{\max} & \longrightarrow & \mathbf{B}_{\max} \oplus (\mathbf{B}_{\mathrm{dR}}/\mathbf{B}_{\mathrm{dR}}^+) & \longrightarrow & 0 \\ & & & & x & \longmapsto & ((1-\varphi)x, x) & & \end{array}$$

of topological $\mathbb{Q}_p$*–vector spaces called the* fundamental exact sequence of p–adic Hodge theory.

Proof. See proposition III.3.1 of Colmez's article [15]. □

Definition 5. Let V be a p–adic representation of G_K. We define the crystalline Dieudonné module

$$\mathbf{D}_{\mathrm{crys},K}(V) = (\mathbf{B}_{\max} \otimes_{\mathbb{Q}_p} V)^{G_K},$$

which is a filtered φ–module over K of dimension $\leq \dim_{\mathbb{Q}_p} V$. We say that V is a *crystalline representation* if $\dim_F \mathbf{D}_{\mathrm{crys},K}(V) = \dim_{\mathbb{Q}_p} V$.

One key property of this theory is that, if V is a crystalline representation, we can recover V from $\mathbf{D}_{\mathrm{crys},K}(V)$:

$$V = \left[\mathrm{Fil}^0\big(\mathbf{B}_{\mathrm{crys}} \otimes_F \mathbf{D}_{\mathrm{crys},K}(V)\big)\right]^{\varphi=1}.$$

This is a consequence of the fact that $(\mathrm{Fil}^0 \mathbf{B}_{\mathrm{crys}})^{\varphi=1} = \mathbb{Q}_p$.

Theorem 6. *Every crystalline representation V is de Rham and*

$$\mathbf{D}_{\mathrm{dR},K}(V) = \mathbf{D}_{\mathrm{crys},K}(V) \otimes_F K.$$

2.6 The Robba ring

Let $s > 0$ as before. For every $s' \geq s$, the Banach valuation $\mathrm{w}_{s'}$ is well-defined on $\widetilde{\mathbf{B}}^{\dagger,s}$. We define $\widetilde{\mathbf{B}}^{\dagger,s}_{\mathrm{rig}}$ to be the Fréchet completion of $\widetilde{\mathbf{B}}^{\dagger,s}$ with respect to the valuations $(\mathrm{w}_{s'})_{s'>s}$ (more precisely, we can choose a sequence $(s_n)_{n\geq 1}$ going to infinity and use the family $(\mathrm{w}_{s_n})_{n\geq 1}$ to define the Fréchet topology). The actions of φ and G_K on $\widetilde{\mathbf{B}}^{\dagger,s}$ extend by continuity to $\widetilde{\mathbf{B}}^{\dagger,s}_{\mathrm{rig}}$. As usual, we set $\widetilde{\mathbf{B}}^{\dagger,s}_{\mathrm{rig},K} = (\widetilde{\mathbf{B}}^{\dagger,s}_{\mathrm{rig}})^{H_K}$.

Similarly, the *Robba ring* $\mathbf{B}^{\dagger,s}_{\mathrm{rig},K}$ is the Fréchet completion of $\mathbf{B}^{\dagger,s}_K$ with respect to the family of valuations $(\mathrm{w}_{s'})_{s'>s}$. We obtain actions of φ and Γ_K on $\mathbf{B}^{\dagger,s}_{\mathrm{rig},K}$ by continuity. If $s \gg 0$, we can identify

$$\mathbf{B}^{\dagger,s}_{\mathrm{rig},K} = \Big\{ \sum_{k\in\mathbb{Z}} a_k \pi_K^k \in \mathbf{B} : a_k \in F' \text{ for all } k \in \mathbb{Z} \\ \text{and } \lim_{k\to-\infty} \mathrm{v}_p(a_k) + \frac{k}{e_K s} = \infty \Big\}$$

(i.e., these are unbounded Laurent series with possibly unbounded coefficients in F' and convergent on the half-open annulus $0 < \mathrm{v}_p(T) \leq 1/(e_K s)$ evaluated at

$T = \pi_K$; thus, $\mathbf{B}^{\dagger,s}_{\mathrm{rig},K}$ corresponds to the ring $\mathcal{R}^{r_0}(\pi_K)$ of Kedlaya–Pottharst–Xiao's article [26] for $r_0 = (p-1)/(e_K s)$ or something similar).

Remark. The Frobenius morphism $\varphi\colon \mathbf{B}^{\dagger,s}_{\mathrm{rig},K} \to \mathbf{B}^{\dagger,ps}_{\mathrm{rig},K}$ is not surjective but makes $\mathbf{B}^{\dagger,ps}_{\mathrm{rig},K}$ into a free $\mathbf{B}^{\dagger,s}_{\mathrm{rig},K}$–module of rank p. Therefore, we can define a left inverse $\psi\colon \mathbf{B}^{\dagger,ps}_{\mathrm{rig},K} \to \mathbf{B}^{\dagger,s}_{\mathrm{rig},K}$ as in section 2.4.1.

Set $s_0 = (p-1)/p$ and $s_n = p^n s_0 = p^{n-1}(p-1)$ for every $n \geq 0$. As mentioned in a remark in section 2.2, there are injective morphisms $\varphi^{-n}\colon \widetilde{\mathbf{B}}^{\dagger,s_n}_K \hookrightarrow \mathbf{B}^+_{\mathrm{dR}}$. Recall that $\mathbf{B}^{\dagger,s_n}_K = \widetilde{\mathbf{B}}^{\dagger,s_n}_K \cap \mathbf{B}$ and observe that there is a copy of $K_n[\![t]\!]$ inside $\mathbf{B}^+_{\mathrm{dR}}$. By proposition III.2.1 of Cherbonnier–Colmez's article [14], for $n \gg 0$ we obtain

$$\begin{array}{ccc} \widetilde{\mathbf{B}}^{\dagger,s_n}_K & \xhookrightarrow{\varphi^{-n}} & \mathbf{B}^+_{\mathrm{dR}} \\ \cup & & \cup \\ \mathbf{B}^{\dagger,s_n}_K & \dashrightarrow & K_n[\![t]\!] \end{array}$$

by restriction. On the other hand, we have inclusions $\mathbf{B}^{\dagger,s_n}_K \subset \mathbf{B}^{\dagger,s_n}_{\mathrm{rig},K}$ and these morphisms extend by continuity to injective morphisms $\varphi^{-n}\colon \mathbf{B}^{\dagger,s_n}_{\mathrm{rig},K} \hookrightarrow \mathbf{B}^+_{\mathrm{dR}}$ for all $n \geq 0$ (cf. propositions 2.11 and 2.12 of Berger's article [4]).

As usual, we are interested in these kinds of rings only for $s \gg 0$, so we define

$$\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}} = \bigcup_{s>0} \widetilde{\mathbf{B}}^{\dagger,s}_{\mathrm{rig}} \quad \text{and} \quad \mathbf{B}^{\dagger}_{\mathrm{rig},K} = \bigcup_{s>0} \mathbf{B}^{\dagger,s}_{\mathrm{rig},K}.$$

The ring $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ is also called *the Robba ring of K*.

2.7 Rings for families

The rings of p–adic Hodge theory introduced up to now are very powerful tools in the study of p–adic representations of G_K (be it $\mathbb{Z}_p$– or $\mathbb{Q}_p$–representations of G_K). But our main goal is to study arithmetic families of such representations: these will be finite free modules over some "nice" topological ring R over $\mathbb{Z}_p$ or A over $\mathbb{Q}_p$ (satisfying certain finiteness conditions at least) endowed with a continuous linear action of G_K and from which we can obtain specializations that are p–adic representations in the classical sense. (The adjective arithmetic here means that there is no Frobenius morphism φ on the base R or A, in contrast to what are known as geometric families.) Therefore, we will need to put the rings of p–adic Hodge theory in families by base change to R or to A.

The base rings that we consider are of one of the two following kinds:

- either an affinoid $\mathbb{Q}_p$–algebra A with an integral model R (if A is reduced, we take $R = A^0$, the subring of power-bounded elements)
- or a *coefficient ring* R in the sense of Mazur.

Definition 7. A *coefficient ring* is a noetherian complete local ring $(R, \mathfrak{m}_R)$ whose residue field κ_R is finite (of characteristic p). In particular, R is a topological $\mathbb{Z}_p$–algebra (with the topology defined by its maximal ideal $\mathfrak{m}_R$).

2.7.1 Integral families

Definition 8. Let R be a coefficient ring. An *R–representation of G_K* (or a *family of representations of G_K over R*) is a finite free R–module $\mathbb{T}$ endowed with a continuous R–linear action of G_K. Let $\mathrm{Rep}_R(G_K)$ denote the category of such representations.

To study families of representations over R, we will need to replace the rings $\mathbf{A}^+$, $\mathbf{A}$, $\mathbf{A}_K^+$ and $\mathbf{A}_K$ with $\mathbf{A}^+ \widehat{\otimes}_{\mathbb{Z}_p} R$, $\mathbf{A} \widehat{\otimes}_{\mathbb{Z}_p} R$, $\mathbf{A}_K^+ \widehat{\otimes}_{\mathbb{Z}_p} R$ and $\mathbf{A}_K \widehat{\otimes}_{\mathbb{Z}_p} R$, respectively. Here, the symbol $\widehat{\otimes}$ denotes the completed tensor product with respect to the $\mathfrak{m}_R$–adic topology on R and the p–adic topologies on $\mathbf{A}$ and the more decorated subrings.

Remarks.

- Recall that $\mathbf{A}$ is a complete discrete valuation ring with residue field $\mathbf{E}$ of characteristic p. Thus, the p–adic topology on $\mathbf{A}$ coincides with the valuation topology.
- Completing $\mathbf{A} \otimes_{\mathbb{Z}_p} R$ with respect to $p\mathbf{A} \otimes_{\mathbb{Z}_p} R + \mathbf{A} \otimes_{\mathbb{Z}_p} \mathfrak{m}_R$ is the same as completing it with respect to $\mathfrak{m}_R$ only, as $p \in \mathfrak{m}_R$.
- We can extend the operators $\varphi, \psi\colon \mathbf{A} \to \mathbf{A}$ to $\mathbf{A} \otimes_{\mathbb{Z}_p} R$ by tensoring with R (i.e., making them act trivially on R). Then both φ and ψ map the ideal $p\mathbf{A} \otimes_{\mathbb{Z}_p} R + \mathbf{A} \otimes_{\mathbb{Z}_p} \mathfrak{m}_R$ to itself and we obtain in this way continuous R–linear maps $\varphi, \psi\colon \mathbf{A} \widehat{\otimes}_{\mathbb{Z}_p} R \to \mathbf{A} \widehat{\otimes}_{\mathbb{Z}_p} R$.
- As the action of G_K on $\mathbf{A}$ is not continuous for the p–adic topology (because the action of G_K on $\mathbf{E}$ is not discrete), one might prefer to work with the weak topology instead. Recall that a basis of neighbourhoods of 0 for the weak topology on $\mathbf{A}$ is given by

$$p^k\mathbf{A} + \pi^n\mathbf{A}^+ \quad \text{for } k, n \geq 0.$$

  One can check that the completion of $\mathbf{A} \otimes_{\mathbb{Z}_p} R$ obtained using this topology coincides with the completion defined above (cf. remark 2.2 of Bellovin–Venjakob's article [3]). Therefore, the induced action of G_K on $\mathbf{A} \widehat{\otimes}_{\mathbb{Z}_p} R$ is continuous.

- Analogous considerations apply to the other rings under consideration.

2.7.2 Rigid families

Definition 9. Let A be an affinoid $\mathbb{Q}_p$–algebra. An *A–representation of G_K* (or a *family of representations of G_K over A*) is a finite free A–module $\mathbb{V}$ endowed with a continuous A–linear action of G_K. Let $\mathrm{Rep}_A(G_K)$ denote the category of such representations.

Let $\mathcal{X}$ be a (quasi-separated) rigid analytic space over $\mathbb{Q}_p$. An *$\mathcal{X}$–representation of G_K* (or a *rigid analytic family of representations of G_K over $\mathcal{X}$*) is a finite locally free $\mathscr{O}_{\mathcal{X}}$–module $\mathscr{V}$ of constant rank d endowed with a continuous $\mathscr{O}_{\mathcal{X}}$–linear action of G_K, in the sense that there exists an admissible affinoid covering $(\mathcal{U}_i)_{i\in I}$ of $\mathcal{X}$ such that $\mathscr{V}(\mathcal{U}_i)$ is an $\mathscr{O}_{\mathcal{X}}(\mathcal{U}_i)$–representation of G_K as above for every $i \in I$.

To study this kind of rigid analytic families of representations, we will need to consider again completed tensor products with the base ring. However, the topologies in this situation are more complicated because most of the rings are only Banach or even Fréchet spaces over $\mathbb{Q}_p$.

Consider two Fréchet spaces C and C' over $\mathbb{Q}_p$ whose topologies are given by families of seminorms $(\rho_i)_{i\in I}$ and $(\rho'_j)_{j\in J}$, respectively. For each pair (ρ_i, ρ'_j) we can define a seminorm $\rho_i \otimes \rho'_j$ on $C \otimes_{\mathbb{Q}_p} C'$ by

$$(\rho_i \otimes \rho'_j)(z) = \inf\Big\{\max_{1\le l\le r}\{\rho_i(x_l)\cdot \rho'_j(y_l)\}\Big\},$$

where the infimum runs over all possible representations

$$z = \sum_{l=1}^{r} x_l \otimes y_l \quad \text{with } x_l \in C \text{ and } y_l \in C' \text{ for all } 1 \le l \le r.$$

Then we write $C \widehat{\otimes}_{\mathbb{Q}_p} C'$ for the completion of $C \otimes_{\mathbb{Q}_p} C'$ with respect to the topology given by the family of seminorms $(\rho_i \otimes \rho'_j)_{(i,j)\in I\times J}$, which is again a Fréchet space.

In fact, the tensor products in our situation are simpler than this because at least the affinoid algebra A is already a Banach space over $\mathbb{Q}_p$. Thus, if B is a Banach algebra over $\mathbb{Q}_p$, we can form the Banach algebra $B \widehat{\otimes}_{\mathbb{Q}_p} A$ with one single tensor product norm. If B is a Fréchet algebra over $\mathbb{Q}_p$ obtained as

$$B = \varprojlim_{n\ge 1} B_n$$

where B_n is a $\mathbb{Q}_p$–Banach algebra for every $n \geq 1$, then

$$B \,\widehat{\otimes}_{\mathbb{Q}_p} A = \varprojlim_{n \geq 1} \big(B_n \,\widehat{\otimes}_{\mathbb{Q}_p} A\big).$$

All in all, we can study A–representations of G_K using $B \,\widehat{\otimes}_{\mathbb{Q}_p} A$, where B can be any of the rings $\mathbf{B}^{\dagger,s}$, $\mathbf{B}_K^{\dagger,s}$, $\widetilde{\mathbf{B}}^{\dagger,s}$, $\widetilde{\mathbf{B}}_K^{\dagger,s}$, $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,s}$, $\widetilde{\mathbf{B}}_{\mathrm{rig},K}^{\dagger,s}$ or $\mathbf{B}_{\mathrm{rig},K}^{\dagger,s}$ for $s \gg 0$ or also $\mathbf{B}_{\mathrm{dR}}^+$ or $\mathbf{B}_{\max}^+$. Furthermore, we set

$$\mathbf{B}_{\mathrm{rig},K}^{\dagger} \,\widehat{\otimes}_{\mathbb{Q}_p} A = \bigcup_{s>0} \big(\mathbf{B}_{\mathrm{rig},K}^{\dagger,s} \,\widehat{\otimes}_{\mathbb{Q}_p} A\big)$$

(and analogously for the other rings which might have s in the notation). Similarly, we write

$$\mathbf{B}_{\mathrm{dR}} \,\widehat{\otimes}_{\mathbb{Q}_p} A = \bigcup_{n \geq 0} \big(t^{-n}\mathbf{B}_{\mathrm{dR}}^+ \,\widehat{\otimes}_{\mathbb{Q}_p} A\big) \quad \text{and} \quad \mathbf{B}_{\max} \,\widehat{\otimes}_{\mathbb{Q}_p} A = \bigcup_{n \geq 0} \big(t^{-n}\mathbf{B}_{\max}^+ \,\widehat{\otimes}_{\mathbb{Q}_p} A\big).$$

All these algebras over A inherit the additional structure (Galois action, Frobenius, filtration...) from the original rings.

Definition 10. Let $\mathbb{V}$ be an A–representation of G_K of rank d.

(1) We define the de Rham Dieudonné module

$$\mathbf{D}_{\mathrm{dR},K}(\mathbb{V}) = \big((\mathbf{B}_{\mathrm{dR}} \,\widehat{\otimes}_{\mathbb{Q}_p} A) \otimes_A \mathbb{V}\big)^{G_K},$$

which is a filtered $(K \otimes_{\mathbb{Q}_p} A)$–module. If it is locally free of rank d, we say that $\mathbb{V}$ is a *de Rham representation*.

(2) We define the crystalline Dieudonné module

$$\mathbf{D}_{\mathrm{crys},K}(\mathbb{V}) = \big((\mathbf{B}_{\max} \,\widehat{\otimes}_{\mathbb{Q}_p} A) \otimes_A \mathbb{V}\big)^{G_K},$$

which is a filtered φ–module over $(K \otimes_{\mathbb{Q}_p} A)$. If it is locally free of rank d over $(F \otimes_{\mathbb{Q}_p} A)$, we say that $\mathbb{V}$ is a *crystalline representation*.

Finally, the previous constructions can be globalized as follows. Let $\mathcal{X}$ be a quasi-separated rigid analytic space and let B denote one of the previous rings of p–adic Hodge theory. We can define a presheaf of topological rings $\mathscr{B}$ on $\mathcal{X}$ by setting

$$\mathscr{B}(\mathcal{U}) = B \,\widehat{\otimes}_{\mathbb{Q}_p} \mathscr{O}_{\mathcal{X}}(\mathcal{U})$$

for every affinoid open $\mathcal{U} = \mathrm{Spm}(A)$ of $\mathcal{X}$.

Lemma 11. *The presheaf $\mathscr{B}$ is actually a sheaf.*

Proof. See lemma 3.3 of Kedlaya–Liu's article [25]. □

Definition 12. Let $\mathscr{V}$ be an $\mathcal{X}$–representation of G_K of rank d.

(1) We define the de Rham Dieudonné sheaf $\mathscr{D}_{\mathrm{dR},K}(\mathscr{V})$ by

$$\mathscr{D}_{\mathrm{dR},K}(\mathscr{V})(\mathcal{U}) = \left(\mathscr{B}_{\mathrm{dR}}(\mathcal{U}) \otimes_{\mathscr{O}_{\mathcal{X}}(\mathcal{U})} \mathscr{V}(\mathcal{U})\right)^{G_K}$$

for every affinoid open $\mathcal{U} = \mathrm{Spm}(A)$ of $\mathcal{X}$. If $\mathscr{D}_{\mathrm{dR},K}(\mathscr{V})$ is a locally free $(K \otimes_{\mathbb{Q}_p} \mathscr{O}_{\mathcal{X}})$–module of rank d, we say that $\mathscr{V}$ is a *de Rham representation*.

(2) We define the crystalline Dieudonné sheaf $\mathscr{D}_{\mathrm{crys},K}(\mathscr{V})$ by

$$\mathscr{D}_{\mathrm{crys},K}(\mathscr{V})(\mathcal{U}) = \left(\mathscr{B}_{\max}(\mathcal{U}) \otimes_{\mathscr{O}_{\mathcal{X}}(\mathcal{U})} \mathscr{V}(\mathcal{U})\right)^{G_K}$$

for every affinoid open $\mathcal{U} = \mathrm{Spm}(A)$ of $\mathcal{X}$. If $\mathscr{D}_{\mathrm{crys},K}(\mathscr{V})$ is a locally free $(F \otimes_{\mathbb{Q}_p} \mathscr{O}_{\mathcal{X}})$–module of rank d, we say that $\mathscr{V}$ is a *crystalline representation*.

3 (φ, Γ)–modules

We keep with the notation introduced in section 2. As a matter of fact, the theory of (φ, Γ)–modules can be considered as part of p–adic Hodge theory. In the same spirit, Fontaine introduced a category of (étale) (φ, Γ_K)–modules, that is a category in the realm of semilinear algebra, and proved that it is equivalent to the category of finitely generated $\mathbb{Z}_p$–modules with a continuous linear action of G_K (including the possibility of torsion).

Therefore, all kinds of constructions with $\mathbb{Z}_p$–representations of G_K should have equivalent versions using (φ, Γ_K)–modules. In particular, Herr studied Galois cohomology with this formalism in his thesis (see the related article [23]) and Cherbonnier and Colmez applied it to Iwasawa theory in their article [14]. In what follows, we explain these results and their versions for families of representations.

3.1 First definitions

Definition 13. A *φ–module* over $\mathbf{A}_K$ is a finite free $\mathbf{A}_K$–module D endowed with a φ–semilinear map $\varphi = \varphi_D \colon D \to D$. We say that the φ–module (D, φ_D) is *étale* if the $\mathbf{A}_K$–linearization $\varphi^*(D) = \mathbf{A}_K \otimes_{\varphi, \mathbf{A}_K} D \to D$ of φ is an isomorphism.

We define *(étale) φ–modules* over other rings such as $\mathbf{B}_K$, $\mathbf{A}_K^{\dagger,s}$, $\mathbf{B}_K^{\dagger,s}$, $\mathbf{A}_K^{\dagger}$, $\mathbf{B}_K^{\dagger}$, $\mathbf{B}_{\mathrm{rig},K}^{\dagger,s}$ or $\mathbf{B}_{\mathrm{rig},K}^{\dagger}$ analogously.

Remark. Sometimes the name φ–module is used to refer to a more general notion in which D is allowed to be any finitely generated $\mathbf{A}_K$–module (not necessarily free). Since I am only interested in free $\mathbb{Z}_p$–representations, this definition works.

Definition 14. A *(φ, Γ_K)–module* over $\mathbf{A}_K$ is a φ–module D endowed with a continuous $\mathbf{A}_K$–semilinear action of Γ_K commuting with φ. We say that D is an *étale* (φ, Γ_K)–module if the underlying φ–module is étale.

We define *(étale) (φ, Γ_K)–modules* over other rings such as $\mathbf{B}_K$, $\mathbf{A}_K^{\dagger,s}$, $\mathbf{B}_K^{\dagger,s}$, $\mathbf{A}_K^{\dagger}$, $\mathbf{B}_K^{\dagger}$, $\mathbf{B}_{\mathrm{rig},K}^{\dagger,s}$ or $\mathbf{B}_{\mathrm{rig},K}^{\dagger}$ analogously.

Definition 15. Let T (resp. V) be a $\mathbb{Z}_p$–representation (resp. $\mathbb{Q}_p$–representation) of G_K. We define the associated (φ, Γ_K)–modules

$$\mathbf{D}(T) = (\mathbf{A} \otimes_{\mathbb{Z}_p} T)^{H_K} \quad (\text{over } \mathbf{A}_K)$$

and

$$\mathbf{D}(V) = (\mathbf{B} \otimes_{\mathbb{Q}_p} V)^{H_K} \quad (\text{over } \mathbf{B}_K).$$

These modules inherit actions of the operators φ and ψ from those on $\mathbf{A}$ and $\mathbf{B}$ and residual actions of $\Gamma_K = G_K/H_K$ from the diagonal Galois actions.

The main result which makes these (φ, Γ_K)–modules useful is the following theorem of Fontaine.

Theorem 16 (Fontaine). *The functor $T \mapsto \mathbf{D}(T)$ defines an equivalence between the categories of $\mathbb{Z}_p$–representations of G_K and of étale (φ, Γ_K)–modules over $\mathbf{A}_K$.*

Similarly, the functor $V \mapsto \mathbf{D}(V)$ defines an equivalence between the categories of $\mathbb{Q}_p$–representations of G_K and of étale (φ, Γ_K)–modules over $\mathbf{B}_K$.

Proof. See theorem 3.4.3 of Fontaine's article [21]. □

In particular, we can recover the representations from the (φ, Γ_K)–modules as follows:

$$T \cong \big(\mathbf{A} \otimes_{\mathbf{A}_K} \mathbf{D}(T)\big)^{\varphi=1} \quad \text{and} \quad V \cong \big(\mathbf{B} \otimes_{\mathbf{B}_K} \mathbf{D}(V)\big)^{\varphi=1}$$

(the Galois actions on the right-hand sides are the diagonal actions of G_K and similarly φ means $\varphi \otimes \varphi_{\mathbf{D}}$).

3.2 Overconvergent versions

Sometimes it is useful to work with smaller (φ, Γ_K)–modules.

Definition 17. Let V be a $\mathbb{Q}_p$–representation of G_K. We define the associated overconvergent (φ, Γ_K)–modules

$$\mathbf{D}^\dagger(V) = (\mathbf{B}^\dagger \otimes_{\mathbb{Q}_p} V)^{H_K} \quad \text{and} \quad \mathbf{D}^{\dagger,s}(V) = (\mathbf{B}^{\dagger,s} \otimes_{\mathbb{Q}_p} V)^{H_K} \text{ for all } s > 0.$$

Then $\dim_{\mathbf{B}^\dagger_K}\big(\mathbf{D}^\dagger(V)\big) \leq \dim_{\mathbb{Q}_p}(V)$ and we say that V is *overconvergent* if these two dimensions are equal.

Similarly, for a $\mathbb{Z}_p$–representation T of G_K we define

$$\mathbf{D}^\dagger(T) = (\mathbf{A}^\dagger \otimes_{\mathbb{Z}_p} T)^{H_K} \quad \text{and} \quad \mathbf{D}^{\dagger,s}(T) = (\mathbf{A}^{\dagger,s} \otimes_{\mathbb{Z}_p} T)^{H_K} \text{ for all } s > 0$$

and we say that T is *overconvergent* if the rank of the free $\mathbf{A}^\dagger_K$–module $\mathbf{D}^\dagger(T)$ coincides with $\operatorname{rank}_{\mathbb{Z}_p}(T)$.

Theorem 18 (Cherbonnier–Colmez). *Every $\mathbb{Z}_p$– or $\mathbb{Q}_p$–representation of G_K is overconvergent.*

Proof. This is the main result of Cherbonnier–Colmez's article [13] (in particular, see their corollary III.5.2). □

Therefore, for any $\mathbb{Q}_p$–representation V of G_K,

$$\mathbf{D}(V) = \mathbf{B}_K \otimes_{\mathbf{B}_K^\dagger} \mathbf{D}^\dagger(V) = \mathbf{B}_K \otimes_{\mathbf{B}_K^{\dagger,s}} \mathbf{D}^{\dagger,s}(V) \quad \text{if } s \gg 0.$$

3.3 Modules over the Robba ring

TODO!!!

3.4 Modules for integral families

Throughout this subsection, let R be a coefficient ring in the sense of definition 7. We recall some results of Dee's article [19] that extend the equivalence of categories from theorem 16 to the case of $\mathrm{Rep}_R(G_K)$.

Definition 19. A *φ–module* over $\mathbf{A}_K \widehat{\otimes}_{\mathbb{Z}_p} R$ is a finite $(\mathbf{A}_K \widehat{\otimes}_{\mathbb{Z}_p} R)$–module D endowed with a φ–semilinear map $\varphi = \varphi_D \colon D \to D$. We say that the φ–module (D, φ_D) is *étale* if the $(\mathbf{A}_K \widehat{\otimes}_{\mathbb{Z}_p} R)$–linearization $\varphi^*(D) \to D$ of φ_D is an isomorphism.

Remark. This definition only requires D to be finitely generated (instead of finite free) for convenience because the proofs of the results for families are based on reducing to the case of finite length by taking projective limits of quotients. This is incoherent with the previous definitions (like definition 13), so I should rewrite things better in the future.

Definition 20. A *(φ, Γ_K)–module* over $\mathbf{A}_K \widehat{\otimes}_{\mathbb{Z}_p} R$ is a φ–module D endowed with a continuous $(\mathbf{A}_K \widehat{\otimes}_{\mathbb{Z}_p} R)$–semilinear action of Γ_K commuting with φ. We say that D is an *étale* (φ, Γ_K)–module if the underlying φ–module is étale.

Definition 21. Let $\mathbb{T}$ be an R–representation of G_K. We define the associated (φ, Γ_K)–module over $\mathbf{A}_K \widehat{\otimes}_{\mathbb{Z}_p} R$

$$\mathbf{D}(\mathbb{T}) = \left((\mathbf{A} \widehat{\otimes}_{\mathbb{Z}_p} R) \otimes_R \mathbb{T}\right)^{H_K}$$

with the actions of the operators φ and ψ inherited from those on $\mathbf{A}$ and the residual action of $\Gamma_K = G_K / H_K$ obtained from the diagonal action of G_K.

Remark. As before, this definition makes sense even if $\mathbb{T}$ is not free as an R–module.

Lemma 22. *Let $\mathbb{T}$ be an R–representation of G_K such that $\mathfrak{m}_R^n \mathbb{T} = 0$ for some $n \in \mathbb{Z}_{\geq 1}$. Write T for $\mathbb{T}$ seen as a $\mathbb{Z}_p$–representation of G_K. Then $\mathbf{D}(\mathbb{T}) = \mathbf{D}(T)$.*

Remark. Since $\kappa_R = R/\mathfrak{m}_R$ is a finite extension of $\mathbb{F}_p$, the hypothesis that $\mathfrak{m}_R^n \mathbb{T} = 0$ implies that indeed T is a finite $\mathbb{Z}_p$–module. The content of the lemma is that the two associated (φ, Γ_K)–modules, one over $\mathbf{A}_K \widehat{\otimes}_{\mathbb{Z}_p} R$ and the other over $\mathbf{A}_K$, are the same.

Proof. The completed tensor product with a finite module coincides with the (algebraic) tensor product. Therefore,

$$(\mathbf{A} \widehat{\otimes}_{\mathbb{Z}_p} R) \otimes_R \mathbb{T} = (\mathbf{A} \widehat{\otimes}_{\mathbb{Z}_p} R) \widehat{\otimes}_R \mathbb{T} \cong \mathbf{A} \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{T} = \mathbf{A} \widehat{\otimes}_{\mathbb{Z}_p} T = \mathbf{A} \otimes_{\mathbb{Z}_p} T.$$

Taking H_K–invariants on both sides we conclude that $\mathbf{D}(\mathbb{T}) = \mathbf{D}(T)$. □

Lemma 23. *Let $\mathbb{T}$ be an R–representation of G_K.*

(1) *For every $n \geq 1$, define $T_n = \mathbb{T}/\mathfrak{m}_R^n \mathbb{T}$. Then*

$$\mathbf{D}(\mathbb{T}) \cong \varprojlim_{n \geq 1} \mathbf{D}(T_n).$$

(2) *The $(\mathbf{A}_K \widehat{\otimes}_{\mathbb{Z}_p} R)$–module $\mathbf{D}(\mathbb{T})$ is complete with respect to the $\mathfrak{m}_R$–adic topology:*

$$\mathbf{D}(\mathbb{T}) \cong \varprojlim_{n \geq 1} \big(\mathbf{D}(\mathbb{T})/\mathfrak{m}_R^n \mathbf{D}(\mathbb{T})\big).$$

Idea of the proof.

(1) It is easy to see that taking H_K–invariants commutes with the projective limit. Moreover,

$$\varprojlim_{n \geq 1} \big((\mathbf{A} \widehat{\otimes}_{\mathbb{Z}_p} R) \otimes_R T_n\big) \cong \varprojlim_{n \geq 1} \big((\mathbf{A} \widehat{\otimes}_{\mathbb{Z}_p} R) \otimes_R \mathbb{T}\big)/\mathfrak{m}_R^n \cong (\mathbf{A} \widehat{\otimes}_{\mathbb{Z}_p} R) \otimes_R \mathbb{T}.$$

See proposition 2.1.8 of Dee's article [19] for more details.

(2) It follows from the previous part and the fact that $\mathbf{D}(T_n) \cong \mathbf{D}(\mathbb{T})/\mathfrak{m}_R^n \mathbf{D}(\mathbb{T})$; see corollary 2.1.10 of Dee's article [19]. □

Lemmata 22 and 23 allow us to reduce the study of $\mathbf{D}(\mathbb{T})$ to the case in which $\mathbb{T}$ is a torsion R–representation and in that situation, regarding $\mathbb{T}$ as a $\mathbb{Z}_p$–representation, one can apply Fontaine's results (cf. theorem 16). One of the main consequences is the following result.

Theorem 24 (Dee). *The functor $\mathbb{T} \mapsto \mathbf{D}(\mathbb{T})$ defines an equivalence between the categories of R–representations of G_K and of étale (φ, Γ_K)–modules over $\mathbf{A}_K \widehat{\otimes}_{\mathbb{Z}_p} R$.*

Proof. See theorems 2.1.27 and 2.2.1 of Dee's article [19]. □

In particular, we can recover the representation from the (φ, Γ_K)–module as follows:

$$\mathbb{T} \cong \big((\mathbf{A} \,\widehat{\otimes}_{\mathbb{Z}_p} R) \otimes_{\mathbf{A}_K \widehat{\otimes}_{\mathbb{Z}_p} R} \mathbf{D}(\mathbb{T})\big)^{\varphi=1}.$$

3.5 Modules for rigid families

TODO !!!

3.6 Herr cohomology (for integral families)

Fontaine and Herr introduced a cohomology theory for (φ, Γ_K)–modules that allows us to compute the cohomology of p–adic Galois representations (see Herr's article [23] based on his thesis). Here we recall a few of their constructions together with results of Cherbonnier–Colmez's article [14] that we will need to study Iwasawa cohomology.

Let R be a coefficient ring as in definition 7 (in particular, we allow the possibility that $R = \mathbb{Z}_p$). Throughout this subsection we assume that K contains μ_p (or μ_4 if $p = 2$), so that K_∞ is the cyclotomic $\mathbb{Z}_p$–extension of K. Fix a topological generator γ of Γ_K.

Definition 25. Let $\mathbb{T}$ be an R–representation of G_K and let $u\colon \mathbf{D}(\mathbb{T}) \to \mathbf{D}(\mathbb{T})$ be an R–linear map commuting with the action of Γ_K (e.g., $u = \varphi$ or ψ). The *Herr complex* $C_{u,\gamma}(K, \mathbb{T})$ is the complex of (φ, Γ_K)–modules

$$C_{u,\gamma}(K,\mathbb{T})\colon\quad 0 \longrightarrow \mathbf{D}(\mathbb{T}) \xrightarrow{(u-1,\gamma-1)} \mathbf{D}(\mathbb{T}) \oplus \mathbf{D}(\mathbb{T}) \xrightarrow{(\gamma-1)\oplus(1-u)} \mathbf{D}(\mathbb{T}) \longrightarrow 0,$$

where the first labelled arrow is the map

$$x \mapsto \big((u-1)x, (\gamma-1)(x)\big)$$

and the second labelled arrow is the map

$$(y, z) \mapsto (\gamma - 1)(y) - (u-1)(z).$$

We write $\mathrm{H}^i_{u,\gamma}(K, \mathbb{T}) = \mathrm{H}^i\big(C_{u,\gamma}(K, \mathbb{T})\big)$ and call it the *i–th (u, γ)–cohomology group of* $\mathbf{D}(\mathbb{T})$.

In particular, a 1–cocycle of $C_{u,\gamma}(K, \mathbb{T})$ is a pair $(x, y) \in \mathbf{D}(\mathbb{T}) \oplus \mathbf{D}(\mathbb{T})$ such that $(\gamma - 1)(x) = (u-1)(y)$. Similarly, a 1–coboundary of $C_{u,\gamma}(K, \mathbb{T})$ is a pair of the form $(x, y) = \big((u-1)(b), (\gamma-1)(b)\big)$ for some $b \in \mathbf{D}(\mathbb{T})$.

Remark. The complex $C_{u,\gamma}(K,\mathbb{T})$ depends on the choice of γ, but its cohomology does not. Indeed, if γ' is another generator of Γ_K, we can express $\gamma' = \gamma^a$ for some $a \in \mathbb{Z}_p^\times$ and so

$$\frac{\gamma' - 1}{\gamma - 1} \in \mathbb{Z}_p[\![\Gamma_K]\!]^\times,$$

as is easily checked identifying γ with $1 + T \in \mathbb{Z}_p[\![T]\!]$:

$$\frac{(1+T)^a - 1}{T} = a + \binom{a}{2} T + \cdots \in \mathbb{Z}_p[\![T]\!]^\times.$$

Then the commutative diagram

$$\begin{array}{lccccccccc}
C_{u,\gamma'}(K,\mathbb{T})\colon & 0 & \longrightarrow & \mathbf{D}(\mathbb{T}) & \longrightarrow & \mathbf{D}(\mathbb{T}) \oplus \mathbf{D}(\mathbb{T}) & \longrightarrow & \mathbf{D}(\mathbb{T}) & \longrightarrow & 0 \\
 & & & \Big\downarrow \frac{\gamma'-1}{\gamma-1} & & \Big\downarrow \frac{\gamma'-1}{\gamma-1} \oplus 1 & & \Big\downarrow 1 & & \\
C_{u,\gamma}(K,\mathbb{T})\colon & 0 & \longrightarrow & \mathbf{D}(\mathbb{T}) & \longrightarrow & \mathbf{D}(\mathbb{T}) \oplus \mathbf{D}(\mathbb{T}) & \longrightarrow & \mathbf{D}(\mathbb{T}) & \longrightarrow & 0
\end{array}$$

induces isomorphisms $\mathrm{H}^i_{u,\gamma'}(K,\mathbb{T}) \cong \mathrm{H}^i_{u,\gamma}(K,\mathbb{T})$.

The following result explains why these Herr cohomology groups are interesting. As one could suspect from the notation, Herr cohomology gives another explicit description of the Galois cohomology of an R–representation.

Theorem 26 (Herr, Dee). *There is an isomorphism of δ–functors*

$$\left(\mathrm{H}^i_{\varphi,\gamma}(K,\mathbb{T})\right)_{i\geq 0} \xrightarrow{\cong} \left(\mathrm{H}^i(K,\mathbb{T})\right)_{i\geq 0}.$$

Idea of the proof. This is proposition 3.1.1 of Dee's article [19]. In loc. cit., Dee reduces the general case to the case in which $R = \mathbb{Z}_p$ by expressing $\mathbf{D}(\mathbb{T})$, $\mathrm{H}^i(K,\mathbb{T})$ and $\mathrm{H}^i_{\varphi,\gamma}(K,\mathbb{T})$ as projective limits of the corresponding objects for $T_n = \mathbb{T}/\mathfrak{m}_R^n\mathbb{T}$.

The case of $\mathbb{Z}_p$–representations was proved by Herr. Theorem 2.1 of Herr's article [23] gives the desired isomorphism restricted to torsion $\mathbb{Z}_p$–representations. In his proof, Herr uses that the usual Galois cohomology is a universal δ–functor and that a certain subcategory of (φ,Γ_K)–modules has enough injectives. Then one can pass to general $\mathbb{Z}_p$–representations by taking projective limits, as is explained in the introduction of ibid. □

Remark. Later we will be interested only in the 1st cohomology groups and we will use an explicit description of the isomorphism $\mathrm{H}^1_{\varphi,\gamma}(K,\mathbb{T}) \cong \mathrm{H}^1(K,\mathbb{T})$ in terms of cocycles due to Cherbonnier and Colmez.

Definition 27. For every $x \in 1 + p\mathbb{Z}_p$, let $r(x) = \mathrm{v}_p\big(\log_p(x)\big)$. We define a map $\log_p^0 \colon 1 + p\mathbb{Z}_p \to \mathbb{Z}_p^\times$ by

$$\log_p^0(x) = \frac{\log_p(x)}{r(x)}.$$

Proposition 28.

(1) *For every $(x, y) \in Z^1_{\varphi,\gamma}(K, \mathbb{T})$, there exists a solution $b \in (\mathbf{A} \mathbin{\widehat{\otimes}}_{\mathbb{Z}_p} R) \otimes_R \mathbb{T}$ to the equation $(\varphi - 1)(b) = x$. Then*

$$\sigma \mapsto c_{x,y}(\sigma) = \frac{\sigma - 1}{\gamma - 1}(y) - (\sigma - 1)(b)$$

defines a 1–cocycle of G_K with values in $\mathbb{T}$ (i.e., an element in $Z^1(K, \mathbb{T})$).

(2) *The map $(x, y) \mapsto \log_p^0\big(\chi_{\mathrm{cyc}}(\gamma)\big)c_{x,y}$ (for some choice of b as above) induces a well-defined isomorphism $\mathrm{H}^1_{\varphi,\gamma}(K, \mathbb{T}) \cong \mathrm{H}^1(K, \mathbb{T})$.*

(3) *This isomorphism of cohomologies is independent of the generator γ of Γ_K in the following sense: if γ' is another generator of Γ_K, these isomorphisms fit into a commutative diagram*

$$\begin{array}{ccc} \mathrm{H}^1_{\varphi,\gamma'}(K, \mathbb{T}) & & \\ \Big\downarrow \frac{\gamma'-1}{\gamma-1} & \searrow^{\cong} & \mathrm{H}^1(K, \mathbb{T}) \\ \mathrm{H}^1_{\varphi,\gamma}(K, \mathbb{T}) & \nearrow_{\cong} & \end{array}$$

(cf. the remark after definition 25).

Remark. Observe that, as γ is a generator of Γ_K and G_K acts on y through Γ_K, the expression

$$\frac{\sigma - 1}{\gamma - 1}(y)$$

makes sense in $\mathbf{D}(\mathbb{T})$. More precisely, replacing σ with its image in Γ_K, the quotient defines an element of $\mathbb{Z}_p[\![\Gamma_K]\!]$.

Proof. This is a combination of proposition I.4.1 and lemma I.4.2 of Cherbonnier–Colmez's article [14].

(1) Recall that $\mathbf{A}$ is a Cohen ring with residue field $\mathbf{E}$ that is separably closed. Thus, any polynomial of the form $X^p - X - \beta$ has a root in $\mathbf{E}$ and these can be lifted to $\mathbf{A}$ by Hensel's lemma. That is, $\varphi - 1 \colon \mathbf{A} \to \mathbf{A}$ is surjective.
Let $(x, y) \in Z^1_{\varphi,\gamma}(K, \mathbb{T})$, so that $(\gamma - 1)(x) = (\varphi - 1)(y)$. The existence of $b \in (\mathbf{A} \mathbin{\widehat{\otimes}}_{\mathbb{Z}_p} R) \otimes_R \mathbb{T}$ such that $(\varphi - 1)(b) = x$ is now clear. To show that

$c_{x,y}(\sigma) \in \mathbb{T}$ (regarded as a subset of $(\mathbf{A} \widehat{\otimes}_{\mathbb{Z}_p} R) \otimes_R \mathbb{T}$), we need to check that it is invariant under φ, as $\mathbf{A}^{\varphi=1} = \mathbb{Z}_p$. Indeed,

$$(\varphi - 1)\Big(\frac{\sigma-1}{\gamma-1}(y) - (\sigma-1)(b)\Big) = \frac{\sigma-1}{\gamma-1}((\gamma-1)(x)) - (\sigma-1)(x) = 0.$$

The fact that $c_{x,y}$ is a 1–cocycle follows easily from its definition.

(2) For this part, we may assume for simplicity that $\log_p^0\big(\chi_{\mathrm{cyc}}(\gamma)\big) = 1$. Also, it is clear that changing b does not modify the cohomology class of $c_{x,y}$.
For the injectivity, suppose that $c_{x,y}$ is a 1–coboundary given by $z \in \mathbb{T}$. Then

$$\frac{\sigma-1}{\gamma-1}(y) - (\sigma-1)(b+z) = 0 \quad \text{for all } \sigma \in G_K.$$

Since H_K acts trivially on y, so does on $b+z$, which implies that $b+z \in \mathbf{D}(\mathbb{T})$. If we take σ to be a lift of γ, we get that $y = (\gamma-1)(b+z)$. On the other hand, $x = (\varphi-1)(b) = (\varphi-1)(b+z)$. All in all, $(x,y) \in B^1_{\varphi,\gamma}(K,\mathbb{T})$.
For the surjectivity, consider $c \in Z^1(K,\mathbb{T})$ and let

$$0 \longrightarrow \mathbb{T} \longrightarrow \mathbb{T}' \longrightarrow R \longrightarrow 0$$

be the corresponding extension. That is, we have $e \in \mathbb{T}'$ lifting $1 \in R$ such that $\sigma(e) = e + c(\sigma)$ for all $\sigma \in G_K$. Take a lift $\widetilde{e} \in \mathbf{D}(\mathbb{T}')$ of $1 \in R \hookrightarrow \mathbf{D}(R)$ and define $x = (\varphi-1)(\widetilde{e})$ and $y = (\gamma-1)(\widetilde{e})$. It is clear by definition that $(\gamma-1)(x) = (\varphi-1)(y)$, which is to say that $(x,y) \in Z^1_{\varphi,\gamma}(K,\mathbb{T})$. Moreover, we can choose $b = \widetilde{e} - e \in (\mathbf{A} \widehat{\otimes}_{\mathbb{Z}_p} R) \otimes_R \mathbb{T}$ with $(\varphi-1)(b) = x$ and then

$$\begin{aligned} c_{x,y}(\sigma) &= \frac{\sigma-1}{\gamma-1}(y) - (\sigma-1)(b) = (\sigma-1)(\widetilde{e}) - (\sigma-1)(\widetilde{e}-e) \\ &= (\sigma-1)(e) = c(\sigma). \end{aligned}$$

(3) Let $(x,y) = Z^1_{\varphi,\gamma}(K,\mathbb{T})$ and $(x',y) = Z^1_{\varphi,\gamma'}(K,\mathbb{T})$ be two cocycles related by $(\gamma'-1)(x) = (\gamma-1)(x)$. Take $b, b' \in (\mathbf{A} \widehat{\otimes}_{\mathbb{Z}_p} R) \otimes_R \mathbb{T}$ as above. We need to see that $\sigma \mapsto c_{x',y} - c_{x,y}$ is a 1–coboundary of G_K with values in $\mathbb{T}$. But we can express it as $\sigma \mapsto (\sigma-1)(z)$ for

$$\begin{aligned} z = &\left[\frac{\log_p^0(\chi_{\mathrm{cyc}}(\gamma'))}{\gamma'-1} - \frac{\log_p^0(\chi_{\mathrm{cyc}}(\gamma))}{\gamma-1}\right](y) - \\ &- \Big[\log_p^0(\chi_{\mathrm{cyc}}(\gamma'))b' - \log_p^0(\chi_{\mathrm{cyc}}(\gamma))b\Big] \end{aligned}$$

and the first term makes sense in $\mathbf{D}(\mathbb{T})$ because

$$\left[\frac{\log_p^0(\chi_{\mathrm{cyc}}(\gamma'))}{\gamma'-1} - \frac{\log_p^0(\chi_{\mathrm{cyc}}(\gamma))}{\gamma-1}\right] \in \mathbb{Z}_p[\![\Gamma_K]\!].$$

Indeed, writing $\gamma' = \gamma^a$ with $a \in \mathbb{Z}_p^\times$ and identifying γ with $1+T \in \mathbb{Z}_p[\![T]\!]$, this factor can be expressed as the product of $\log_p^0(\chi_{\mathrm{cyc}}(\gamma))$ and

$$\begin{aligned}\frac{a}{(1+T)^a-1} - \frac{1}{T} = \frac{1}{T}\left[\frac{1}{1+\frac{a-1}{2}T+\cdots} - 1\right] = \frac{1}{T}\left[-\frac{a-1}{2}T+\cdots\right] \\ = -\frac{a-1}{2} + \cdots \in \mathbb{Z}_p[\![T]\!].\end{aligned}$$

Finally, we can check that $z \in (\mathbf{A} \mathbin{\widehat{\otimes}}_{\mathbb{Z}_p} R) \otimes_R \mathbb{T}$ is invariant under φ (i.e., $(\varphi-1)(z) = 0$) and so in fact $z \in \mathbb{T}$. □

Proposition 29 (Herr). *Let $\mathbb{T}$ be an R–representation of G_K. The vertical maps*

$$\begin{array}{ccccccccc} C_{\varphi,\gamma}(K,\mathbb{T})\colon & 0 & \longrightarrow & \mathbf{D}(\mathbb{T}) & \xrightarrow{(\varphi-1,\gamma-1)} & \mathbf{D}(\mathbb{T})\oplus\mathbf{D}(\mathbb{T}) & \xrightarrow{(\gamma-1)\oplus(1-\varphi)} & \mathbf{D}(\mathbb{T}) & \longrightarrow 0 \\ & & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle -\psi\oplus 1} & & \downarrow{\scriptstyle -\psi} & \\ C_{\psi,\gamma}(K,\mathbb{T})\colon & 0 & \longrightarrow & \mathbf{D}(\mathbb{T}) & \xrightarrow{(\psi-1,\gamma-1)} & \mathbf{D}(\mathbb{T})\oplus\mathbf{D}(\mathbb{T}) & \xrightarrow{(\gamma-1)\oplus(1-\psi)} & \mathbf{D}(\mathbb{T}) & \longrightarrow 0 \end{array}$$

define a quasi-isomorphism from $C_{\varphi,\gamma}(K,\mathbb{T})$ to $C_{\psi,\gamma}(K,\mathbb{T})$.

In particular, $\mathrm{H}^i_{\psi,\gamma}(K,\mathbb{T}) \cong \mathrm{H}^i_{\varphi,\gamma}(K,\mathbb{T}) \cong \mathrm{H}^i(K,\mathbb{T})$ for all $i \geq 0$.

Idea of the proof. This is proposition 4.1 of Herr's article [23]. The result follows from the non-trivial fact that $\gamma - 1$ acts invertibly on $\mathbf{D}(\mathbb{T})^{\psi=0}$ (see theorem 3.8 of ibid.) and because ψ is surjective.

Let us be more explicit for the isomorphism $\mathrm{H}^1_{\varphi,\gamma}(K,\mathbb{T}) \cong \mathrm{H}^1_{\psi,\gamma}(K,\mathbb{T})$, as in lemma I.5.2 of Cherbonnier–Colmez's article [14].

- Surjectivity. Let $(x,y) \in Z^1_{\psi,\gamma}(K,\mathbb{T})$, so that $(\gamma-1)(x) = (\psi-1)(y)$. Then $y = \psi(y) - (\gamma-1)(x)$ and applying φ we get $\varphi(y) = \varphi\psi(y) - (\gamma-1)(\varphi(x))$. Therefore,

  $$\begin{aligned}(\varphi-1)(y) &= \varphi\psi(y) - y - (\gamma-1)(\varphi(x)) \\ &= (\gamma-1)\big(-\varphi(x) + (\gamma-1)^{-1}(\varphi\psi(y)-y)\big),\end{aligned}$$

  which means that $\big(-\varphi(x) + (\gamma-1)^{-1}(\varphi\psi(y)-y), y\big) \in Z^1_{\varphi,\gamma}(K,\mathbb{T})$. A direct computation shows that this is a preimage of (x,y) under $-\psi\oplus 1$, as $\psi\varphi = 1$.

- Injectivity. Let $(x, y) \in Z^1_{\varphi,\gamma}(K, \mathbb{T})$ and assume that

$$\big(-\psi(x), y\big) = \big((\psi-1)(b), (\gamma-1)(b)\big) \quad \text{for some } b \in \mathbf{D}(\mathbb{T}).$$

Now, we compute

$$\psi\big(x - (\varphi-1)(b)\big) = -(\psi-1)(b) - (1-\psi)(b) = 0$$

and

$$(\gamma-1)\big(x - (\varphi-1)(b)\big) = (\varphi-1)(y) - (\varphi-1)(y) = 0.$$

Since $\gamma - 1$ acts invertibly on $\mathbf{D}(\mathbb{T})^{\psi=0}$, we conclude that $x = (\varphi-1)(b)$. This equality and $y = (\gamma-1)(b)$, mean that $(x, y) \in B^1_{\varphi,\gamma}(K, \mathbb{T})$. □

Thus, we can compute $\mathrm{H}^1(K, \mathbb{T})$ using (ψ, Γ_K)–cohomology. Our end goal, however, is to compute $\mathrm{H}^1_{\mathrm{Iw}}(K, \mathbb{T})$ using the formalism of (φ, Γ_K)–modules. This will be possible by replacing K above with K_n for $n \geq 1$ and taking limits. (In particular, the assumption that Γ_K is isomorphic to $\mathbb{Z}_p$ is harmless.) We conclude this subsection with the result that will allow us to study $\mathrm{H}^1_{\mathrm{Iw}}(K, \mathbb{T})$.

Lemma 30. *Let $\mathbb{T}$ be an R–representation of G_K. There is a short exact sequence*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \big(\mathbf{D}(\mathbb{T})^{\psi=1}\big)_{\Gamma_K} & \longrightarrow & \mathrm{H}^1_{\psi,\gamma}(K, \mathbb{T}) & \longrightarrow & \big(\mathbf{D}(\mathbb{T})/(\psi-1)\big)^{\Gamma_K} & \longrightarrow & 0, \\ & & y & \longmapsto & (0, y) & & & & \\ & & & & (x, y) & \longmapsto & x & & \end{array}$$

where $(\,\cdot\,)^{\Gamma_K}$ denotes the Γ_K–invariants and $(\,\cdot\,)_{\Gamma_K}$ denotes the Γ_K–coinvariants. (Since we fixed a topological generator γ of Γ_K, we could have written

$$\big(\mathbf{D}(\mathbb{T})^{\psi=1}\big)_{\Gamma_K} = \big(\mathbf{D}(\mathbb{T})^{\psi=1}\big)/(\gamma-1)$$

and

$$\big(\mathbf{D}(\mathbb{T})/(\psi-1)\big)^{\Gamma_K} = \big(\mathbf{D}(\mathbb{T})/(\psi-1)\big)^{\gamma=1}$$

alternatively.)

Proof. This is lemma I.5.5 of Cherbonnier–Colmez's article [14]. It follows formally from the definition of $\mathrm{H}^1_{\psi,\gamma}(K, \mathbb{T})$. □

Remark. Combining the first map of lemma 30 with proposition 28 and (the proof of the surjectivity in) proposition 29, we get a morphism $\left(\mathbf{D}(\mathbb{T})^{\psi=1}\right)_{\Gamma_K} \to \mathrm{H}^1(K, \mathbb{T})$ that can be described as follows. To (the class of) $y \in \mathbf{D}(\mathbb{T})^{\psi=1}$ we attach (the class of) $(x, y) \in Z^1_{\varphi,\gamma}(K, \mathbb{T})$, where

$$x = (\gamma - 1)^{-1}(\varphi - 1)(y)$$

(this makes sense because $(\varphi - 1)(y) \in \mathbf{D}(\mathbb{T})^{\psi=0}$ and $\gamma - 1$ acts invertibly there). After choosing $b \in (\mathbf{A} \mathbin{\widehat{\otimes}}_{\mathbb{Z}_p} R) \otimes_R \mathbb{T}$ such that $(\varphi - 1)(b) = x$, the class of (x, y) corresponds to the element in $\mathrm{H}^1(K, \mathbb{T})$ given by the 1–cocycle

$$\sigma \mapsto \log_p^0\big(\chi_{\mathrm{cyc}}(\gamma)\big)\left[\frac{\sigma - 1}{\gamma - 1}(y) - (\sigma - 1)(b)\right].$$

4 Some Iwasawa theory

Continuing with the notation from sections 2 and 3, we want to recall next the results of Cherbonnier and Colmez to construct interesting maps of Iwasawa theory using (φ, Γ_K)–modules, as well as the extensions of such results to families by Dee, Kedlaya, Liu, Pottharst, Xiao... TODO!!!

This part follows mostly sections II and IV of Cherbonnier–Colmez's article [14].

4.1 Iwasawa cohomology

Definition 31.

(1) Let $\mathbb{T}$ be an R–representation of G_K. We define the *Iwasawa cohomology groups*

$$\mathrm{H}^i_{\mathrm{Iw}}(K, \mathbb{T}) = \mathrm{H}^i_{\mathrm{Iw}}(K_\infty/K, \mathbb{T}) = \varprojlim_{n \geq 1} \mathrm{H}^i(K_n, \mathbb{T}),$$

where the projective limit is taken with respect to the corestriction maps.

(2) Let V be a $\mathbb{Q}_p$–representation of G_K and let T be a stable $\mathbb{Z}_p$–lattice of V. We define the *Iwasawa cohomology groups*

$$\mathrm{H}^i_{\mathrm{Iw}}(K, V) = \mathrm{H}^i_{\mathrm{Iw}}(K_\infty/K, V) = \mathrm{H}^i_{\mathrm{Iw}}(K, T) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

(This is independent of the choice of lattice T.)

Remark. We are only interested in $\mathrm{H}^1_{\mathrm{Iw}}(K, \mathbb{T})$. The corestriction maps in the cyclotomic tower can be described explicitly in terms of 1–cocycles as follows.

In general, consider a subgroup H of finite index of a profinite group G and let M be a (continuous) G–module. Fix a system of representatives X of G/H and write $\widehat{g}$ for the representative in X of $g \in G$. Given a cohomology class $[c] \in \mathrm{H}^1(H, M)$ represented by a 1–cocycle c, its corestriction $\mathrm{cor}([c]) \in \mathrm{H}^1(G, M)$ is represented by

$$g \mapsto \sum_{x \in X} \widehat{gx} \cdot c\big((\widehat{gx})^{-1} gx\big).$$

Consider the Iwasawa algebra $\Lambda(\Gamma_K) = \mathbb{Z}_p[\![\Gamma_K]\!]$, which can be interpreted as the $\mathbb{Z}_p$–algebra of measures on Γ_K with values in $\mathbb{Z}_p$. There is also an isomorphism $\Lambda(\Gamma_K) \cong \mathbb{Z}_p[\![T]\!]$ (that we have already used in section 3) given by $\gamma \mapsto 1 + T$. On $\Lambda(\Gamma_K)$ there is a natural action of G_K given by

$$\sigma \cdot \gamma = \bar{\sigma}\gamma, \quad \text{where } \bar{\sigma} \text{ is the image of } \sigma \text{ under } G_K \twoheadrightarrow \Gamma_K.$$

More generally, let R be a coefficient ring in the sense of definition 7. We define

$$\Lambda_R(\Gamma_K) = \Lambda(\Gamma_K) \mathbin{\widehat{\otimes}}_{\mathbb{Z}_p} R,$$

where the completed tensor product is computed with respect to the maximal ideals of $\Lambda(\Gamma_K)$ and R.

If $\mathbb{T}$ is an R–representation of G_K and $[\mu] \in \mathrm{H}^1\big(K, \Lambda_R(\Gamma_K) \otimes_R \mathbb{T}\big)$, we regard the 1–cocycle μ as a family of measures: for every $\sigma \in G_K$, $\mu(\sigma)$ corresponds to a measure on Γ_K with values in $\mathbb{T}$. Thus, for any continuous map $f\colon \Gamma_K \to R$, we will write

$$\int_{\Gamma_K} f(x)\, \mu(x)$$

for the 1–cocycle

$$\sigma \mapsto \int_{\Gamma_K} f(x)\, \big(\mu(\sigma)\big)(x).$$

Lemma 32 (Shapiro). *Let $\mathbb{T}$ be an R–representation of G_K. There is a canonical isomorphism*

$$\mathrm{H}^1\big(K, \Lambda_R(\Gamma_K) \otimes_R \mathbb{T}\big) \cong \mathrm{H}^1_{\mathrm{Iw}}(K, \mathbb{T})$$

that sends $[\mu] \in \mathrm{H}^1\big(K, \Lambda_R(\Gamma_K) \otimes_R \mathbb{T}\big)$ to the class in $\mathrm{H}^1_{\mathrm{Iw}}(K, \mathbb{T})$ represented by the compatible family of 1–cocycles

$$\int_{\Gamma_{K_n}} 1\, \mu(x) \in Z^1(K_n, \mathbb{T}) \quad \textit{for } n \geq 1.$$

Proof. This is a "limit" of the more usual version of Shapiro's lemma (using that in finite index the induced and coinduced modules coincide).Indeed, we can write

$$\Lambda_R(\Gamma_K) \otimes_R \mathbb{T} = \Lambda(\Gamma_K) \mathbin{\widehat{\otimes}}_{\mathbb{Z}_p} \mathbb{T} = \varprojlim_{n\geq 1} \Big(\mathbb{Z}_p\big[G_K/G_{K_n}\big] \otimes_{\mathbb{Z}_p} \mathbb{T} \Big)$$

and Shapiro's lemma gives isomorphisms

$$\mathrm{H}^1\big(G_K, \mathbb{Z}_p\big[G_K/G_{K_n}\big] \otimes_{\mathbb{Z}_p} \mathbb{T}\big) \cong \mathrm{H}^1(G_{K_n}, \mathbb{T})$$

for all $n \geq 1$. More precisely, $\mathrm{Hom}_{\mathbb{Z}_p}\big(\mathbb{Z}_p[G_K/G_{K_n}], \mathbb{T}\big) \cong \mathbb{Z}_p\big[G_K/G_{K_n}\big] \otimes_{\mathbb{Z}_p} \mathbb{T}$ as $\mathbb{Z}_p[G_K]$–modules (where G_K acts on the coinduced module by conjugation) via

$$\Big(f\colon \mathbb{Z}_p[G_K/G_{K_n}] \to \mathbb{T} \Big) \mapsto \sum_{\alpha \in G_K/G_{K_n}} \alpha \otimes f(\alpha)$$

and there is an isomorphism $\mathrm{H}^1\left(G_K, \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p[G_K/G_{K_n}], \mathbb{T})\right) \cong \mathrm{H}^1(G_{K_n}, \mathbb{T})$ that in terms of 1–cocycles is

$$\Big(c\colon G_K \to \mathrm{Hom}_{\mathbb{Z}_p}\big(\mathbb{Z}_p[G_K/G_{K_n}], \mathbb{T}\big)\Big) \longmapsto \Big(\sigma \mapsto \big(c(\sigma)\big)(1)\Big)$$

(i.e., restricting cocycles to G_{K_n} and evaluating their images at 1).

From the description of corestrictions given in the previous remark, one can check that the transition maps in both projective limits are compatible with these isomorphisms.

Finally, one proves that

$$\mathrm{H}^1\big(K, \Lambda_R(\Gamma_K) \otimes_R \mathbb{T}\big) \cong \varprojlim_{n\geq 1} \mathrm{H}^1\big(K, \mathbb{Z}_p[G_K/G_{K_n}] \otimes_{\mathbb{Z}_p} \mathbb{T}\big)$$

checking a Mittag–Leffler condition for the H^0 groups. See proposition II.1.1 of Colmez's article [15] for more details.

Interpreting the elements of $\mathbb{Z}_p\big[G_K/G_{K_n}\big] \otimes_{\mathbb{Z}_p} \mathbb{T} \cong \mathbb{Z}_p\big[\Gamma_K/\Gamma_{K_n}\big] \otimes_{\mathbb{Z}_p} \mathbb{T}$ as measures on Γ_K, we can trace the definitions of the isomorphisms and see that the evaluation at 1 corresponds to taking the measure of Γ_{K_n}, whence the last formula follows. □

Remark. More generally, Shapiro's lemma provides an isomorphism of δ–functors. Nevertheless, we will only use the version stated above.

Corollary 33. *Let $\mathbb{T}$ be an R–representation of G_K and let $k \in \mathbb{Z}$. There is a canonical isomorphism*

$$\mathrm{H}^1\big(K, \Lambda_R(\Gamma_K) \otimes_R \mathbb{T}\big) \cong \varprojlim_{n\geq 1} \mathrm{H}^1\big(K_n, \mathbb{T}(k)\big)$$

given in terms of 1–cocycles by

$$\mu \mapsto \left(\Big(\int_{\Gamma_{K_n}} \chi_{\mathrm{cyc}}^k(x)\, \mu(x)\Big)(k)\right)_{n\geq 1},$$

where the notation (k) means the k–th Tate twist.

In particular, $\mathrm{H}^1_{\mathrm{Iw}}\big(K, \mathbb{T}(k)\big) \cong \mathrm{H}^1_{\mathrm{Iw}}(K, \mathbb{T})$.

Proof. This is proposition II.1.8 of Colmez's article [15].

It follows from the isomorphism $\Lambda_R(\Gamma_K) \otimes_R \mathbb{T} \cong \Lambda_R(\Gamma_K) \otimes_R \mathbb{T}(k)$ induced by

$$\Lambda_R(\Gamma_K) \cong \Lambda_R(\Gamma_K)(k)$$

$$\gamma \longmapsto \chi_{\mathrm{cyc}}^k(\gamma)\gamma$$

and lemma 32 applied to $\mathbb{T}(k)$. □

4.2 The regulator map

First of all, we introduce some notation for the rest of the section. Fix a topological generator γ_1 of Γ_{K_1} and set $\gamma_n = \gamma_1^{[K_n:K_1]}$, which is a topological generator of Γ_{K_n}.

Let $\mathbb{T}$ be an R–representation of G_K. For each $n \geq 1$, we regard $\mathbb{T}$ as a representation of G_{K_n} by restriction. Then lemma 30 (and the remark after it) give a short exact sequence

$$0 \longrightarrow \frac{\mathbf{D}(\mathbb{T})^{\psi=1}}{(\gamma_n - 1)} \longrightarrow \mathrm{H}^1(K_n, \mathbb{T}) \longrightarrow \left(\frac{\mathbf{D}(\mathbb{T})}{(\psi - 1)}\right)^{\gamma_n=1} \longrightarrow 0$$

constructed in terms of (ψ, Γ_{K_n})–cohomology. Observe that the φ–module $\mathbf{D}(\mathbb{T})$ does not depend on n because $H_{K_n} = H_K$. We will see that these sequences are compatible for varying $n \geq 1$.

Lemma 34. *Let $\mathbb{T}$ be an R–representation of G_K and let $n \geq 1$. The corestriction map*

$$\begin{array}{ccc} \mathrm{H}^1(K_{n+1}, \mathbb{T}) & \xrightarrow{\mathrm{cor}} & \mathrm{H}^1(K_n, \mathbb{T}) \\ \wr\| & & \wr\| \\ \mathrm{H}^1_{\psi,\gamma}(K_{n+1}, \mathbb{T}) & \dashrightarrow & \mathrm{H}^1_{\psi,\gamma}(K_n, \mathbb{T}) \end{array}$$

can be described on (ψ, Γ)–cohomology in terms of 1–cocycles as

$$(x, y) \mapsto (x', y) = \Big(\frac{\gamma_{n+1} - 1}{\gamma_n - 1}(x), y\Big).$$

Proof. This is lemma II.2.1 of Cherbonnier–Colmez's article [14].

We can compute the corestriction using the remark after definition 31. More concretely, set $G = G_{K_n}$, $H = G_{K_{n+1}}$ and $X = \{1, \widetilde{\gamma}_n, \ldots, \widetilde{\gamma}_n^{p-1}\}$ for a fixed lift $\widetilde{\gamma}_n \in G_{K_n}$ of $\gamma_n \in \Gamma_{K_n}$. For $c \in Z^1(K_{n+1}, \mathbb{T})$, we can express $\mathrm{cor}(c) \in Z^1(K_n, \mathbb{T})$ as

$$\sigma \mapsto \sum_{i=0}^{p-1} \widehat{\sigma\gamma_n^i} \cdot c\big((\widehat{\sigma\gamma_n^i})^{-1}\sigma\widetilde{\gamma}_n^i\big).$$

On the other hand, to every $(x, y) \in Z^1_{\psi,\gamma_{n+1}}(K_{n+1}, \mathbb{T})$ equipped with a solution $b \in (\mathbf{A} \mathbin{\widehat{\otimes}}_{\mathbb{Z}_p} R) \otimes_R \mathbb{T}$ to

$$(\varphi - 1)(b) = -\varphi(x) + (\gamma_{n+1} - 1)^{-1}\big(\varphi\psi(y) - y\big)$$

we attach $c_{(x,y)} \in Z^1(K_{n+1}, \mathbb{T})$ defined by

$$c_{(x,y)}(\sigma) = \log_p^0\big(\chi_{\mathrm{cyc}}(\gamma_{n+1})\big)\left[\frac{\sigma - 1}{\gamma_{n+1} - 1}(y) - (\sigma - 1)(b)\right].$$

Working in $\mathrm{Frac}\big(R[\![G_{K_n}]\!]\big) \otimes_{R[\![G_{K_n}]\!]} (\mathbf{A} \widehat{\otimes}_{\mathbb{Z}_p} R) \otimes_R \mathbb{T}$, we can define

$$a_{(x,y)} = \frac{y}{\widetilde{\gamma}_{n+1} - 1} - b \quad \text{where } \widetilde{\gamma}_{n+1} = \widetilde{\gamma}_n^p$$

and then $c_{(x,y)}$ is the 1–coboundary

$$c_{(x,y)}(\sigma) = \log_p^0\big(\chi_{\mathrm{cyc}}(\gamma_{n+1})\big)(\sigma - 1)\big(a_{(x,y)}\big).$$

We can rearrange the sum defining $\mathrm{cor}\big(c_{(x,y)}\big)$ as follows:

$$\begin{aligned}\sum_{i=0}^{p-1} \widehat{\sigma\gamma_n^i} \cdot \big((\widehat{\sigma\gamma_n^i})^{-1}\sigma\widetilde{\gamma}_n^i - 1\big)\big(a_{(x,y)}\big) &= \sum_{i=0}^{p-1} \sigma\widetilde{\gamma}_n^i\big(a_{(x,y)}\big) - \sum_{i=0}^{p-1} \widehat{\sigma\gamma_n^i}\big(a_{(x,y)}\big) \\ &= \sum_{i=0}^{p-1} \sigma\widetilde{\gamma}_n^i\big(a_{(x,y)}\big) - \sum_{i=0}^{p-1} \widetilde{\gamma}_n^i\big(a_{(x,y)}\big) = (\sigma - 1)\left[\sum_{i=0}^{p-1} \widetilde{\gamma}_n^i\big(a_{(x,y)}\big)\right]\end{aligned}$$

But

$$\sum_{i=0}^{p-1} \widetilde{\gamma}_n^i\big(a_{(x,y)}\big) = \frac{\widetilde{\gamma}_{n+1} - 1}{\widetilde{\gamma}_n - 1}\left[\frac{y}{\widetilde{\gamma}_{n+1} - 1} - b\right] = \frac{y}{\widetilde{\gamma}_n - 1} - \frac{\widetilde{\gamma}_{n+1} - 1}{\widetilde{\gamma}_n - 1}b$$

and it is clear that

$$(\varphi - 1)\Big(\frac{\widetilde{\gamma}_{n+1} - 1}{\widetilde{\gamma}_n - 1}b\Big) = -\varphi(x') + (\gamma_n - 1)^{-1}\big(\varphi\psi(y) - y\big).$$

All in all, we have checked that

$$\begin{aligned}\mathrm{cor}\big(c_{(x,y)}\big)(\sigma) &= \log_p^0\big(\chi_{\mathrm{cyc}}(\gamma_{n+1})\big)(\sigma - 1)\big(a_{(x',y)}\big) \\ &= \log_p^0\big(\chi_{\mathrm{cyc}}(\gamma_n)\big)(\sigma - 1)\big(a_{(x',y)}\big) = c_{(x',y)}(\sigma)\end{aligned}$$

where $\log_p^0\big(\chi_{\mathrm{cyc}}(\gamma_{n+1})\big) = \log_p^0\big(\chi_{\mathrm{cyc}}(\gamma_n)\big)$ because $\gamma_{n+1} = \gamma_n^p$. □

Therefore, we obtain commutative diagrams

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \dfrac{\mathbf{D}(\mathbb{T})^{\psi=1}}{(\gamma_{n+1}-1)} & \longrightarrow & \mathrm{H}^1(K_{n+1},\mathbb{T}) & \longrightarrow & \left(\dfrac{\mathbf{D}(\mathbb{T})}{(\psi-1)}\right)^{\gamma_{n+1}=1} & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle 1} & & \Big\downarrow{\scriptstyle \mathrm{cor}} & & {\scriptstyle \frac{\gamma_{n+1}-1}{\gamma_n-1}}\Big\downarrow & & \\
0 & \longrightarrow & \dfrac{\mathbf{D}(\mathbb{T})^{\psi=1}}{(\gamma_{n}-1)} & \longrightarrow & \mathrm{H}^1(K_{n},\mathbb{T}) & \longrightarrow & \left(\dfrac{\mathbf{D}(\mathbb{T})}{(\psi-1)}\right)^{\gamma_{n}=1} & \longrightarrow & 0
\end{array}$$

with exact rows for all $n \geq 1$. We can take projective limits.

Proposition 35. *Let $\mathbb{T}$ be an R–representation of G_K.*

(1) *The natural map*

$$\mathbf{D}(\mathbb{T})^{\psi=1} \longrightarrow \varprojlim_{n\geq 1} \frac{\mathbf{D}(\mathbb{T})^{\psi=1}}{(\gamma_n - 1)}$$

(where the transition maps are induced by the identity on $\mathbf{D}(\mathbb{T})$) is an isomorphism.

(2) *We have*

$$\varprojlim_{n\geq 1} \left(\frac{\mathbf{D}(\mathbb{T})}{(\psi-1)}\right)^{\gamma_n=1} = 0,$$

where the transition maps are given by multiplication by $(\gamma_{n+1}-1)/(\gamma_n-1)$.

Proof. If $R = \mathbb{Z}_p$, this is proposition II.3.1 of Cherbonnier–Colmez's article [14]. The general case follows from this by regarding $\mathbb{T}$ as a projective limit of the $\mathbb{Z}_p$–representations $T_n = \mathbb{T}/\mathfrak{m}_R^n\mathbb{T}$ for $n \geq 1$ and using the techniques of Dee's article [19]. □

Putting everything together, we obtain the following result.

Theorem 36 (Fontaine). *Let $\mathbb{T}$ be an R–representation of G_K. The map*

$$\mathrm{Log}^*_{\mathbb{T}^*(1)} \colon \mathbf{D}(\mathbb{T})^{\psi=1} \to \mathrm{H}^1_{\mathrm{Iw}}(K,\mathbb{T})$$

that sends $y \in \mathbf{D}(\mathbb{T})^{\psi=1}$ to the system of cohomology classes in $\mathrm{H}^1_{\mathrm{Iw}}(K,\mathbb{T})$ represented by the cocycles

$$\sigma \mapsto \log_p^0(\chi_{\mathrm{cyc}}(\gamma_n))\left[\frac{\sigma-1}{\gamma_n-1}(y) - (\sigma-1)(b_n)\right],$$

where $b_n \in (\mathbf{A}\,\widehat{\otimes}_{\mathbb{Z}_p} R) \otimes_R \mathbb{T}$ is a solution to

$$(\varphi-1)(b_n) = (\gamma_n-1)^{-1}(\varphi-1)(y),$$

is an isomorphism.

Remark. This is theorem II.1.3 of Cherbonnier–Colmez's article [14] for $R = \mathbb{Z}_p$ and proposition III.3.2 of Dee's article [19] in the general case. Cherbonnier and Colmez attribute it to Fontaine (even though he did not publish it).

Definition 37. Let $\mathbb{T}$ be an R–representation of G_K. We define

$$\mathrm{Exp}^*_{\mathbb{T}^*(1)} \colon \mathrm{H}^1_{\mathrm{Iw}}(K, \mathbb{T}) \to \mathbf{D}(\mathbb{T})^{\psi=1}$$

to be the inverse of the isomorphism $\mathrm{Log}^*_{\mathbb{T}^*(1)}$ described in theorem 36.

As a matter of fact, the whole Iwasawa cohomology can be computed in this way in terms of (φ, Γ_K)–modules.

Theorem 38. *Let $\mathbb{T}$ be an R–representation of G_K. The Iwasawa cohomology groups $\mathrm{H}^i_{\mathrm{Iw}}(K, \mathbb{T})$ for $i \geq 0$ are computed by the complex*

$$0 \longrightarrow \mathbf{D}(\mathbb{T}) \xrightarrow{\psi-1} \mathbf{D}(\mathbb{T}) \longrightarrow 0$$

concentrated in degrees 1 *and* 2.

Remark. This is theorem 3.3.4 of Dee's article [19].

4.3 Bloch–Kato's exponential maps

In their article [10], Bloch and Kato introduced certain maps relating the first cohomology groups of representations and their Dieudonné modules.

Definition 39.

(1) Let V be a $\mathbb{Q}_p$–representation of G_K. We define the following subgroups of $\mathrm{H}^1(K, V)$:
- the *exponential part* $\mathrm{H}^1_{\mathrm{e}}(K, V) = \mathrm{Ker}\big(\mathrm{H}^1(K, V) \to \mathrm{H}^1(K, \mathbf{B}_{\mathrm{crys}}^{\varphi=1} \otimes_{\mathbb{Q}_p} V)\big)$;
- the *finite part* $\mathrm{H}^1_{\mathrm{f}}(K, V) = \mathrm{Ker}\big(\mathrm{H}^1(K, V) \to \mathrm{H}^1(K, \mathbf{B}_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} V)\big)$, and
- the *geometric part* $\mathrm{H}^1_{\mathrm{g}}(K, V) = \mathrm{Ker}\big(\mathrm{H}^1(K, V) \to \mathrm{H}^1(K, \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)\big)$.

(2) Let T be a $\mathbb{Z}_p$–representation of G_K and consider $V = T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, so that T is a stable $\mathbb{Z}_p$–lattice of V. Let $i \colon \mathrm{H}^1(K, T) \to \mathrm{H}^1(K, V)$ be the morphism induced by the inclusion $T \hookrightarrow V$. For $* \in \{\mathrm{e}, \mathrm{f}, \mathrm{g}\}$, define the subgroup

$$\mathrm{H}^1_*(K, T) = i^{-1}\big(\mathrm{H}^1_*(K, V)\big) \subset \mathrm{H}^1(K, T).$$

Let V be a $\mathbb{Q}_p$–representation of G_K. The fundamental exact sequence from proposition 4 can be rewritten as

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow \mathbf{B}_{\max}^{\varphi=1} \longrightarrow \mathbf{B}_{\mathrm{dR}}/\mathbf{B}_{\mathrm{dR}}^{+} \longrightarrow 0.$$

Tensoring it with V, we obtain a short exact sequence

$$0 \longrightarrow V \longrightarrow \mathbf{B}_{\max} \otimes_{\mathbb{Q}_p} V \longrightarrow (\mathbf{B}_{\mathrm{dR}}/\mathbf{B}_{\mathrm{dR}}^{+}) \otimes_{\mathbb{Q}_p} V \longrightarrow 0$$

whose long exact sequence of cohomology is

$$0 \longrightarrow V^{G_K} \longrightarrow \mathbf{D}_{\mathrm{crys}}^{\varphi=1}(V) \longrightarrow \frac{\mathbf{D}_{\mathrm{dR}}(V)}{\mathrm{Fil}^0\,\mathbf{D}_{\mathrm{dR}}(V)} \longrightarrow$$
$$\longrightarrow \mathrm{H}^1(K,V) \longrightarrow \mathrm{H}^1(K,\mathbf{B}_{\max}^{\varphi=1} \otimes_{\mathbb{Q}_p} V) \longrightarrow \cdots$$

Therefore, the connecting morphism induces an isomorphism

$$\frac{\mathbf{D}_{\mathrm{dR}}(V)}{\mathrm{Fil}^0\,\mathbf{D}_{\mathrm{dR}}(V) + \mathbf{D}_{\mathrm{crys}}^{\varphi=1}(V)} \cong \mathrm{H}^1_{\mathrm{e}}(K,V)$$

Definition 40. Let V be a $\mathbb{Q}_p$–representation of G_K. We define *Bloch–Kato's exponential map*

$$\exp_V\colon \frac{\mathbf{D}_{\mathrm{dR}}(V)}{\mathrm{Fil}^0\,\mathbf{D}_{\mathrm{dR}}(V) + \mathbf{D}_{\mathrm{crys}}^{\varphi=1}(V)} \longrightarrow \mathrm{H}^1_{\mathrm{e}}(K,V)$$

to be the isomorphism induced by the connecting morphism of the fundamental exact sequence as in the paragraph above. Its inverse is *Bloch–Kato's logarithm map*

$$\log_V\colon\ \mathrm{H}^1_{\mathrm{e}}(K,V) \longrightarrow \frac{\mathbf{D}_{\mathrm{dR}}(V)}{\mathrm{Fil}^0\,\mathbf{D}_{\mathrm{dR}}(V) + \mathbf{D}_{\mathrm{crys}}^{\varphi=1}(V)}.$$

Remark. By abuse of notation, we often write

$$\exp_V\colon \frac{\mathbf{D}_{\mathrm{dR}}(V)}{\mathrm{Fil}^0\,\mathbf{D}_{\mathrm{dR}}(V)} \longrightarrow \mathrm{H}^1(K,V)$$

or even $\exp_V\colon \mathbf{D}_{\mathrm{dR}}(V) \to \mathrm{H}^1(K,V)$ for the maps induced by the isomorphism version of $\exp_V$ in the obvious way.

Consider the Kummer dual $V^*(1) = \mathrm{Hom}_{\mathbb{Q}_p}(V,\mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(1)$ of V. The cup product gives a perfect pairing

$$\smile\colon\ \mathrm{H}^1(K,V)\times\mathrm{H}^1\big(K,V^*(1)\big)\to\mathrm{H}^2\big(K,\mathbb{Q}_p(1)\big)\cong\mathbb{Q}_p$$

which allows us to identify $\mathrm{H}^1\big(K,V^*(1)\big)$ with the dual of $\mathrm{H}^1(K,V)$. On the other hand, our fixed choice of ε provides an isomorphism between $\mathbf{D}_{\mathrm{dR}}(\mathbb{Q}_p(1))=t^{-1}K$ and K. Since $\mathbf{D}_{\mathrm{dR}}$ preserves tensor products, we get a perfect pairing

$$\mathbf{D}_{\mathrm{dR}}(V)\otimes_K\mathbf{D}_{\mathrm{dR}}\big(V^*(1)\big)\cong\mathbf{D}_{\mathrm{dR}}\big(V\otimes_{\mathbb{Q}_p}V^*(1)\big)\longrightarrow\mathbf{D}_{\mathrm{dR}}\big(\mathbb{Q}_p(1)\big)\cong K\xrightarrow{\mathrm{Tr}_{K/\mathbb{Q}_p}}\mathbb{Q}_p.$$

This in turn induces the perfect pairing

$$[\,\cdot\,,\,\cdot\,]_{\mathbf{D}_{\mathrm{dR}}(V)}\colon\ \mathrm{Fil}^0\,\mathbf{D}_{\mathrm{dR}}(V)\times\frac{\mathbf{D}_{\mathrm{dR}}\big(V^*(1)\big)}{\mathrm{Fil}^0\,\mathbf{D}_{\mathrm{dR}}\big(V^*(1)\big)}\longrightarrow\mathbb{Q}_p$$

by means of which we identify $\mathbf{D}_{\mathrm{dR}}\big(V^*(1)\big)/\mathrm{Fil}^0$ with the dual of $\mathrm{Fil}^0\,\mathbf{D}_{\mathrm{dR}}(V)$.

From the Bloch–Kato exponential map for $V^*(1)$

$$\exp_{V^*(1)}\colon\mathbf{D}_{\mathrm{dR}}\big(V^*(1)\big)\twoheadrightarrow\frac{\mathbf{D}_{\mathrm{dR}}\big(V^*(1)\big)}{\mathrm{Fil}^0\,\mathbf{D}_{\mathrm{dR}}\big(V^*(1)\big)}\to\mathrm{H}^1_{\mathrm{e}}\big(K,V^*(1)\big)\hookrightarrow\mathrm{H}^1\big(K,V^*(1)\big)$$

we obtain by duality a morphism

$$\exp^*_{V^*(1)}\colon\ \mathrm{H}^1\big(K,V\big)\to\mathrm{Fil}^0\,\mathbf{D}_{\mathrm{dR}}(V)\hookrightarrow\mathbf{D}_{\mathrm{dR}}(V).$$

Explicitly,

$$\begin{array}{ccc}
\phi & \dashrightarrow & \exp^*_{V^*(1)}(\phi) \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
\mathrm{H}^1(K,V) & \xrightarrow{\exp^*_{V^*(1)}} & \mathrm{Fil}^0\,\mathbf{D}_{\mathrm{dR}}(V) \\
\wr\| & & \wr\| \\
\mathrm{H}^1\big(K,V^*(1)\big)^* & \longrightarrow & \left(\dfrac{\mathbf{D}_{\mathrm{dR}}\big(V^*(1)\big)}{\mathrm{Fil}^0\,\mathbf{D}_{\mathrm{dR}}\big(V^*(1)\big)}\right)^* \\
\rotatebox{90}{$\ni$} & & \rotatebox{90}{$\ni$} \\
\phi\smile\cdot & \longmapsto & \phi\smile\exp_{V^*(1)}(\,\cdot\,)
\end{array}$$

and so the relation between $\exp_{V^*(1)}$ and $\exp^*_{V^*(1)}$ is given by

$$[\exp^*_{V^*(1)}(\phi),\,\cdot\,]_{\mathbf{D}_{\mathrm{dR}}(V)}=\phi\smile\exp_{V^*(1)}(\,\cdot\,).$$

Remark. Suppose that V is de Rham. Let $k\in\mathbb{Z}$. If $k\gg0$, then $\mathrm{Fil}^0\,\mathbf{D}_{\mathrm{dR}}\big(V(k)\big)=0$

and $\mathrm{Fil}^0\,\mathbf{D}_{\mathrm{dR}}\big(V(-k)\big) = \mathbf{D}_{\mathrm{dR}}\big(V(-k)\big)$. In fact,

$$\exp^*_{V^*(1+k)}\colon\ \mathrm{H}^1\big(K, V(-k)\big) \to \mathbf{D}_{\mathrm{dR}}\big(V(-k)\big)$$

is an isomorphism for $k \gg 0$.

Let us explain how to compute with the dual exponential maps. But before that, we need to introduce some operators.

For each $m \geq 1$, we define $\mathrm{Tr}_m\colon K_\infty \to K_m$ and $\mathrm{pr}_{K_m}\colon K_\infty \to K_m$ as follows: given $x \in K_\infty$, we can choose $n \gg 0$ such that $x \in K_n$ and then

$$\mathrm{Tr}_m(x) = \frac{1}{p^n}\,\mathrm{Tr}_{K_n/K_m}(x) \quad \text{and} \quad \mathrm{pr}_{K_m}(x) = \frac{1}{[K_n : K_m]}\,\mathrm{Tr}_{K_n/K_m}(x)$$

(these definitions are independent of the choice of n). For $m \gg 0$, so that K_{m+1}/K_m is a cyclic extension of degree p, we have $\mathrm{pr}_{K_m} = p^m\,\mathrm{Tr}_m$.

Recall that $\mathbf{B}_{\mathrm{dR}}$ contains $\overline{K}$ and the distinguished element t, on which G_K acts through χ_{cyc}. In particular, $K_\infty((t)) \subset \mathbf{B}_{\mathrm{dR}}^{H_K}$. We extend the previous maps to $\mathrm{Tr}_m, \mathrm{pr}_{K_m}\colon K_\infty((t)) \to K_m((t))$ by $t \mapsto t$.

Proposition 41. *The subfield $K_\infty((t))$ is dense in $\mathbf{B}_{\mathrm{dR}}^{H_K}$. Therefore, for each $m \geq 1$, we can extend Tr_m and pr_{K_m} by continuity to $\mathbb{Q}_p$–linear maps $\mathbf{B}_{\mathrm{dR}}^{H_K} \to K_m((t))$. Furthermore,*

$$\lim_{m\to\infty} \mathrm{pr}_{K_m}(x) = x \quad \textit{for all } x \in \mathbf{B}_{\mathrm{dR}}^{H_K}.$$

Proof. This is proposition IV.1.1 of Cherbonnier–Colmez's article [14]. □

Let V be a de Rham $\mathbb{Q}_p$–representation of G_K. For every $m \geq 1$, we extend the maps Tr_m and pr_{K_m} by $\mathbf{D}_{\mathrm{dR}}(V)$–linearity and get

$$\mathrm{Tr}_m, \mathrm{pr}_{K_m}\colon \mathbf{B}_{\mathrm{dR}}^{H_K} \otimes_K \mathbf{D}_{\mathrm{dR}}(V) \to K_m((t)) \otimes_K \mathbf{D}_{\mathrm{dR}}(V).$$

We also define a "projection"

$$\partial_{V(-k)}\colon K_\infty((t)) \otimes_K \mathbf{D}_{\mathrm{dR}}(V) \to K_\infty \otimes_K \mathbf{D}_{\mathrm{dR}}\big(V(-k)\big)$$

for each $k \in \mathbb{Z}$ as follows. Every $x \in K_\infty((t)) \otimes_K \mathbf{D}_{\mathrm{dR}}(V)$ has a unique expansion

$$x = \sum_{i \gg -\infty} t^i x_i \quad \text{with } x_i \in K_\infty \otimes_K \mathbf{D}_{\mathrm{dR}}(V) \text{ for all } i \in \mathbb{Z}$$

and we set $\partial_{V(-k)} = t^k x_k \in K_\infty \otimes_K \mathbf{D}_{\mathrm{dR}}\big(V(-k)\big)$.

TODO: I think that the last part is explained more systematically in something called Sen–Tate theory. It might be worth learning what $\mathbf{D}_{\mathrm{dif}}$ is (it links (φ, Γ_K)–modules with $\mathbf{D}_{\mathrm{dR}}$).

Since V is de Rham, the natural map

$$\mathbf{B}_{\mathrm{dR}} \otimes_K \mathbf{D}_{\mathrm{dR}}(V) \to \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V$$

(given by multiplication in $\mathbf{B}_{\mathrm{dR}}$) is an isomorphism. Taking H_K–invariants on both sides, we obtain $\mathbf{B}_{\mathrm{dR}}^{H_K} \otimes_K \mathbf{D}_{\mathrm{dR}}(V) \cong \left(\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V\right)^{H_K}$.

Proposition 42. *Let V be a de Rham $\mathbb{Q}_p$–representation of G_{K_n} and let $k \in \mathbb{Z}$. Let $n \in \mathbb{Z}_{\geq 1}$ and take $\alpha \in \mathrm{H}^1\big(K_n, V(-k)\big)$. We have an inflation-restriction exact sequence*

$$0 \longrightarrow \mathrm{H}^1\big(\Gamma_{K_n}, (\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V(-k))^{H_K}\big) \xrightarrow{\inf} \mathrm{H}^1\big(G_{K_n}, \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V(-k)\big) -$$
$$\xrightarrow{\mathrm{res}} \mathrm{H}^1\big(H_K, \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V(-k)\big) = 0.$$

Then, given $[c] \in \mathrm{H}^1\big(\Gamma_{K_n}, (\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V(-k))^{H_K}\big)$, represented by a 1–cocycle

$$c\colon \Gamma_{K_n} \longrightarrow \big(\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V(-k)\big)^{H_K} \cong \mathbf{B}_{\mathrm{dR}}^{H_K} \otimes_K \mathbf{D}_{\mathrm{dR}}\big(V(-k)\big),$$

such that the image of α under $\mathrm{H}^1\big(G_{K_n}, V(-k)\big) \to \mathrm{H}^1\big(G_{K_n}, \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V(-k)\big)$ coincides with $\inf\big([c]\big)$,

$$\exp^*_{V^*(1+k)}(\alpha) = (\partial_{V(-k)} \circ \mathrm{pr}_{K_n})\left(\frac{c(\sigma)}{\log_p\big(\chi_{\mathrm{cyc}}(\sigma)\big)}\right)$$

for any $\sigma \in \Gamma_{K_n}$ with $\log_p\big(\chi_{\mathrm{cyc}}(\sigma)\big) \neq 0$.

Proof. This is proposition IV.1.2 of Cherbonnier–Colmez's article [14]. TODO: give a better reference. I think this is (related to?) a result of Kato. □

4.4 Reciprocity laws

In this subsection, let V be a de Rham $\mathbb{Q}_p$–representation of G_K. We want to describe $\mathrm{Exp}^*_{V^*(1)}\colon \mathrm{H}^1_{\mathrm{Iw}}(K, V) \to \mathbf{D}(V)^{\psi=1}$ and relate its image to $\mathbf{D}_{\mathrm{dR}}(V)$.

Lemma 43. *For every $n \in \mathbb{Z}_{\geq 1}$, let $s_n = p^{n-1}(p-1)$ (as in the remark of section 2.2.1). For $n \gg 0$ we have $\mathbf{D}(V)^{\psi=1} \subset \mathbf{D}^{\dagger, s_n}(V)$.*

Proof. See proposition III.3.2 of Cherbonnier–Colmez's article [14]. □

In particular, given $[\mu] \in \mathrm{H}^1_{\mathrm{Iw}}(K, V)$, we can view

$$\varphi^{-n}\big(\mathrm{Exp}^*_{V^*(1)}([\mu])\big) \in \big(\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V\big)^{H_K} \cong \mathbf{B}_{\mathrm{dR}}^{H_K} \otimes_K \mathbf{D}_{\mathrm{dR}}(V)$$

and so we can apply the operators Tr_m and pr_{K_m} to it.

Theorem 44 (Cherbonnier–Colmez). *Let V be a de Rham $\mathbb{Q}_p$–representation of G_K and let $m \in \mathbb{Z}_{\geq 1}$. Let $[\mu] \in \mathrm{H}^1_{\mathrm{Iw}}(K, V) \cong \mathrm{H}^1\big(K, \Lambda(\Gamma_K) \otimes_{\mathbb{Z}_p} V\big)$.*

(1) *The element*

$$\mathrm{Exp}^*_{V^*(1),K_m}([\mu]) = \mathrm{Tr}_m\Big(\varphi^{-n}\big(\mathrm{Exp}^*_{V^*(1)}([\mu])\big)\Big) \in K_m((t)) \otimes_K \mathbf{D}_{\mathrm{dR}}(V)$$

(which makes sense for $n \gg 0$ so that we can apply lemma 43 and $n \geq m$) is independent of n.

(2) *For every $k \in \mathbb{Z}$, consider*

$$[\mu_{m,k}] = \left[\Big(\int_{\Gamma_{K_m}} \chi_{\mathrm{cyc}}^{-k}(x)\, \mu(x)\Big)(-k)\right] \in \mathrm{H}^1\big(K_m, V(-k)\big)$$

and

$$\begin{aligned} e_{m,k} = \exp^*_{V^*(1+k)}\big([\mu_{m,k}]\big) \in \mathbf{D}_{\mathrm{dR},K_m}\big(V(-k)\big) &= t^{-k}\mathbf{D}_{\mathrm{dR},K_m}(V) \\ &\subset K_m((t)) \otimes_K \mathbf{D}_{\mathrm{dR},K}(V) \subset \mathbf{B}_{\mathrm{dR}}^{H_K} \otimes_K \mathbf{D}_{\mathrm{dR},K}(V). \end{aligned}$$

Then

$$\mathrm{Exp}^*_{V^*(1),K_m}([\mu]) = \sum_{k\in\mathbb{Z}} e_{m,k}.$$

(3) *If $m \gg 0$, then*

$$\mathrm{Exp}^*_{V^*(1),K_m}([\mu]) = p^{-m}\varphi^{-m}\big(\mathrm{Exp}^*_{V^*(1)}([\mu])\big)$$

or, equivalently,

$$\mathrm{pr}_{K_m}\Big(\varphi^{-m}\big(\mathrm{Exp}^*_{V^*(1)}([\mu])\big)\Big) = \varphi^{-m}\big(\mathrm{Exp}^*_{V^*(1)}([\mu])\big).$$

Proof. This is theorem IV.2.1 of Cherbonnier–Colmez's article [14]. TODO: explain some of the ideas. □

This important theorem allows us to recover the dual exponential maps on Tate twists of a de Rham $\mathbb{Q}_p$–representation V of G_K (and not only its restrictions to G_{K_m} for $m \gg 0$) using the following fact:

Fact. Let L/E be finite extensions of K. The diagram

$$\begin{array}{ccc} \mathrm{H}^1(L,V) & \xrightarrow{\exp^*_{V^*(1)}} & \mathbf{D}_{\mathrm{dR},L}(V) \cong L \otimes_K \mathbf{D}_{\mathrm{dR}}(V) \\ {\scriptstyle \mathrm{cor}}\downarrow & & \downarrow{\scriptstyle \mathrm{Tr}_{L/E} \otimes \mathrm{id}_{\mathbf{D}_{\mathrm{dR}}(V)}} \\ \mathrm{H}^1(E,V) & \xrightarrow[\exp^*_{V^*(1)}]{} & \mathbf{D}_{\mathrm{dR},E}(V) \cong E \otimes_K \mathbf{D}_{\mathrm{dR}}(V) \end{array}$$

is commutative.

Consider $m \in \mathbb{Z}_{\geq 1}$ large enough (depending only on V) to apply part (3) of theorem 44. We define

$$\mathrm{Sp}_{\mathrm{cyc},V} \colon \mathbf{D}(V)^{\psi=1} \xrightarrow{p^{-m}\varphi^{-m}} K_m((t)) \otimes_K \mathbf{D}_{\mathrm{dR}}(V) \xrightarrow{\mathrm{Tr}_{K_m/K} \otimes \mathrm{id}_{\mathbf{D}_{\mathrm{dR}}(V)}} K((t)) \otimes_K \mathbf{D}_{\mathrm{dR}}(V)$$

(which is independent of m by part (1) of theorem 44 and because of the equality $\mathrm{Tr}_m = \mathrm{Tr}_{K_n/K_m} \circ \mathrm{Tr}_n$ for $n \geq m$). Given $[\mu] \in \mathrm{H}^1_{\mathrm{Iw}}(K,V)$, the fact above and part (2) of theorem 44 imply that

$$\mathrm{Sp}_{\mathrm{cyc},V}\big(\mathrm{Exp}^*_{V^*(1)}([\mu])\big) = \sum_{k \in \mathbb{Z}} \exp^*_{V^*(1+k)}([\mu_{0,k}])t^k,$$

where

$$[\mu_{0,k}] = \left[\Big(\int_{\Gamma_K} \chi_{\mathrm{cyc}}^{-k}(x)\,\mu(x)\Big)(-k)\right] \in \mathrm{H}^1(K,V) \quad \text{for all } k \in \mathbb{Z}.$$

We can recover the coefficient of t^k by composing with $\partial_{V(-k)}$.

TODO: Should I express this power series as a measure on Γ_K? Right now, I have a Laurent series whose coefficients should correspond to $\int_{\mathbb{Z}_p^\times} x^k\,\nu(x)$ and from this I could recover the Amice transform $\int_{\mathbb{Z}_p^\times} \binom{x}{k}\,\nu(x)$.

4.5 Interpolation in an integral family

Throughout this subsection, let R be a finite flat $\Lambda(\Gamma_K)$–algebra that is a coefficient ring in the sense of Mazur (cf. definition 7).

Definition 45. An *arithmetic point* of R is a continuous $\mathbb{Z}_p$–algebra homomorphism $\nu \colon R \to \overline{\mathbb{Q}}_p$ with the property that the composition

$$\Gamma_K \to R \xrightarrow{\nu} \overline{\mathbb{Q}}_p$$

is of the form $\gamma \mapsto \nu_0(\gamma)\gamma^{k-2}$ for some integer $k \geq 2$ and some finite-order character $\nu_0\colon \Gamma_K \to \mu_{p^\infty}$. In this situation, we say that ν has *weight* (k, ν_0) and call $\mathrm{Ker}(\nu)$ an *arithmetic prime* of R. Let F_ν denote the residue field of $\mathrm{Ker}(\nu)$ and let $\mathscr{O}_\nu$ be the ring of integers of F_ν.

We write $\mathfrak{X}^{\mathrm{arith}}(R)$ for the set of arithmetic points of R.

Let $\mathbb{T}$ be an R–representation of G_K and let $\nu \in \mathfrak{X}^{\mathrm{arith}}(R)$. We define

$$T_\nu = \mathbb{T} \otimes_{R,\nu} \mathscr{O}_\nu \quad \text{and} \quad V_\nu = T_\nu \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Then, the natural map $\mathbb{T} \to V_\nu$ induces specialization maps

$$\mathrm{H}^1_{\mathrm{Iw}}(K, \mathbb{T}) \xrightarrow{\mathrm{Sp}_\nu} \mathrm{H}^1_{\mathrm{Iw}}(K, V_\nu) \quad \text{and} \quad \mathbf{D}(\mathbb{T})^{\psi=1} \xrightarrow{\mathrm{Sp}_\nu} \mathbf{D}(V_\nu)^{\psi=1}.$$

Combining this with the results of section 4.4, for every $\nu \in \mathfrak{X}^{\mathrm{arith}}(R)$ and every $k \in \mathbb{Z}$, the diagram

$$\begin{array}{ccc}
\mathrm{H}^1_{\mathrm{Iw}}(K, \mathbb{T}) & \xrightarrow{\mathrm{Exp}^*_{\mathbb{T}^*(1)}} & \mathbf{D}(\mathbb{T})^{\psi=1} \\
\downarrow{\scriptstyle \mathrm{Sp}_\nu} & & \downarrow{\scriptstyle \mathrm{Sp}_\nu} \\
\mathrm{H}^1_{\mathrm{Iw}}(K, V) & \xrightarrow{\mathrm{Exp}^*_{V^*(1)}} & \mathbf{D}(V_\nu)^{\psi=1} \\
\downarrow{\scriptstyle \int_{\Gamma_K} \chi_{\mathrm{cyc}}^{-k}} & & \downarrow{\scriptstyle \partial_{V(-k)} \circ \mathrm{Sp}_{\mathrm{cyc}, V_\nu}} \\
\mathrm{H}^1\big(K, V_\nu(-k)\big) & \xrightarrow{\exp^*_{V_\nu^*(1+k)}} & \mathbf{D}_{\mathrm{dR}}\big(V_\nu(-k)\big)
\end{array}$$

is commutative.

Part II
The theory for (relative) Lubin–Tate extensions

The previous sections all deal with the most classical and well-understood situation where K_∞/K is a cyclotomic extension. It is natural to hope that a similar theory can be developed for other kinds of extensions. Unfortunately, the theory of fields of norms of Fontaine and Wintenberger, which is essential to define the base rings for (φ,Γ)–modules, imposes certain restrictions on the extension K_∞/K. Specifically on the ramification properties of such extension.

The fundamental article [28] of Kisin and Ren, following ideas that had appeared in work of Colmez and others, initiated the systematic study of p–adic Galois representations through (φ,Γ)–modules when K_∞/K is the extension obtained from the torsion points of a Lubin–Tate formal group. There has been a lot of progress in this subject since then, thanks to the work of Berger, Fourquaux, Schneider, Venjakob, Xie…

The objective of this second part is to (slightly) generalize the p–adic Hodge theory and the Iwasawa theory of Lubin–Tate extensions to allow for what are known as *relative* Lubin–Tate formal groups, which often appear more naturally.

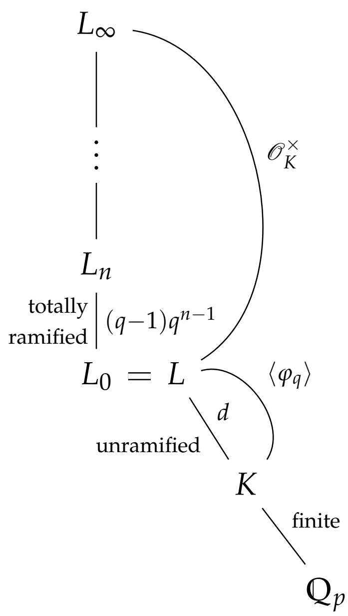
For the first time, we replace some of the notation of the previous sections. More specifically, the same symbols will now denote more general versions of the objects that appeared before. One can recover the previous theory by regarding the cyclotomic extension as obtained from the torsion points of the multiplicative formal group.

5 Relative Lubin–Tate groups

Throughout this section, let K denote a fixed finite extension of $\mathbb{Q}_p$ and write $\mathscr{O}_K$ for its ring of integers and $k = \mathbb{F}_q$ for its residue field.

To begin with, we recall the basic theory of Lubin–Tate formal groups *relative to an unramified extension L/K*, as introduced in de Shalit's short article [36]. A more detailed exposition with proofs is included in the first chapter of de Shalit's book [37] or in Schneider's course notes [32]. Some results will not be stated in their most general form here.

Let L be the finite unramified extension of degree d over K, with ring of integers $\mathscr{O}_L$ and residue field k_L. The Galois group $\mathrm{Gal}(L/K)$ is generated by the Frobenius element φ_q, which lifts the q–th power map on k_L. Our goal is to describe *Lubin–Tate towers of extensions* of the following shape:



5.1 The formal module

Definition 46. Let π_L be a uniformizer of $\mathscr{O}_L$. A *Frobenius power series for π_L* is a formal power series $\phi(Z) \in \mathscr{O}_L[\![Z]\!]$ satisfying that

(i) $\phi(Z) \equiv \pi_L Z \mod Z^2$ and

(ii) $\phi(Z) \equiv Z^q \mod \pi_L$.

Fix once and for all a uniformizer π_L and a Frobenius power series $\phi(Z)$ for π_L. We will write $\xi_K = \mathrm{N}_{L/K}(\pi_L) \in \mathscr{O}_K$. As in the original theory of Lubin and Tate, we can attach a formal group law to these objects.

Lemma 47. *Let $n \in \mathbb{Z}_{\geq 1}$. For every linear homogeneous polynomial*

$$F_1(Z_1, \dots, Z_n) = a_1 Z_1 + \cdots + a_n Z_n \in \mathscr{O}_K[Z_1, \dots, Z_n]$$

(with coefficients in $\mathscr{O}_K$), there exists a unique power series

$$F(Z_1, \dots, Z_n) \in \mathscr{O}_L[\![Z_1, \dots, Z_n]\!]$$

(with coefficients in $\mathscr{O}_L$, not necessarily in $\mathscr{O}_K$) such that

(i) $F(Z_1, \dots, Z_n) \equiv F_1(Z_1, \dots, Z_n) \mod (Z_1, \dots, Z_n)^2$ *and*

(ii) $\phi\big(F(Z_1, \dots, Z_n)\big) = F^{\varphi_q}\big(\phi(Z_1), \dots, \phi(Z_n)\big)$.

Proof. See lemma I.1.4 of de Shalit's book [37]. □

As an immediate application of lemma 47, we obtain the following result.

Theorem 48.

(1) *There exists a unique formal group law $\mathfrak{F} = \mathfrak{F}_\phi(X, Y) \in \mathscr{O}_L[\![X, Y]\!]$ for which the Frobenius power series $\phi(Z)$ defines a homomorphism $\mathfrak{F} \to \mathfrak{F}^{\varphi_q}$ of formal groups over $\mathscr{O}_L$ (i.e., $\phi\big(\mathfrak{F}(X, Y)\big) = \mathfrak{F}^{\varphi_q}\big(\phi(X), \phi(Y)\big)$). Here, $\mathfrak{F}^{\varphi_q}$ denotes the power series obtained by applying φ_q to the coefficients of $\mathfrak{F}$.*

(2) *There is an injective morphism of rings $[\,\cdot\,]_\phi \colon \mathscr{O}_K \to \mathrm{End}_{\mathscr{O}_L}(\mathfrak{F}_\phi)$ defined as follows: for every $a \in \mathscr{O}_K$, the power series $[a]_\phi(Z) \in \mathscr{O}_L[\![Z]\!]$ is characterized by the properties*

$$[a]_\phi(Z) \equiv aZ \mod Z^2 \quad \text{and} \quad \phi\big([a]_\phi(Z)\big) = [a]_\phi^{\varphi_q}\big(\phi(Z)\big).$$

Proof. See theorem I.1.3 and proposition I.1.5 of de Shalit's book [37]. □

Remark. Lemma 47 (resp. the second part of theorem 48) is a special case of lemma I.1.4 (resp. proposition I.1.5) of de Shalit's book [37]. The more general results allow one to relate the formal group laws obtained from different uniformizers and Frobenius power series. For our applications, it is sufficient to note that $\phi^{\varphi_q}(Z)$ is a Frobenius power series for the uniformizer $\varphi_q(\pi_L)$ and then $\mathfrak{F}_\phi^{\varphi_q} = \mathfrak{F}_{\phi^{\varphi_q}}$ and $[a]_\phi^{\varphi_q} = [a]_{\phi^{\varphi_q}}$. Thus, we obtain at most d formal groups corresponding to the Frobenius iterates of π_L, all of which have the same norm ξ_K.

Definition 49. The formal group $\mathfrak{F}_\phi$ from theorem 48 is the *Lubin–Tate formal group (relative to the extension L/K)* associated with the Frobenius power series ϕ.

5.2 Torsion points

Define $\phi_1 = \phi$ and $\phi_{n+1} = \phi^{\varphi_q^n} \circ \phi_n = \phi^{\varphi_q^n} \circ \phi^{\varphi_q^{n-1}} \circ \cdots \circ \phi^{\varphi_q} \circ \phi$ for $n \geq 1$. That is, $\phi_n \in \operatorname{Hom}_{\mathscr{O}_L}(\mathfrak{F}_\phi, \mathfrak{F}_\phi^{\varphi_q^n})$ for all $n \in \mathbb{Z}_{\geq 1}$. (In particular, $\phi_d = [\xi_K]_\phi$.) To these power series we attach the torsion modules

$$\mathfrak{F}_{\phi,n} = \mathfrak{F}_\phi[\phi_n] = \{\, z \in \mathfrak{m}_{\mathbb{C}_p} : \phi_n(z) = 0 \,\}.$$

If π_K is any uniformizer of $\mathscr{O}_K$, one can prove (using Weierstrass's preparation theorem and counting points) that

$$\mathfrak{F}_{\phi,n} = \{\, z \in \mathfrak{m}_{\mathbb{C}_p} : [\pi_K^n]_\phi(z) = 0 \,\} = \{\, z \in \mathfrak{m}_{\mathbb{C}_p} : [a]_\phi(z) = 0 \text{ for all } a \in \mathfrak{m}_K^n \,\}$$

Set $L_n = L(\mathfrak{F}_{\phi,n})$ for every $n \geq 1$.

Proposition 50. *For every $n \in \mathbb{Z}_{\geq 1}$, the set $\mathfrak{F}_{\phi,n}$ becomes a free $(\mathscr{O}_K/\mathfrak{m}_K)$–module of rank 1 with the addition given by $\mathfrak{F}_\phi$ and the multiplication by scalars induced by $[\,\cdot\,]_\phi$.*

Proof. See proposition 3.3 of Schneider's notes [32]. □

Proposition 51. *Let $n \in \mathbb{Z}_{\geq 1}$. The extension L_n/L is finite and Galois with a canonical isomorphism $\operatorname{Gal}(L_n/L) \cong (\mathscr{O}_K/\mathfrak{m}_K^n)^\times$. Furthermore, L_n/L is totally ramified and any generator z_n of $\mathfrak{F}_{\phi,n}$ generates $\mathscr{O}_{L_n}$ over $\mathscr{O}_L$.*

Proof. See proposition 3.5 of Schneider's notes [32]. □

Definition 52. Let

$$L_\infty = \bigcup_{n \geq 1} L_n = L\Big(\bigcup_{n \geq 1} \mathfrak{F}_{\phi,n} \Big).$$

The *Lubin–Tate character of* $\mathfrak{F}_\phi$ is the isomorphism $\chi_{\xi_K} = \chi_\phi \colon \operatorname{Gal}(L_\infty/L) \to \mathscr{O}_K^\times$ characterized by

$$\sigma(z) = [\chi_\phi(\sigma)]_\phi(z) \quad \text{for all } z \in \bigcup_{n \geq 1} \mathfrak{F}_{\phi,n} \text{ and all } \sigma \in \operatorname{Gal}(L_\infty/L).$$

Remark. One can prove that the fields L_n for $n \in \mathbb{Z}_{\geq 1}$ and the character χ_{ξ_K} depend only on $\xi_K = \mathrm{N}_{L/K}(\pi_L)$ (cf. proposition I.1.8 of de Shalit's book [37]). In particular, we may replace $\mathfrak{F}_\phi$ with $\mathfrak{F}_\phi^{\varphi_q^n} = \mathfrak{F}_{\phi^{\varphi_q^n}}$ for any $n \in \mathbb{Z}$. We write χ_{ξ_K} instead of χ_ϕ whenever we want to stress this independence of ϕ.

Theorem 53. *Let $\mathrm{v}_K \colon K \twoheadrightarrow \mathbb{Z} \cup \{\infty\}$ denote the normalized valuation of K. Recall that $d = [L : K]$.*

(1) *The compositum of L_∞ and the maximal unramified extension K^{ur} of K is the maximal abelian extension K^{ab} of K. Consequently,*

$$\mathrm{Gal}(K^{\mathrm{ab}}/L) = \mathrm{Gal}(L_\infty/L) \times \mathrm{Gal}(K^{\mathrm{ur}}/L).$$

(2) *The map* $\mathrm{v}_K^{-1}(d\mathbb{Z}) = \mathscr{O}_K^\times \cdot \xi_K^{\mathbb{Z}} \to \mathrm{Gal}(K^{\mathrm{ab}}/L)$ *defined by*

$$u \cdot \xi_K^j \mapsto \big([u^{-1}]_\phi, \varphi_q^{dj}\big) \in \mathrm{Gal}(L_\infty/L) \times \mathrm{Gal}(K^{\mathrm{ur}}/L)$$

(where $[u^{-1}]_\phi$ *denotes the element of* $\mathrm{Gal}(L_\infty/L)$ *corresponding to* u^{-1} *via the Lubin–Tate character* χ_ϕ*) is the restriction of the (local) Artin reciprocity map* $\mathrm{rec}_K \colon K^\times \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$.

Proof. See proposition 5.8, theorems 5.9 and 5.26 and corollary 5.12 of Schneider's notes [32]. □

For our applications, we are going to need some more constructions and notations. Throughout this section, let $G_L = \mathrm{Gal}(\overline{K}/L)$, $H_L = \mathrm{Gal}(\overline{K}/L_\infty)$ and $\Gamma_L = G_L/H_L \cong \mathrm{Gal}(L_\infty/L)$. We sometimes identify Γ_L with $\mathscr{O}_K^\times$ via the Lubin–Tate character χ_{ξ_K}.

Definition 54. The *Tate module of the (relative) Lubin–Tate group* $\mathfrak{F}_\phi$ is

$$\mathrm{T}_\phi \, \mathfrak{F}_\phi = \varprojlim_{n \geq 0} \mathfrak{F}_{\phi^{\varphi_q^{-n}}, n} = \varprojlim_{n \geq 0} \mathfrak{F}_\phi^{\varphi_q^{-n}} \big[(\phi^{\varphi_q^{-n}})_n \big] = \varprojlim_{n \geq 0} \mathrm{Ker}\big(\phi^{\varphi_q^{-1}} \circ \phi^{\varphi_q^{-2}} \circ \cdots \circ \phi^{\varphi_q^{-n}} \big),$$

where the projective limit is taken with respect to the transition maps given by $\phi^{\varphi_q^{-n}} : \mathfrak{F}_\phi^{\varphi_q^{-n}} \to \mathfrak{F}_\phi^{\varphi_q^{-n+1}}$.

Remark. The set $\mathrm{T}_\phi \, \mathfrak{F}_\phi$ inherits the structure of an $\mathscr{O}_K^\times$–module and an action of Γ_L from the respective structures on each $\mathfrak{F}_{\phi^{\varphi_q^{-n}}, n}$. Propositions 50 and 51 imply that $\mathrm{T}_\phi \, \mathfrak{F}_\phi \cong \mathscr{O}_K(\chi_{\xi_K})$. That is, for every $(z_n)_{n \geq 0} \in \mathrm{T}_\phi \, \mathfrak{F}_\phi$ and every $\sigma \in \Gamma_L$,

$$\sigma\big((z_n)_{n \geq 0}\big) = \big([\chi_{\xi_K}(\sigma)]_{\phi^{\varphi_q^{-n}}}(z_n)\big)_{n \geq 0}.$$

5.3 The formal logarithm

Definition 55. Let $\omega(Z) = (1 + \cdots)\, dZ \in \mathscr{O}_L[\![Z]\!]\, dZ$ be the *(normalized) invariant differential of the formal group* $\mathfrak{F}_\phi$, characterized by

$$\omega \circ \mathfrak{F}_\phi(X, Y) = \omega(X) + \omega(Y).$$

The *formal logarithm of the formal group* $\mathfrak{F}_\phi$ is the formal integral

$$\log_\phi(Z) = \log_{\mathfrak{F}_\phi}(Z) = \int \omega(Z) = Z + \cdots \in Z \cdot L[\![Z]\!].$$

Remark. The formal logarithm defines an isomorphism $\log_\phi \colon \mathfrak{F}_\phi \to \widehat{\mathbb{G}}_a$ of formal groups over L. Its inverse is the formal exponential $\exp_\phi \colon \widehat{\mathbb{G}}_a \to \mathfrak{F}_\phi$. In particular, $\log_\phi\big(\mathfrak{F}_\phi(X,Y)\big) = \log_\phi(X) + \log_\phi(Y)$ and $\log_\phi\big([a]_\phi(Z)\big) = a \cdot \log_\phi(Z)$ for every $a \in \mathscr{O}_K$.

Lemma 56. *The formal logarithm of $\mathfrak{F}_\phi$ can also be computed as*

$$\log_\phi(Z) = \lim_{n\to\infty} \frac{\phi_n(Z)}{\varphi_q^{n-1}(\pi_L)\cdots\varphi_q(\pi_L)\pi_L},$$

where the limit is taken with respect to the (π_L, Z)–adic topology of $L[\![Z]\!]$.

Proof. This is similar to part of the proof of lemma 9.8 of Colmez's article [16]. We reproduce the relevant parts here for the convenience of the reader.

Let

$$l_n(Z) = \frac{\phi_n(Z)}{\varphi_q^{n-1}(\pi_L)\cdots\varphi_q(\pi_L)\pi_L} = \frac{\phi^{\varphi_q^{n-1}} \circ \cdots \circ \phi^{\varphi_q} \circ \phi(Z)}{\varphi_q^{n-1}(\pi_L)\cdots\varphi_q(\pi_L)\pi_L}.$$

We need to check that the sequence $(l_n)_{n\geq 1}$ converges in $L[\![Z]\!]$.

We can write

$$l_{n+1}(Z) - l_n(Z) = \frac{\phi^{\varphi_q^n} \circ \phi_n(Z) - \varphi_q^n(\pi_L)\phi_n(Z)}{\varphi_q^n(\pi_L)\cdots\varphi_q(\pi_L)\pi_L} = \frac{\rho^{\varphi_q^n}(\phi_n(Z))}{\varphi_q^n(\pi_L)\cdots\varphi_q(\pi_L)\pi_L}$$

with $\rho(Z) = \phi(Z) - \pi_L Z \in Z^2\mathscr{O}_L[\![Z]\!]$. But the action of φ_q does not change the valuations and $\phi(Z) \equiv Z^q \bmod \pi_L$. Thus, the coefficient of Z^k in $l_{n+1}(Z) - l_n(Z)$ has π_L–adic valuation at least

$$2\big(n - \lfloor \log_q(k) \rfloor\big) - (n+1) = n - 1 - \lfloor \log_q(k) \rfloor \xrightarrow[n\to\infty]{} +\infty.$$

The previous estimate proves the convergence of the sequence $(l_n(Z))_{n\geq 1}$; let $l(Z)$ denote its limit in $L[\![Z]\!]$. We can check that it satisfies the defining properties of $\log_\phi$:

- By definition, $l_n(Z) = Z + \cdots$. Therefore, the coefficient of Z in $l(Z)$ is 1.
- We claim that $l(Z)$ defines a homomorphism of formal groups $\mathfrak{F}_\phi \to \widehat{\mathbb{G}}_a$.

Indeed,

$$\begin{aligned} l(\mathfrak{F}_\phi(X,Y)) &= \lim_{n\to\infty} \frac{\phi_n(\mathfrak{F}_\phi(X,Y))}{\varphi_q^{n-1}(\pi_L)\cdots\varphi_q(\pi_L)\pi_L} = \lim_{n\to\infty} \frac{\mathfrak{F}_\phi^{\varphi_q^n}(\phi_n(X),\phi_n(Y))}{\varphi_q^{n-1}(\pi_L)\cdots\varphi_q(\pi_L)\pi_L} \\ &= \lim_{n\to\infty} \frac{\phi_n(X)+\phi_n(Y)}{\varphi_q^{n-1}(\pi_L)\cdots\varphi_q(\pi_L)\pi_L} = l(X)+l(Y), \end{aligned}$$

where in the third equality we used that $\mathfrak{F}_\phi^{\varphi_q^n}(X,Y) \equiv X+Y \bmod (X,Y)^2$ and so the contribution of the higher order terms tends to 0 (as can be seen from a rough estimate as above). □

Corollary 57. *The formal logarithm of $\mathfrak{F}_\phi$ satisfies the equation $\log_\phi^{\varphi_q} \circ \phi = \pi_L \cdot \log_\phi$.*

Corollary 58. *The zeros of the formal logarithm $\log_\phi(Z)$ are exactly the torsion points of $\mathfrak{F}_\phi$, namely*

$$\bigcup_{n\geq 1} \mathfrak{F}_\phi[\phi_n],$$

each with multiplicity 1*.*

Definition 59. The *invariant derivation of the formal group $\mathfrak{F}_\phi$* is

$$\partial_\phi = \frac{d}{d\big(\log_\phi(Z)\big)}.$$

Remark. Write $\mathrm{dlog}_\phi(Z) = g_\phi(Z)\,dZ$. By the definition of ∂_ϕ, for every formal series $f(Z) \in L[\![Z,Z^{-1}]\!]$ we get $f'(Z)\,dZ = \partial_\phi(f)(Z)\,\mathrm{dlog}_\phi(Z)$ or, equivalently,

$$\partial_\phi(f)(Z) = \frac{f'(Z)}{g_\phi(Z)}.$$

6 (Modified) period rings

Since the extension L_∞/L is no longer cyclotomic, we have to modify the construction of some of the period rings of p–adic Hodge theory. The exposition in this section is mostly based on the very detailed presentation in Schneider and Venjakob's article [34], even if the main constructions were already present in Kisin and Ren's previous work [28].

6.1 Rings of formal series

Define $\mathbf{E}'_L = k_L((Z))$ and $\mathbf{A}'^{,+}_L = \mathscr{O}_L[\![Z]\!]$. Let $\mathbf{A}'_L$ be the π_L–adic completion of $\mathscr{O}_L((Z))$ and let $\mathbf{B}'_L$ be its field of fractions. In more concrete terms,

$$\mathbf{A}'_L = \left\{ \sum_{k \in \mathbb{Z}} a_k Z^k \in \mathscr{O}_L[\![Z, Z^{-1}]\!] : \lim_{k \to -\infty} |a_k|_p = 0 \right\}$$

and

$$\mathbf{B}'_L = \left\{ \sum_{k \in \mathbb{Z}} a_k Z^k \in L[\![Z, Z^{-1}]\!] : \sup_{k \in \mathbb{Z}} |a_k|_p < \infty \text{ and } \lim_{k \to -\infty} |a_k|_p = 0 \right\}.$$

Then $\mathbf{A}'_L$ is a Cohen ring with residue field $\mathbf{E}'_L$ and $\mathbf{B}'_L = \mathbf{A}'_L[\pi_L^{-1}]$. We endow $\mathbf{A}'_L$ with the weak topology, for which the $\mathscr{O}_L$–submodules

$$\pi_L^k \mathbf{A}'_L + Z^n \mathbf{A}'^{,+}_L \quad \text{for } k, n \in \mathbb{Z}_{\geq 0}$$

form a basis of open neighbourhoods of 0, and

$$\mathbf{B}'_L = \bigcup_{n \geq 0} \pi_L^{-n} \mathbf{A}'_L$$

with the direct limit topology.

We also consider the ring

$$\mathbf{B}^+_{\mathrm{rig},L} = \left\{ \sum_{k \geq 0} a_k Z^k \in L[\![Z]\!] : \lim_{k \to \infty} |a_k|_p r^k = 0 \text{ for all } r \in [0,1) \right\}$$

of rigid analytic functions on the open unit disc $\mathcal{B}$. Since the rigid variety $\mathcal{B}$ over L is quasi-Stein (namely, it is the rising union of closed discs $\mathcal{B}[r]$ of increasing radius r), the ring $\mathbf{B}^+_{\mathrm{rig},L}$ is a Fréchet L–algebra with respect to the norms

$$\Big\|\sum_{k\geq 0} a_k Z^k\Big\|_{\mathcal{B}[r]} = \max_{k\geq 0}\big(|a_k|_p r^k\big) \quad \text{for } r \in p^{\mathbb{Q}} \text{ with } r < 1.$$

6.1.1 Operators

Definition 60. The *Coleman Frobenius operator* (associated with the relative Lubin–Tate formal group $\mathfrak{F}_\phi$) is the morphism of K–algebras $\varphi_q \colon \mathbf{A}'^{,+}_L \to \mathbf{A}'^{,+}_L$ defined by $\varphi_q\big(f(Z)\big) = f^{\varphi_q}\big(\phi(Z)\big)$.

Lemma 61. *The morphism $\varphi_q \colon \mathbf{A}'^{,+}_L \to \mathbf{A}'^{,+}_L$ is injective and*

$$\mathrm{Im}(\varphi_q) = \{\, f(Z) \in \mathbf{A}'^{,+}_L : f\big(\mathfrak{F}_\phi(Z, v_1)\big) = f(Z) \textit{ for all } v_1 \in \mathfrak{F}_{\phi,1} \,\}.$$

Proof. See lemma 4.1 of Schneider's notes [32] (and observe that applying φ_q on the coefficients of power series is an automorphism). □

Remark. The Coleman Frobenius operator can be used to define Coleman norm and trace operators and a Coleman map on norm-compatible systems of units as in the classical theory (cf. section 4 of Schneider's notes [32] or section 2 of Schneider–Venjakob's article [34]).

Since $\varphi_q(Z) = \phi(Z) \equiv Z^q \bmod \pi_L$ is a unit in $\mathbf{A}'_L$, we can extend φ_q to a morphism $\mathscr{O}_L(\!(Z)\!) \to \mathbf{A}'_L$ and by continuity to $\varphi_q \colon \mathbf{A}'_L \to \mathbf{A}'_L$ and $\varphi_q \colon \mathbf{B}'_L \to \mathbf{B}'_L$. Similarly, we obtain $\varphi_q \colon \mathbf{B}^+_{\mathrm{rig},L} \to \mathbf{B}^+_{\mathrm{rig},L}$. Analogously, there are actions of Γ_L on $\mathbf{A}'_L$, $\mathbf{B}'_L$ and $\mathbf{B}^+_{\mathrm{rig},L}$ defined by

$$\big(\gamma, f(Z)\big) \mapsto f\big([\chi_\phi(\gamma)]_\phi(Z)\big).$$

Remark. The formal logarithm $\log_\phi(Z)$ converges on the open unit disc and so is an element of $\mathbf{B}^+_{\mathrm{rig},L}$. Corollary 57 says that φ_q acts on $\log_\phi(Z)$ as multiplication by π_L.

Lemma 62. *The $\varphi_q(\mathbf{A}'_L)$–module $\mathbf{A}'_L$ (resp. the $\varphi_q(\mathbf{B}'_L)$–module $\mathbf{B}'_L$) is free with basis $1, Z, \ldots, Z^{q-1}$.*

Proof. See proposition 1.7.3 of Schneider's book [31] (where the result is stated for the classical Lubin–Tate case, but the proof works verbatim for the *relative* Lubin–Tate situation.) □

Definition 63. The operator ψ_q on $\mathbf{A}'_L$ is the unique additive endomorphism of $\mathbf{A}'_L$ satisfying that

$$\varphi_q \circ \psi_q = \frac{1}{\pi_L} \operatorname{Tr}_{\mathbf{A}'_L/\varphi_q(\mathbf{A}'_L)}.$$

We define ψ_q on $\mathbf{B}'_L$ and $\mathbf{B}^+_{\mathrm{rig},L}$ by the same formula.

Remarks.

(1) The fact that ψ_q on $\mathbf{A}'_L$ is well-defined is not obvious, but it can be proved exactly as in remark 3.2.i of Schneider–Venjakob's article [34].

(2) By definition, ψ_q is *almost* a left inverse of φ_q:

$$\psi_q \circ \varphi_q = \frac{q}{\varphi_q^{-1}(\pi_L)}.$$

The reason to normalize ψ_q in this way instead of making it an actual left inverse of φ_q is that, with this definition, the operators φ_q and ψ_q are adjoint via a certain Pontryagin duality, as Schneider and Venjakob showed in their article [34].

(3) There is the *projection formula*

$$\psi_q(\varphi_q(f)g) = f\psi_q(g) \quad \text{for all } f, g \in \mathbf{B}'_L \text{ (resp. } \mathbf{B}^+_{\mathrm{rig},L}).$$

Lemma 64. *The actions of φ_q, ψ_q and Γ_L on $\mathbf{A}'_L$ (resp. $\mathbf{B}'_L$, $\mathbf{B}^+_{\mathrm{rig},L}$) are continuous.*

Proof. See proposition 1.7.8 of Schneider's book [31] and proposition 2.4.(b) of Fourquaux–Xie's article [22] for the continuity on $\mathbf{A}'_L$ and $\mathbf{B}'_L$ (where the result is stated for the classical Lubin–Tate case, but the proofs work verbatim for the *relative* Lubin–Tate situation.)

TODO: find references for $\mathbf{B}^+_{\mathrm{rig},L}$ □

Lemma 65. *The operators φ_q, ψ_q and ∂_ϕ (on either of the rings $\mathbf{A}'_L$, $\mathbf{B}'_L$ or $\mathbf{B}^+_{\mathrm{rig},L}$) satisfy the relations*

$$\partial_\phi \circ \varphi_q = \pi_L \varphi_q \circ \partial_\phi \quad \textit{and} \quad \varphi_q \circ \psi_q \circ \partial_\phi = \partial_\phi \circ \varphi_q \circ \psi_q.$$

Proof. Write $\operatorname{dlog}_\phi(Z) = g_\phi(Z)\,dZ$. Differentiating both sides of the identity $\varphi_q(\log_\phi(Z)) = \pi_L \log_\phi(Z)$, we see that $\varphi_q(g_\phi(Z))\phi'(Z) = \pi_L g_\phi(Z)$. That is,

$$\frac{\phi'(Z)}{g_\phi(Z)} = \frac{\pi_L}{\varphi_q(g_\phi(Z))}.$$

Therefore,

$$\partial_\phi \circ \varphi_q(f) = \frac{\varphi_q(f')\phi'}{g_\phi} = \frac{\varphi_q(f')\pi_L}{\varphi_q(g_\phi)} = \pi_L \varphi_q\left(\frac{f'}{g_\phi}\right) = \pi_L \varphi_q \circ \partial_\phi(f).$$

On the other hand, one checks that

$$\mathrm{Tr}_{\mathbf{A}'_L/\varphi_q(\mathbf{A}'_L)}(f) = \sum_{v_1 \in \mathfrak{F}_{\phi,1}} f\big(\mathfrak{F}_\phi(v_1, \cdot\,)\big)$$

(see the proof of remark 3.2.ii of Schneider–Venjakob's article [34] for more details). But ∂_ϕ is invariant under $\mathfrak{F}_\phi$, which implies that

$$\partial_\phi\big(f(\mathfrak{F}_\phi(v_1, \cdot\,))\big) = \big(\partial_\phi(f)\big)\big(\mathfrak{F}_\phi(v_1, \cdot\,)\big) \quad \text{for all } v_1 \in \mathfrak{F}_{\phi,1}.$$

All in all, ∂_ϕ commutes with $\varphi_q \circ \psi_q$. □

6.2 Constructions of p–adic Hodge theory

The rings introduced in the previous subsection are very simple but have the disadvantage that the actions of φ_q and G_L seem to be defined in a very *ad hoc* way in comparison to their analogues for the usual period rings of p–adic Hodge theory. In this subsection, we give other constructions that resemble the ones introduced by Fontaine (cf. sections 2.1 to 2.4).

6.2.1 Perfect rings of characteristic p

Consider the rings

$$\widetilde{\mathbf{E}}^+ = \varprojlim_{x \mapsto x^q} \mathscr{O}_{\mathbb{C}_p} \cong \varprojlim_{x \mapsto x^q} \big(\mathscr{O}_{\mathbb{C}_p}/\pi_L \mathscr{O}_{\mathbb{C}_p}\big) \quad \text{and} \quad \widetilde{\mathbf{E}} = \varprojlim_{x \mapsto x^q} \mathbb{C}_p,$$

with the addition and multiplication laws defined as follows: for $x = \big(x^{(n)}\big)_{n \geq 0}$ and $y = \big(y^{(n)}\big)_{n \geq 0}$ in $\widetilde{\mathbf{E}}$, the elements $x + y$ and xy of $\widetilde{\mathbf{E}}$ are given by

$$(x+y)^{(n)} = \lim_{m \to \infty} \big(x^{(n+m)} + y^{(n+m)}\big)^{q^m} \quad \text{and} \quad (xy)^{(n)} = x^{(n)} y^{(n)}.$$

One can show that $\widetilde{\mathbf{E}}^+$ is a valuation ring with fraction field $\widetilde{\mathbf{E}}$ of characteristic p. We write φ_q for the q–th power Frobenius endomorphism $x \mapsto x^q$ of $\widetilde{\mathbf{E}}$.

6.2.2 Perfect rings of characteristic 0

To lift the constructions to characteristic 0, one can use rings of Witt vectors. In fact, since we allow K to have ramification over $\mathbb{Q}_p$, it is more convenient to work with *ramified* Witt vectors. (See section 1.1 of Schneider's book [31] for a systematic account of ramified Witt vectors.)

Let $F = \mathrm{W}(k)[p^{-1}]$ be the maximal absolutely unramified subfield of K. Consider the rings $\widetilde{\mathbf{A}}^+ = \mathrm{W}_K(\widetilde{\mathbf{E}}^+) = \mathrm{W}(\widetilde{\mathbf{E}}^+) \otimes_{\mathscr{O}_F} \mathscr{O}_K$ and $\widetilde{\mathbf{A}} = \mathrm{W}_K(\widetilde{\mathbf{E}}) = \mathrm{W}(\widetilde{\mathbf{E}}) \otimes_{\mathscr{O}_F} \mathscr{O}_K$ and let $\widetilde{\mathbf{B}}^+ = \widetilde{\mathbf{A}}^+[\pi_L^{-1}]$ and $\widetilde{\mathbf{B}} = \widetilde{\mathbf{A}}[\pi_L^{-1}]$. We endow $\widetilde{\mathbf{A}}$ with the weak topology, which is the product topology coming from the valuation topology on $\widetilde{\mathbf{E}}$. Alternatively, if $\widetilde{\pi}_L$ is an element of $\widetilde{\mathbf{E}}$ with $\widetilde{\pi}_L^{(0)} = \pi_L$ and $[\widetilde{\pi}_L]$ is its Teichmüller representative in $\widetilde{\mathbf{A}}$, the sets

$$\pi_L^k \widetilde{\mathbf{A}} + [\widetilde{\pi}_L]^n \widetilde{\mathbf{A}}^+ \quad \text{for } k, n \geq 0$$

form a basis of neighbourhoods of 0 for the weak topology on $\widetilde{\mathbf{A}}$. Then the weak topologies on $\widetilde{\mathbf{A}}^+$, $\widetilde{\mathbf{B}}^+$ and $\widetilde{\mathbf{B}}$ are the induced ones regarding

$$\widetilde{\mathbf{A}}^+ \subset \widetilde{\mathbf{A}}, \qquad \widetilde{\mathbf{B}} = \bigcup_{n\geq 0} \pi_L^{-n} \widetilde{\mathbf{A}}, \qquad \widetilde{\mathbf{B}}^+ \subset \widetilde{\mathbf{B}}.$$

By the functoriality of the Witt vectors constructions, the q–th power Frobenius endomorphism φ_q and the action of the Galois group G_L naturally lift to continuous actions on the rings $\widetilde{\mathbf{A}}^+$, $\widetilde{\mathbf{A}}$, $\widetilde{\mathbf{B}}^+$ and $\widetilde{\mathbf{B}}$ in characteristic 0 (cf. lemma 1.5.3 of Schneider's book [31]).

6.2.3 Imperfect rings of characteristic p

Since $\phi^{\varphi_q^{-n}}(Z) \equiv Z^q \bmod \pi_L$, reduction modulo π_L at each level yields a well-defined map

$$\begin{aligned} \iota \colon \ \mathrm{T}_\phi \mathfrak{F}_\phi &\longrightarrow \widetilde{\mathbf{E}}^+ \\ (z_n)_{n\geq 0} &\longmapsto (z_n \mod \pi_L)_{n\geq 0} \end{aligned}$$

(not a morphism in any clear way). As a matter of fact, the image of ι lies in the maximal ideal of $\widetilde{\mathbf{E}}^+$.

Fix once and for all a generator t_0 of $\mathrm{T}_\phi \mathfrak{F}_\phi$ (as an $\mathscr{O}_K$–module). We obtain an embedding of $\mathbf{E}_L'$ into $\widetilde{\mathbf{E}}$ given by $Z \mapsto \iota(t_0)$. One can prove that its image is independent of the choice of t_0. Let $\mathbf{E}_L = \mathrm{Im}\big(\mathbf{E}_L' \hookrightarrow \widetilde{\mathbf{E}}\big)$. If $\iota(t_0) = (\overline{z}_n)_{n\geq 0}$, there is an induced action of Γ_L on $\mathbf{E}_L$ defined by

$$\gamma(f(\iota(t_0))) = f(\iota(\gamma(t_0))) = f\big(\big([\chi_{\xi_K}(\gamma)]_{\phi^{\varphi_q^{-n}}}(\overline{z}_n)\big)_{n\geq 0}\big)$$

(where the power series $[\chi_{\xi_K}(\gamma)]_{\phi^{\varphi_q^{-n}}}$ is reduced modulo π_L). There is also a q–th power Frobenius morphism:

$$f(\iota(t_0))^q = f^{\varphi_q}\big(\iota(t_0)^q\big) = f^{\varphi_q}(\phi(\iota(t_0)))$$

(where the power series ϕ is reduced modulo π_L). Next, we want to lift these constructions to the rings of characteristic 0.

6.2.4 Imperfect rings of characteristic 0

Lemma 66. *There is a unique map $\{\,\cdot\,\}\colon \widetilde{\mathbf{E}}^+ \to \widetilde{\mathbf{A}}^+$ (not a morphism in any clear sense) such that, for every $x \in \widetilde{\mathbf{E}}^+$, $\{x\}$ is a lift of x with the property that $\varphi_q(\{x\}) = \phi(\{x\})$. Moreover, $\{\,\cdot\,\}$ respects the action of G_L and commutes with $[a]_\phi$ for all $a \in \mathscr{O}_K$.*

Proof. This result is the analogue of lemma 1.2 of Kisin–Ren's article [28], which in turn is based on lemma 9.3 of Colmez's article [16]. We adapt it here to the *relative* Lubin–Tate situation for the convenience of the reader.

Let $\widetilde{x}$ be an arbitrary lift of x in $\widetilde{\mathbf{A}}^+$. We want to define

$$\{x\} = \lim_{n\to\infty} (\varphi_q^{-1} \circ \phi)^n(\widetilde{x})$$

(where the exponent n denotes the composition of $\varphi_q^{-1} \circ \phi$ with itself n times). If we can prove that this limit exists and is independent of the choice of $\widetilde{x}$, it will clearly satisfy the defining properties of $\{x\}$.

Observe that the set of lifts of x is precisely $\widetilde{x} + \pi_L\widetilde{\mathbf{A}}^+$. Since $\varphi_q^{-1}(\pi_L)/\pi_L$ is a unit and $\phi(\pi_L^k\widetilde{\mathbf{A}}^+) \subset \pi_L^{k+1}\widetilde{\mathbf{A}}^+$ (as $\phi(Z) \equiv Z^q \bmod \pi_L$), the map $\varphi_q^{-1} \circ \phi$ is contractive on $\widetilde{x} + \pi_L\widetilde{\mathbf{A}}^+$. But $\widetilde{x} + \pi_L\widetilde{\mathbf{A}}^+$ is complete with respect to the π_L–adic topology. Therefore, there is a unique fixed point that must be $\{x\}$.

Next let $\sigma \in G_L$. We can write

$$\sigma \circ (\varphi_q^{-1} \circ \phi)^n(\widetilde{x}) = \phi^{\varphi_q^{-1}} \circ \phi^{\varphi_q^{-2}} \circ \cdots \circ \phi^{\varphi_q^{-n}}(\sigma\varphi_q^{-n}(\widetilde{x})).$$

Since $\sigma \circ \varphi_q^{-1} = \varphi_q^{-1} \circ \sigma$ on $\widetilde{\mathbf{A}}^+$, we deduce that $\sigma(\{x\})$ is a lift of $\sigma(x)$ and that $\varphi_q\big(\sigma(\{x\})\big) = \phi\big(\sigma(\{x\})\big)$. Therefore, $\sigma(\{x\}) = \{\sigma(x)\}$.

Finally, for $a \in \mathscr{O}_K$, we can write

$$\begin{aligned}[a]_\phi \circ (\varphi_q^{-1} \circ \phi)^n(\widetilde{x}) &= \phi^{\varphi_q^{-1}} \circ \phi^{\varphi_q^{-2}} \circ \cdots \circ \phi^{\varphi_q^{-n}} \big([a]_\phi^{\varphi_q^{-n}} \circ \varphi_q^{-n}(\widetilde{x})\big) \\ &= \phi^{\varphi_q^{-1}} \circ \phi^{\varphi_q^{-2}} \circ \cdots \circ \phi^{\varphi_q^{-n}} \big(\varphi_q^{-n} \circ [a]_\phi(\widetilde{x})\big)\end{aligned}$$

and we conclude that $[a]_\phi(\{x\}) = \{[a]_\phi(x)\}$ by the same argument. □

Proposition 67. *The map*

$$\begin{aligned}\iota_\phi \colon \mathrm{T}_\phi \mathfrak{F}_\phi &\longrightarrow \widetilde{\mathbf{A}}^+ \\ t &\longmapsto \{\iota(t)\}\end{aligned}$$

satisfies the following properties:

(1) $[a]_\phi(\iota_\phi(t)) = \iota_\phi(a \cdot t)$ *for every* $a \in \mathscr{O}_K$*;*
(2) $\varphi_q(\iota_\phi(t)) = \phi(\iota_\phi(t))$*, and*
(3) $\sigma(\iota_\phi(t)) = \iota_\phi(\sigma(t)) = \iota_\phi\big([\chi_\phi(\sigma)]_\phi(t)\big) = [\chi_\phi(\sigma)]_\phi(\iota_\phi(t))$ *for every* $\sigma \in G_L$*.*

Proof. These properties follow immediately from lemma 66 and the corresponding properties of ι. □

Recall that we fixed a generator t_0 of $\mathrm{T}_\phi \mathfrak{F}_\phi$. Define $\omega_\phi = \iota_\phi(t_0) \in \widetilde{\mathbf{A}}^+$. By analogy with the situation in characteristic p, we can define an embedding of $\mathscr{O}_L$–algebras $\mathbf{A}_L'^{,+} \hookrightarrow \widetilde{\mathbf{A}}^+$ by $Z \mapsto \omega_\phi$. We observe that ω_ϕ is a unit in the local ring $\widetilde{\mathbf{A}}$, as its reduction in $\widetilde{\mathbf{E}}$ is $\iota(t_0) \neq 0$. Thus, and by continuity with respect to the π_L–adic topologies, the embedding can be extended to

$$\mathbf{A}_L' \hookrightarrow \widetilde{\mathbf{A}} \quad \text{and} \quad \mathbf{B}_L' \hookrightarrow \widetilde{\mathbf{B}}.$$

Proposition 67 implies that these maps are compatible with the Frobenius operators φ_q and the actions of G_L. One can prove that the images of these embeddings are independent of the choice of t_0 (cf. remark 2.1.17 of Schneider's book [31]). Let $\mathbf{A}_L^+ = \mathrm{Im}\big(\mathbf{A}_L'^{,+} \hookrightarrow \widetilde{\mathbf{A}}^+\big)$, $\mathbf{A}_L = \mathrm{Im}\big(\mathbf{A}_L' \hookrightarrow \widetilde{\mathbf{A}}\big)$ and $\mathbf{B}_L = \mathrm{Im}\big(\mathbf{B}_L' \hookrightarrow \widetilde{\mathbf{B}}\big)$. The main result that we are going to use is the following:

Proposition 68. *The morphism of* $\mathscr{O}_L$*–algebras* $\mathbf{A}_L' \to \mathbf{A}_L$ *defined by* $Z \mapsto \omega_\phi$ *is an isomorphism of topological* $\mathscr{O}_L$*–algebras (with respect to the weak topologies) compatible with the continuous actions of the Frobenius operator* φ_q *and the Galois group* Γ_L*.*

Proof. Only the fact that this map is a homeomorphism with respect to the weak topologies is unclear, but that is the content of proposition 2.1.16.(i) of Schneider's book [31]. □

This result allows us to work more concretely in terms of formal series and then translate the constructions to work over $\mathbf{A}_L$, as is done in the article [34] of Schneider and Venjakob.

6.2.5 Larger rings of periods

Let $\mathbf{E}$ denote the separable closure of $\mathbf{E}_L$ inside $\widetilde{\mathbf{E}}$. Let $\mathbf{B}$ be the p–adic completion of the maximal unramified extension of $\mathbf{B}_L$ inside $\widetilde{\mathbf{B}}$ and put $\mathbf{A} = \mathbf{B} \cap \widetilde{\mathbf{A}}$ and $\mathbf{A}^+ = \mathbf{B} \cap \widetilde{\mathbf{A}}^+$. One can prove that $\mathbf{A}$ is a complete discrete valuation ring with field of fractions $\mathbf{B}$ and residue field $\mathbf{E}$. Moreover, the theory of fields of norms of Fontaine and Wintenberger gives natural isomorphisms

$$\mathrm{Gal}(\mathbf{B}/\mathbf{B}_L) \cong \mathrm{Gal}(\mathbf{A}/\mathbf{A}_L) \cong \mathrm{Gal}(\mathbf{E}/\mathbf{E}_L) \cong H_L,$$

where for the characteristic 0 rings $\mathrm{Gal}(\,\cdot\,/\,\cdot\,)$ means continuous automorphisms (cf. lemma 1.4 of Kisin–Ren's article [28]).

Lemma 69. *The sequence*

$$0 \longrightarrow \mathscr{O}_K \longrightarrow \mathbf{A} \xrightarrow{\varphi_q - 1} \mathbf{A} \longrightarrow 0$$

is exact.

Proof. This is analogous to remark 5.1 of Schneider–Venjakob's article [34]. We recall the main idea of the proof here.

The sequence of the statement can be expressed as the projective limit of the sequences

$$0 \longrightarrow \mathscr{O}_K/\pi_K^n\mathscr{O}_K \longrightarrow \mathbf{A}/\pi_K^n\mathbf{A} \xrightarrow{\varphi_q - 1} \mathbf{A}/\pi_K^n\mathbf{A} \longrightarrow 0$$

for $n \geq 1$, so it suffices to prove that each of those is exact (as the first terms satisfy the Mittag–Leffler condition). An induction argument on n reduces the assertion to the case of $n = 1$, but the exactness of

$$0 \longrightarrow \mathbb{F}_q \longrightarrow \mathbf{E} \xrightarrow{\varphi_q - 1} \mathbf{E} \longrightarrow 0$$

is clear. □

Recall that $\mathbf{A}_{\mathrm{crys}}$ is the p–adic completion of a divided power envelope of $\mathrm{W}(\widetilde{\mathbf{E}}^+)$. Consider $\mathbf{A}_{\mathrm{crys},K} = \mathbf{A}_{\mathrm{crys}} \otimes_F K$. We have embeddings $\mathbf{A}_L^+ \subseteq \widetilde{\mathbf{A}}^+ \subseteq \mathbf{A}_{\mathrm{crys},K}$.

Recall also that $\mathbf{B}_{\mathrm{crys}}$ is constructed from $\mathbf{A}_{\mathrm{crys}}$ by inverting a period that is usually called t ("a p–adic analogue of $2\pi i$"). Consider $\mathbf{B}_{\mathrm{crys},K} = \mathbf{B}_{\mathrm{crys}} \otimes_F K$. By (the analogues of) proposition 9.10 and lemma 9.17 of Colmez's article [16], we obtain a period $t_\phi = \log_\phi(\omega_\phi) \in \mathbf{B}_{\mathrm{crys},K}^\times$ that can play the role of t in the (relative) Lubin–Tate situation. Since t_ϕ is a unit, we see that ω_ϕ is also a unit in $\mathbf{B}_{\mathrm{crys},K}$.

It turns out that the inclusion $\mathbf{A}_L'^{,+} \hookrightarrow \mathbf{A}_{\mathrm{crys},K}$ given by $Z \mapsto \omega_\phi$ extends to a continuous ring homomorphism $\mathbf{B}_{\mathrm{rig},L}^+ \to \mathbf{A}_{\mathrm{crys},K}[\pi_L^{-1}]$, where we consider the natural Fréchet topology on $\mathbf{B}_{\mathrm{rig},L}^+$ and the π_L–adic topology on $\mathbf{A}_{\mathrm{crys},K}[\pi_L^{-1}]$ (cf. lemma 1.4 of Schneider–Venjakob's preprint [35]).

6.3 Rings of functions on annuli

Recall that we defined $\mathbf{B}_{\mathrm{rig},L}^+$ to be the ring of global (rigid analytic) functions on $\mathcal{B}$, the p–adic unit disc over L centred at the origin. We can obtain other rings if we consider annuli inside $\mathcal{B}$.

Consider $r, s \in p^{\mathbb{Q}}$ with $r \le s < 1$ (resp. $r < s \le 1$). The closed disc $\mathcal{B}[r]$ of radius r is the affinoid subdomain of $\mathcal{B}$ defined by the inequality $|Z|_p \le r$. The closed annulus $\mathcal{B}[r,s]$ (resp. the half-open annulus $\mathcal{B}[r,s)$) is the affinoid subdomain (resp. admissible open) of $\mathcal{B}$ defined by the inequalities $r \le |Z|_p \le s$ (resp. $r \le |Z|_p < s$). Observe that

$$\mathcal{B}[r,s) = \bigcup_{r<s'<s} \mathcal{B}[r,s']$$

and so we may view $\mathcal{B}[r,s)$ as a quasi-Stein rigid space.

Given a rigid analytic space $\mathcal{Y}$ over L and a complete extension L' of L, we write $\mathscr{O}(\mathcal{Y}/L')$ for the ring of global (rigid analytic) functions on the base change $\mathcal{Y}_{L'}$ and $\mathscr{O}^{\mathrm{bd}}(\mathcal{Y}/L')$ for the subring of $\mathscr{O}(\mathcal{Y}/L')$ consisting of those functions that are bounded.

For every closed interval $[r,s] \subset (0,1)$ as above, we define the Banach algebras $\mathbf{B}_{\mathrm{rig},L}^{\dagger,[r,s]} = \mathscr{O}(\mathcal{B}[r,s]/L)$ and $\mathbf{B}_L^{\dagger,[r,s]} = \mathscr{O}^{\mathrm{bd}}(\mathcal{B}[r,s]/L)$. Then, for every half-open interval $[r,1) \subset (0,1)$ as above, we define the Fréchet algebras

$$\mathbf{B}_{\mathrm{rig},L}^{\dagger,[r,1)} = \mathscr{O}(\mathcal{B}[r,1)/L) = \varprojlim_{r<s<1} \mathbf{B}_{\mathrm{rig},L}^{\dagger,[r,s]} \quad \text{and} \quad \mathbf{B}_L^{\dagger,[r,1)} = \mathscr{O}(\mathcal{B}[r,1)/L) = \varprojlim_{r<s<1} \mathbf{B}_L^{\dagger,[r,s]}.$$

Finally, we define the LF algebras

$$\mathbf{B}_{\mathrm{rig},L}^{\dagger} = \varinjlim_{r<1} \mathbf{B}_{\mathrm{rig},L}^{\dagger,[r,1)} \quad \text{and} \quad \mathbf{B}_L^{\dagger} = \varinjlim_{r<1} \mathbf{B}_L^{\dagger,[r,1)}.$$

The ring $\mathbf{B}^{\dagger}_{\mathrm{rig},L}$ is called *the Robba ring of L*, while $\mathbf{B}^{\dagger}_L$ is the *subring of overconvergent elements of* $\mathbf{B}'_L$. One can check that $\mathbf{B}^{\dagger}_{\mathrm{rig},L}$ is a Bézout domain and that $\mathbf{B}^{\dagger}_L$ is a field whose subring of functions that are bounded by 1 is local (even henselian), thus giving rise to another topology on $\mathbf{B}^{\dagger}_L$. The completion of $\mathbf{B}^{\dagger}_L$ with respect to that topology coincides with $\mathbf{B}'_L$.

In more concrete terms, we can write

$$\mathbf{B}^{\dagger}_L = \Big\{ \sum_{k\in\mathbb{Z}} a_k Z^k \in L[\![Z, Z^{-1}]\!] : \sup_{k\in\mathbb{Z}} |a_k|_p < \infty \text{ and } \lim_{k\to-\infty} |a_k|_p r^k = 0 \text{ for some } r \in (0,1) \Big\}$$

and

$$\mathbf{B}^{\dagger}_{\mathrm{rig},L} = \Big\{ \sum_{k\in\mathbb{Z}} a_k Z^k \in L[\![Z, Z^{-1}]\!] : \lim_{k\to-\infty} |a_k|_p r^k = 0 \text{ for some } r \in (0,1) \Big\}.$$

6.3.1 Operators

We can extend the operators φ_q and $\gamma \in \Gamma_L$ from $\mathbf{A}'^{,+}_L$ to $\mathbf{B}^{\dagger}_L$ and $\mathbf{B}^{\dagger}_{\mathrm{rig},L}$ by continuity, as was done for $\mathbf{B}'_L$ in section 6.1.1.

More geometrically, one checks as in lemma 2.6 of Fourquaux–Xie's article [22] that, if $r, s \in p^{\mathbb{Q}}$ satisfy that $p^{-1/e(q-1)} < r \le s < 1$, then every $\gamma \in \Gamma_L$ defines a bijective morphism $\mathcal{B}[r,s] \to \mathcal{B}[r,s]$ given on points by $z \mapsto [\chi_\phi(\gamma)]_\phi(z)$ and that φ_q defines a surjective morphism $\mathcal{B}[r,s] \to L \otimes_{\varphi_q, L} \mathcal{B}[r^q, s^q]$ given on points by $z \mapsto \phi(z)$. Furthermore, the induced morphisms $\gamma\colon \mathbf{B}^{\dagger,[r,s]}_{\mathrm{rig},L} \to \mathbf{B}^{\dagger,[r,s]}_{\mathrm{rig},L}$ and $\varphi_q\colon \mathbf{B}^{\dagger,[r^q,s^q]}_{\mathrm{rig},L} \to \mathbf{B}^{\dagger,[r,s]}_{\mathrm{rig},L}$ are isometries with respect to the supremum norms.

To extend ψ_q, observe that each $v_1 \in \mathfrak{F}_{\phi,1}$ defines an isomorphism of affinoids $\mathcal{B}[r,s] \to \mathcal{B}[r,s]$ given on points by $z \mapsto \mathfrak{F}_\phi(v_1, z)$. We define $\mathrm{Tr}_{\varphi_q}\colon \mathbf{B}^{\dagger,[r,s]}_{\mathrm{rig},L} \to \mathbf{B}^{\dagger,[r,s]}_{\mathrm{rig},L}$ by

$$\mathrm{Tr}_{\varphi_q}(f) = \sum_{v_1\in\mathfrak{F}_{\phi,1}} f\big(\mathfrak{F}_\phi(v_1, \cdot\,)\big)$$

(cf. the proof of lemma 65) and claim that its image is contained in the image of φ_q. Indeed, by continuity it suffices to prove it for $\mathrm{Tr}_{\varphi_q}(Z^n)$ for $n \in \mathbb{Z}$. If $n \ge 0$, this is a consequence of the analogous statement for $\mathbf{A}'^{,+}_L$ (see the proof of remark 3.2.ii of Schneider–Venjakob's article [34]). If $n < 0$, we adapt the calculation of page 37 of Schneider–Venjakob's preprint [35] using the previous case:

$$\varphi_q\Big(Z^n \pi_L \psi_q\Big(\frac{\phi(Z)^{-n}}{Z^{-n}}\Big)\Big) = \varphi_q(Z^n)\,\mathrm{Tr}_{\varphi_q}\Big(\frac{\phi(Z)^{-n}}{Z^{-n}}\Big) = \phi(Z)^n \sum_{v_1\in\mathfrak{F}_{\phi,1}} \frac{\phi(\mathfrak{F}_\phi(v_1,Z))^{-n}}{\mathfrak{F}_\phi(v_1,Z)^{-n}}$$

$$= \phi(Z)^n \sum_{v_1 \in \mathfrak{F}_{\phi,1}} \frac{\phi(Z)^{-n}}{\mathfrak{F}_\phi(v_1, Z)^{-n}} = \sum_{v_1 \in \mathfrak{F}_{\phi,1}} \mathfrak{F}_\phi(v_1, Z)^n = \mathrm{Tr}_{\varphi_q}(Z^n).$$

All in all, we can define

$$\psi_q = \varphi_q^{-1}\Big(\frac{1}{\pi_L}\,\mathrm{Tr}_{\varphi_q}(\,\cdot\,)\Big),$$

first as an operator $\psi_q\colon \mathbf{B}^{\dagger,[r,s]}_{\mathrm{rig},L} \to \mathbf{B}^{\dagger,[r^q,s^q]}_{\mathrm{rig},L}$ and then by taking projective and inductive limits as an operator $\psi_q\colon \mathbf{B}^{\dagger}_{\mathrm{rig},L} \to \mathbf{B}^{\dagger}_{\mathrm{rig},L}$.

7 (φ_q, Γ_L)–modules

In this section we recall the definitions and results from Kisin and Ren's article [28] but adapted to our situation in which the Lubin–Tate formal group is *relative*. We provide the proofs whenever they are not the same as in the original references.

7.1 Modules over $\mathbf{A}_L$ or $\mathbf{B}_L$

Definition 70.

(1) An *étale φ_q–module over* $\mathbf{A}_L$ is a finitely generated $\mathbf{A}_L$–module M endowed with a φ_q–semilinear morphism $\varphi_q = \varphi_M \colon M \to M$ whose $\mathbf{A}_L$–linearization $\varphi_q^*(M) = \mathbf{A}_L \otimes_{\varphi_q, \mathbf{A}_L} M \to M$ is an isomorphism.

(2) An *étale φ_q–module over* $\mathbf{B}_L$ is a finitely generated $\mathbf{B}_L$–module M endowed with a φ_q–semilinear morphism $\varphi_q = \varphi_M \colon M \to M$ admitting a φ_q–stable $\mathbf{A}_L$–lattice N that is an étale φ_q–module over $\mathbf{A}_L$ with respect to $\varphi_N = \varphi_M|_N$.

We write $\varphi_q\text{–}\mathrm{Mod}^{\text{ét}}_{\mathbf{A}_L}$ (resp. $\varphi_q\text{–}\mathrm{Mod}^{\text{ét}}_{\mathbf{B}_L}$) for the category of étale φ_q–modules over $\mathbf{A}_L$ (resp. over $\mathbf{B}_L$).

Remark. The Frobenius endomorphism φ_q of an étale φ_q–module is automatically continuous (see remark 3.8 of Schneider–Venjakob's article [34]).

Definition 71. Let M be an étale φ_q–module (over $\mathbf{A}_L$ or $\mathbf{B}_L$). We define the endomorphism

$$\begin{array}{rcccl} \psi_q = \psi_M \colon M & \xleftarrow{\cong} & \varphi_q^*(M) & \longrightarrow & M \\ f\varphi_q(m) & \longleftarrow & f \otimes m & \longmapsto & \psi_q(f)m \end{array}$$

characterized by

$$\psi_q \circ \varphi_q = \frac{q}{\varphi_q^{-1}(\pi_L)} \, \mathrm{id}_M \, .$$

Remark. The endomorphism ψ_q is automatically continuous (see remark 3.8 of Schneider–Venjakob's article [34]) and we have the projection formulae

$$\psi_q(f\varphi_q(m)) = \psi_q(f)m \quad \text{and} \quad \psi_q(\varphi_q(f)m) = f\psi_q(m)$$

by construction.

Definition 72. An *étale (φ_q, Γ_L)–module over* $\mathbf{A}_L$ *or* $\mathbf{B}_L$ is an étale φ_q–module with an $\mathbf{A}_L$– or $\mathbf{B}_L$–semilinear continuous action of Γ_L commuting with φ_q. We write $(\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{\text{ét}}_{\mathbf{A}_L}$ (resp. $(\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{\text{ét}}_{\mathbf{B}_L}$) for the category of étale (φ_q, Γ_L)–modules over $\mathbf{A}_L$ (resp. over $\mathbf{B}_L$).

7.1.1 Equivalence with representations

Let $\mathrm{Rep}_{\mathscr{O}_K}(G_L)$ (resp. $\mathrm{Rep}_K(G_L)$) denote the category of finite $\mathscr{O}_K$–modules (resp. K–vector spaces) endowed with a continuous $\mathscr{O}_K$–linear (resp. K–linear) action of the Galois group G_L. Let $\mathrm{Rep}_{\mathscr{O}_K,\mathrm{fr}}(G_L)$ (resp. $\mathrm{Rep}_{\mathscr{O}_K,\mathrm{tor}}(G_L)$) denote the full subcategory of objects of $\mathrm{Rep}_{\mathscr{O}_K}(G_L)$ that are free (resp. killed by π_K^n for some uniformizer π_K of K and some $n \geq 1$) as $\mathscr{O}_K$–modules.

Definition 73.

(1) The (φ_q, Γ_L)–module (over $\mathbf{A}_L$) associated with $T \in \mathrm{Ob}(\mathrm{Rep}_{\mathscr{O}_K}(G_L))$ is

$$\mathbf{D}(T) = (\mathbf{A} \otimes_{\mathscr{O}_K} T)^{H_L}.$$

(2) The (φ_q, Γ_L)–module (over $\mathbf{B}_L$) associated with $V \in \mathrm{Ob}(\mathrm{Rep}_K(G_L))$ is

$$\mathbf{D}(V) = (\mathbf{B} \otimes_K V)^{H_L}.$$

(3) The ($\mathscr{O}_K$–linear) representation associated with $M \in \mathrm{Ob}\big((\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{\text{ét}}_{\mathbf{A}_L}\big)$ is

$$\mathbf{V}(M) = (\mathbf{A} \otimes_{\mathbf{A}_L} M)^{\varphi_q = 1}.$$

(4) The (K–linear) representation associated with $M \in \mathrm{Ob}\big((\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{\text{ét}}_{\mathbf{B}_L}\big)$ is

$$\mathbf{V}(M) = (\mathbf{B} \otimes_{\mathbf{B}_L} M)^{\varphi_q = 1}.$$

Theorem 74 (Kisin–Ren). *The functors*

$$\mathrm{Rep}_{\mathscr{O}_K}(G_L) \underset{\mathbf{V}}{\overset{\mathbf{D}}{\rightleftarrows}} (\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{\text{ét}}_{\mathbf{A}_L}$$

(resp.

$$\mathrm{Rep}_K(G_L) \underset{\mathbf{V}}{\overset{\mathbf{D}}{\rightleftarrows}} (\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{\text{ét}}_{\mathbf{B}_L})$$

are exact quasi-inverse equivalences of categories that are compatible with tensor products and duality.

Proof. For the non-relative Lubin–Tate case, this is theorem 1.6 of Kisin–Ren's article [28], which in turn uses the same arguments of sections A1.2 and A3.4 of Fontaine's article [21] for the cyclotomic case. (Alternatively, section 3 of Brinon–Conrad's notes [11] contains all the details of the proof once we know that $\mathrm{Gal}(\mathbf{E}/\mathbf{E}_L) \cong H_L$.) Exactly the same arguments work in the *relative* Lubin–Tate situation too. Here we just summarize the general strategy.

The proof of this theorem can be seen as a series of reductions to simpler cases.

- The statement for $\mathrm{Rep}_K(G_L)$ and (φ_q, Γ_L)–$\mathrm{Mod}^{\text{ét}}_{\mathbf{B}_L}$ can be reduced to the statement for $\mathrm{Rep}_{\mathscr{O}_K}(G_L)$ and (φ_q, Γ_L)–$\mathrm{Mod}^{\text{ét}}_{\mathbf{A}_L}$ by choosing $\mathbb{Z}_p$–lattices and $\mathbf{A}_L$–lattices.
- The objects of $\mathrm{Rep}_{\mathscr{O}_K}(G_L)$ (resp. of (φ_q, Γ_L)–$\mathrm{Mod}^{\text{ét}}_{\mathbf{A}_L}$) that are finite free as $\mathscr{O}_K$–modules (resp. as $\mathbf{A}_L$–modules) can be written as the projective limit of their quotients by powers of π_K. Therefore, one can reduce to the torsion case (i.e., to objects that are killed by some power of π_K).
- The case of objects of $\mathrm{Rep}_{\mathscr{O}_K}(G_L)$ and (φ_q, Γ_L)–$\mathrm{Mod}^{\text{ét}}_{\mathbf{A}_L}$ that are killed by π_K^n for some $n \in \mathbb{Z}_{\geq 1}$ can be reduced to the case by an induction argument on n in which objects are killed by π_K.
- In the end the proof boils down to showing that (in the killed-by-π_K case) the maps

$$\mathbf{E} \otimes_{\mathbf{E}_L} \mathbf{D}(T) \to \mathbf{E} \otimes_{\mathbb{F}_q} T \quad \text{and} \quad \mathbf{E} \otimes_{\mathbb{F}_q} \mathbf{V}(M) \to \mathbf{E} \otimes_{\mathbf{E}_L} M$$

  induced by multiplication in $\mathbf{E}$ are isomorphisms. Then one can deduce that, in the most general cases of the statement of the theorem, the analogous maps

$$\mathbf{A} \otimes_{\mathbf{A}_L} \mathbf{D}(T) \to \mathbf{A} \otimes_{\mathscr{O}_K} T \quad \text{and} \quad \mathbf{A} \otimes_{\mathscr{O}_K} \mathbf{V}(M) \to \mathbf{A} \otimes_{\mathbf{A}_L} M$$

  (resp.

$$\mathbf{B} \otimes_{\mathbf{B}_L} \mathbf{D}(V) \to \mathbf{B} \otimes_K V \quad \text{and} \quad \mathbf{B} \otimes_K \mathbf{V}(M) \to \mathbf{B} \otimes_{\mathbf{B}_L} M)$$

  are isomorphisms. □

7.1.2 Overconvergent representations

Definition 75. An *étale φ_q–module over $\mathbf{B}_L^\dagger$* is a finite $\mathbf{B}_L^\dagger$–vector space M endowed with a φ_q–semilinear morphism $\varphi_q = \varphi_M \colon M \to M$ whose matrix (in some basis) is invertible. We write φ_q–$\mathrm{Mod}^{\text{ét}}_{\mathbf{B}_L^\dagger}$ for the category of étale φ_q–modules over $\mathbf{B}_L^\dagger$.

Definition 76. An *étale (φ_q, Γ_L)–module over $\mathbf{B}_L^\dagger$* is an étale φ_q–module over $\mathbf{B}_L^\dagger$ with a $\mathbf{B}_L^\dagger$–semilinear continuous action of Γ_L that commutes with φ_q. We write (φ_q, Γ_L)–$\mathrm{Mod}^{\text{ét}}_{\mathbf{B}_L^\dagger}$ for the category of étale (φ_q, Γ_L)–modules over $\mathbf{B}_L^\dagger$.

Base change from $\mathbf{B}_L^\dagger$ to $\mathbf{B}_L$ (via the natural inclusion $\mathbf{B}_L^\dagger \hookrightarrow \mathbf{B}_L'$ and the isomorphism $\mathbf{B}_L' \cong \mathbf{B}_L$ given by $Z \mapsto \omega_\phi$; cf. proposition 68) induces a functor

(φ_q, Γ_L)–$\mathrm{Mod}^{\text{ét}}_{\mathbf{B}_L^\dagger} \to (\varphi_q, \Gamma_L)$–$\mathrm{Mod}^{\text{ét}}_{\mathbf{B}_L}$. Its essential image can be related to other rings introduced in section 6.

Definition 77.

(1) An étale (φ_q, Γ_L)–module M is called *overconvergent* if it admits a $\mathbf{B}_L$–basis in terms of which the matrices of φ_q and of every $\gamma \in \Gamma_L$ have coefficients in (the image of) $\mathbf{B}_L^\dagger$. Such a basis generates $M^\dagger \in \mathrm{Ob}\big((\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{\text{ét}}_{\mathbf{B}_L^\dagger}\big)$ with the property that $\mathbf{B}_L \otimes_{\mathbf{B}_L^\dagger} M^\dagger \cong M$.

(2) A representation $V \in \mathrm{Rep}_K(G_L)$ is called *overconvergent* if its associated (φ_q, Γ_L)–module $\mathbf{D}(V)$ is overconvergent. In that case, we write $\mathbf{D}^\dagger(V)$ for the corresponding module in (φ_q, Γ_L)–$\mathrm{Mod}^{\text{ét}}_{\mathbf{B}_L^\dagger}$.

We write (φ_q, Γ_L)–$\mathrm{Mod}^{\dagger,\text{ét}}_{\mathbf{B}_L}$ for the full subcategory of overconvergent modules in (φ_q, Γ_L)–$\mathrm{Mod}^{\text{ét}}_{B_L}$ and $\mathrm{Rep}^\dagger_K(G_L)$ for the full subcategory of overconvergent representations in $\mathrm{Rep}_K(G_L)$.

7.2 Modules over $\mathbf{B}^\dagger_{\mathrm{rig},L}$

Definition 78. A *free φ_q–module over* $\mathbf{B}^\dagger_{\mathrm{rig},L}$ is a free $\mathbf{B}^\dagger_{\mathrm{rig},L}$–module $\mathscr{M}$ of finite rank endowed with a φ_q–semilinear endomorphism $\varphi_q = \varphi_{\mathscr{M}} \colon \mathscr{M} \to \mathscr{M}$ such that the $\mathbf{B}^\dagger_{\mathrm{rig},L}$–linearization $1 \otimes \varphi_q \colon \varphi_q^*(\mathscr{M}) \to \mathscr{M}$ is an isomorphism. We write φ_q–$\mathrm{Mod}_{\mathbf{B}^\dagger_{\mathrm{rig},L},\mathrm{fr}}$ for the category of such modules.

Definition 79. A *free (φ_q, Γ_L)–module over* $\mathbf{B}^\dagger_{\mathrm{rig},L}$ is a free φ_q–module over $\mathbf{B}^\dagger_{\mathrm{rig},L}$ endowed with a $\mathbf{B}^\dagger_{\mathrm{rig},L}$–semilinear continuous action of Γ_L commuting with φ_q. We write (φ_q, Γ_L)–$\mathrm{Mod}_{\mathbf{B}^\dagger_{\mathrm{rig},L},\mathrm{fr}}$ for the category of such modules.

Proposition 80. *For each $\mathscr{M} \in \mathrm{Ob}\big((\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}_{\mathbf{B}^\dagger_{\mathrm{rig},L},\mathrm{fr}}\big)$, there exist a radius $r_0 \in p^{\mathbb{Q}}$ such that $p^{-1/(q-1)e} < r_0 < 1$ and a finite free $\mathbf{B}^{\dagger,[r_0,1)}_{\mathrm{rig},L}$–module $\mathscr{M}_0$ endowed with*

(i) a φ_q–semilinear continuous morphism

$$\varphi_q \colon \mathscr{M}_0 \to \mathbf{B}^{\dagger,[r_0^{1/q},1)}_{\mathrm{rig},L} \otimes_{\mathbf{B}^{\dagger,[r_0,1)}_{\mathrm{rig},L}} \mathscr{M}_0$$

such that the $\mathbf{B}^{\dagger,[r_0^{1/q},1)}_{\mathrm{rig},L}$–linearization

$$1 \otimes \varphi_q \colon \mathbf{B}^{\dagger,[r_0^{1/q},1)}_{\mathrm{rig},L} \otimes_{\varphi_q, \mathbf{B}^{\dagger,[r_0,1)}_{\mathrm{rig},L}} \mathscr{M}_0 \to \mathbf{B}^{\dagger,[r_0^{1/q},1)}_{\mathrm{rig},L} \otimes_{\mathbf{B}^{\dagger,[r_0,1)}_{\mathrm{rig},L}} \mathscr{M}_0$$

is an isomorphism and

(ii) a $\mathbf{B}^{\dagger,[r_0,1)}_{\mathrm{rig},L}$–semilinear continuous action of Γ_L commuting with φ_q with the property that

$$\mathscr{M} = \mathbf{B}^{\dagger}_{\mathrm{rig},L} \otimes_{\mathbf{B}^{\dagger,[r_0,1)}_{\mathrm{rig},L}} \mathscr{M}_0$$

compatibly with the actions of φ_q and Γ_L.

Proof. See proposition 2.24 of Berger–Schneider–Xie's article [8] for a proof of a more general result. □

In the notation of proposition 80, we may view $\mathscr{M}_0$ as the global sections of a coherent sheaf on the annulus $\mathcal{B}[r_0, 1)$. What is more, $\mathcal{B}[r_0, 1)$ is a quasi-Stein space and so its coherent sheaves are uniquely determined by their global sections. Given $r, s \in p^{\mathbb{Q}}$ with $r_0 \leq r < s < 1$, we write

$$\mathscr{M}|_{\mathcal{B}[r,s]} = \mathbf{B}^{\dagger,[r,s]}_{\mathrm{rig},L} \otimes_{\mathbf{B}^{\dagger,[r_0,1)}_{\mathrm{rig},L}} \mathscr{M}_0 \quad \text{and} \quad \mathscr{M}|_{\mathcal{B}[r,1)} = \mathbf{B}^{\dagger,[r,1)}_{\mathrm{rig},L} \otimes_{\mathbf{B}^{\dagger,[r_0,1)}_{\mathrm{rig},L}} \mathscr{M}_0.$$

In particular,

$$\mathscr{M}|_{\mathcal{B}[r,1)} = \varprojlim_{r<s'<1} \mathscr{M}|_{\mathcal{B}[r,s']} \quad \text{and} \quad \mathscr{M} = \varinjlim_{r'<1} \mathscr{M}|_{\mathcal{B}[r',1)}.$$

7.2.1 Slope filtrations

In this subsection we briefly recall the theory of slope filtrations on φ_q–modules over the Robba ring. The general theory is explained in sections 1.4 to 1.7 of Kedlaya's article [24].

Definition 81. Consider $\mathscr{M} \in \mathrm{Ob}\big(\varphi_q\text{–}\mathrm{Mod}_{\mathbf{B}^{\dagger}_{\mathrm{rig},L},\mathrm{fr}}\big)$ of rank m over $\mathbf{B}^{\dagger}_{\mathrm{rig},L}$.

(1) Choose a basis element

$$x \in \bigwedge^m \mathscr{M}$$

and consider $\alpha \in \big(\mathbf{B}^{\dagger}_{\mathrm{rig},L}\big)^{\times}$ such that $\varphi_q(x) = \alpha x$. The *degree of $\mathscr{M}$* is

$$\deg(\mathscr{M}) = \mathrm{v}_{\mathbf{B}^{\dagger}_L}(\alpha),$$

where $\mathrm{v}_{\mathbf{B}^{\dagger}_L}$ is the normalized discrete valuation on $\mathbf{B}^{\dagger}_L$ given by the subring of functions that are bounded by 1.

(2) If $\mathscr{M}$ is non-zero, we define the *slope of $\mathscr{M}$* to be

$$\mu(\mathscr{M}) = \frac{\deg(\mathscr{M})}{m}.$$

(3) Let $s \in \mathbb{Q}$. We say that $\mathscr{M}$ is *pure of slope* s if $\mu(\mathscr{M}) = s$ and $\mu(\mathscr{N}) \geq s$ for all non-trivial subobjects $\mathscr{N}$ of $\mathscr{M}$ in $\varphi_q\text{–Mod}_{\mathbf{B}^\dagger_{\mathrm{rig},L},\mathrm{fr}}$.

We write $\varphi_q\text{–Mod}^s_{\mathbf{B}^\dagger_{\mathrm{rig},L},\mathrm{fr}}$ (resp. $(\varphi_q,\Gamma_L)\text{–Mod}^s_{\mathbf{B}^\dagger_{\mathrm{rig},L},\mathrm{fr}}$) for the full subcategory of modules that are pure of slope s in $\varphi_q\text{–Mod}_{\mathbf{B}^\dagger_{\mathrm{rig},L},\mathrm{fr}}$ (resp. in $(\varphi_q,\Gamma_L)\text{–Mod}_{\mathbf{B}^\dagger_{\mathrm{rig},L},\mathrm{fr}}$).

Theorem 82 (Kedlaya). *Every* $\mathscr{M} \in \mathrm{Ob}\big(\varphi_q\text{–Mod}_{\mathbf{B}^\dagger_{\mathrm{rig},L},\mathrm{fr}}\big)$ *admits a unique filtration*

$$0 = \mathscr{M}_0 \subset \mathscr{M}_1 \subset \cdots \subset \mathscr{M}_r = \mathscr{M}$$

by saturated φ_q*–submodules such that the successive quotients are pure of slopes*

$$\mu(\mathscr{M}_1/\mathscr{M}_0) < \cdots < \mu(\mathscr{M}_r/\mathscr{M}_{r-1}).$$

This filtration is known as Kedlaya's slope filtration of $\mathscr{M}$.

Proof. See proposition 1.4.15 in Kedlaya's article [24]. □

Definition 83. We say that $\mathscr{M} \in \mathrm{Ob}\big(\varphi_q\text{–Mod}_{\mathbf{B}^\dagger_{\mathrm{rig},L},\mathrm{fr}}\big)$ is *étale* if it admits a lattice $\mathscr{N}$ over the subring of functions bounded by 1 in $\mathbf{B}^\dagger_L$ (i.e., of series with coefficients in $\mathscr{O}_L$) with the property that φ_q induces an isomorphism $\varphi_q^*(\mathscr{N}) \to \mathscr{N}$.

Theorem 84 (Kedlaya). *A free* φ_q*–module* $\mathscr{M}$ *over* $\mathbf{B}^\dagger_{\mathrm{rig},L}$ *is étale if and only if it is pure of slope* 0.

Proof. See theorems 1.6.10 and 1.7.1 of Kedlaya's article [24], noting that our definitions are different from the (equivalent) definitions 1.4.6 and 1.6.1 in ibid. □

7.2.2 Equivalence with K–analytic representations

Definition 85. The (φ_q,Γ_L)–module over $\mathbf{B}^\dagger_{\mathrm{rig},L}$ associated with an overconvergent representation $V \in \mathrm{Ob}\big(\mathrm{Rep}^\dagger_K(G_L)\big)$ is

$$\mathbf{D}^\dagger_{\mathrm{rig}}(V) = \mathbf{B}^\dagger_{\mathrm{rig},L} \otimes_{\mathbf{B}^\dagger_L} \mathbf{D}^\dagger(V).$$

One can check that $\mathbf{D}^\dagger_{\mathrm{rig}}$ defines a functor $\mathrm{Rep}^\dagger_K(G_L) \to (\varphi_q,\Gamma_L)\text{–Mod}^0_{\mathbf{B}^\dagger_{\mathrm{rig},L},\mathrm{fr}}$, but it is not an equivalence of categories unless we restrict to certain subcategories.

Definition 86. We say that a representation $V \in \mathrm{Ob}\big(\mathrm{Rep}_K(G_L)\big)$ is K*–analytic* if the $\mathbb{C}_p$–semilinear representations $\mathbb{C}_p \otimes_{\sigma,K} V$ of G_L are isomorphic to the trivial representation for all the embeddings $\sigma\colon K \hookrightarrow \mathbb{C}_p$ other than the identity.

Theorem 87 (Berger). *Every* $V \in \mathrm{Ob}\big(\mathrm{Rep}_K^{\mathrm{an}}(G_L)\big)$ *is overconvergent.*

Proof. This is theorem C (or theorem 10.1) of Berger's article [6]. □

Next we describe the essential image of $\mathrm{Rep}_K^{\mathrm{an}}(G_L)$ under $\mathbf{D}_{\mathrm{rig}}^{\dagger}$. More precisely, we define a notion of analyticity of (φ_q, Γ_L)–modules based on the differential of the action of Γ_L.

Lemma 88. *Let* $\mathscr{M} \in \mathrm{Ob}\big((\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}_{\mathbf{B}_{\mathrm{rig},L}^{\dagger},\mathrm{fr}}\big)$.

(1) *For every* $r, s \in p^{\mathbb{Q}}$ *with* $r \leq s < 1$ *and for* $\gamma \in \Gamma_L$ *sufficiently close to* 1 *(depending on r and s), the series*

$$\log(\gamma) = \sum_{k \geq 1} (-1)^{k-1} \frac{(\gamma - 1)^k}{k}$$

induces a well-defined operator on $\mathscr{M}|_{\mathcal{B}[r,s]}$.

(2) *Let* $\mathrm{Lie}(\Gamma_L)$ *be the Lie algebra of* Γ_L *(regarded as a p–adic Lie group) and let* $\exp_{\Gamma_L} \colon \mathrm{Lie}(\Gamma_L) \dashrightarrow \Gamma_L$ *be the corresponding exponential map. There is a well-defined* $\mathbb{Z}_p$*–linear map of Lie algebras*

$$\begin{aligned} d\Gamma_L \colon \mathrm{Lie}(\Gamma_L) &\longrightarrow \mathrm{End}_L(\mathscr{M}) \\ \mathfrak{x} &\longmapsto \log\big(\exp_{\Gamma_L}(\mathfrak{x})\big) \end{aligned}$$

such that, for every $\mathfrak{x} \in \mathrm{Lie}(\Gamma_L)$*,*

$$\big(d\Gamma_L(\mathfrak{x})\big)(fm) = \big(d\Gamma_L(\mathfrak{x})\big)(f) \cdot m + f \cdot \big(d\Gamma_L(\mathfrak{x})\big)(m)$$

for all $f \in \mathbf{B}_{\mathrm{rig},L}^{\dagger}$ *and all* $m \in \mathscr{M}$.

Proof. See section 1.3 (until equation 1.2) of Fourquaux–Xie's article [22], which in turn adapts the calculations of lemma 2.1.2 of Kisin–Ren's article [28]. The arguments work exactly in the same way for the *relative* Lubin–Tate situation too. □

Definition 89. A module $\mathscr{M} \in \mathrm{Ob}\big((\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}_{\mathbf{B}_{\mathrm{rig},L}^{\dagger},\mathrm{fr}}\big)$ is called $\mathscr{O}_K$*–analytic* if the $\mathbb{Z}_p$–linear map $d\Gamma_L \colon \mathrm{Lie}(\Gamma_L) \to \mathrm{End}_L(\mathscr{M})$ from lemma 88 is in fact $\mathscr{O}_K$–linear. We write $(\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{\mathrm{an}}_{\mathbf{B}_{\mathrm{rig},L}^{\dagger},\mathrm{fr}}$ (resp. $(\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{0,\mathrm{an}}_{\mathbf{B}_{\mathrm{rig},L}^{\dagger},\mathrm{fr}}$) for the full subcategory of $\mathscr{O}_K$–analytic objects in $(\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}_{\mathbf{B}_{\mathrm{rig},L}^{\dagger},\mathrm{fr}}$ (resp. $(\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{0}_{\mathbf{B}_{\mathrm{rig},L}^{\dagger},\mathrm{fr}}$).

Lemma 90. *Let* $\mathscr{M} \in \mathrm{Ob}\big((\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{\mathrm{an}}_{\mathbf{B}_{\mathrm{rig},L}^{\dagger},\mathrm{fr}}\big)$.

(1) *For every* $r, s \in p^{\mathbb{Q}}$ *with* $r \leq s < 1$, *the operator*

$$N_\nabla = \frac{\log(\gamma)}{\log(\chi_\phi(\gamma))} \quad \text{on } \mathscr{M}|_{\mathcal{B}[r,s]}$$

is well-defined (for $\gamma \in \Gamma_L$ *sufficiently close to* 1 *but* $\neq 1$*) and is independent of* γ.

(2) *Gluing these operators for varying* r *and* s, *we obtain an* L*–linear differential operator* $N_\nabla \colon \mathscr{M} \to \mathscr{M}$ *that commutes with* φ_q. *In particular,*

$$N_\nabla(fm) = N_\nabla(f) \cdot m + f \cdot N_\nabla(m)$$

for all $f \in \mathbf{B}^\dagger_{\mathrm{rig},L}$ *and all* $m \in \mathscr{M}$.

(3) *There is a singular connection* ∇ *on* $\mathscr{M}$ *with simple poles at the non-zero torsion points of* $\mathfrak{F}_\phi$ *(i.e., the zeros of* $\phi_n(Z)$ *for* $n \geq 1$ *other than* 0*) such that*

$$N_\nabla = \langle \nabla, \log_\phi(Z)\partial_\phi \rangle.$$

Proof. See equation 1.3 of Fourquaux–Xie's article [22] and the calculations of lemma 2.1.4 of Kisin–Ren's article [28] over $\mathbf{B}^+_{\mathrm{rig},L}$. The arguments work exactly in the same way for the *relative* Lubin–Tate situation. □

Theorem 91 (Berger). *The functor*

$$\mathbf{D}^\dagger_{\mathrm{rig}} \colon \mathrm{Rep}^{\mathrm{an}}_K(G_L) \longrightarrow (\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{0,\mathrm{an}}_{\mathbf{B}^\dagger_{\mathrm{rig},L},\mathrm{fr}}$$

is an exact equivalence of categories that is compatible with tensor products and duality.

Proof. This is theorem D (or theorem 10.4) of Berger's article [6]. □

7.3 Modules over $\mathbf{B}^+_{\mathrm{rig},L}$

Let $Q(Z) = \phi(Z)/Z = \pi_L + \cdots \in \mathbf{A}'^{,+}_L \subset \mathbf{B}^+_{\mathrm{rig},L}$, which by definition satisfies that $\varphi_q(\omega_\phi) = Q(\omega_\phi) \cdot \omega_\phi$.

Definition 92. A *free* φ_q*–module over* $\mathbf{B}^+_{\mathrm{rig},L}$ is a free $\mathbf{B}^+_{\mathrm{rig},L}$–module $\mathscr{M}$ of finite rank endowed with a φ_q–semilinear morphism $\varphi_q = \varphi_{\mathscr{M}} \colon \mathscr{M} \to \mathscr{M}[Q^{-1}]$ such that the linearization $1 \otimes \varphi_q \colon \varphi_q^*(\mathscr{M})[Q^{-1}] \to \mathscr{M}[Q^{-1}]$ is an isomorphism. We write φ_q–$\mathrm{Mod}_{\mathbf{B}^+_{\mathrm{rig},L},\mathrm{fr}}$ for the category of such modules.

Remark. The module $\mathscr{M}$, being finite free over the ring $\mathbf{B}^+_{\mathrm{rig},L}$ of global rigid analytic functions on $\mathcal{B}$, corresponds to a coherent sheaf on the quasi-Stein space $\mathcal{B}$. Given

$r \in p^{\mathbb{Q}}$ with $r < 1$, we write $\mathscr{M}|_{\mathcal{B}[r]}$ for the sections of that sheaf on the affinoid subdomain $\mathcal{B}[r]$ (the closed disc of radius r).

Definition 93. A *free* (φ_q, Γ_L)*–module over* $\mathbf{B}^+_{\mathrm{rig},L}$ is a free φ_q–module $\mathscr{M}$ with a $\mathbf{B}^+_{\mathrm{rig},L}$–semilinear continuous action of Γ_L commuting with φ_q and such that the induced action on $\mathscr{M}/Z\mathscr{M}$ is trivial. We write (φ_q, Γ_L)–$\mathrm{Mod}_{\mathbf{B}^+_{\mathrm{rig},L},\mathrm{fr}}$ for the category of such modules.

Example 94. The ring $\mathbf{B}^+_{\mathrm{rig},L}$ itself, with the actions of φ_q and Γ_L induced by the relative Lubin–Tate structures associated with ϕ as in section 6.1, is an object of (φ_q, Γ_L)–$\mathrm{Mod}_{\mathbf{B}^+_{\mathrm{rig},L},\mathrm{fr}}$ (cf. lemma 2.1.1 of Kisin–Ren's article [28]).

7.3.1 Differential operators

Lemma 95. *Let* $\mathscr{M} \in \mathrm{Ob}\big((\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}_{\mathbf{B}^+_{\mathrm{rig},L},\mathrm{fr}}\big)$.

(1) *For every* $r \in p^{\mathbb{Q}}$ *with* $r < 1$ *and for* $\gamma \in \Gamma_L$ *sufficiently close to* 1 *(depending on* r*), the series*

$$\log(\gamma) = \sum_{k \geq 1} (-1)^{k-1} \frac{(\gamma - 1)^k}{k}$$

induces a well-defined operator on $\mathscr{M}|_{\mathcal{B}[r]}$.

(2) *Let* $\mathrm{Lie}(\Gamma_L)$ *be the Lie algebra of* Γ_L *(regarded as a* p*–adic Lie group) and let* $\exp_{\Gamma_L} \colon \mathrm{Lie}(\Gamma_L) \dashrightarrow \Gamma_L$ *be the corresponding exponential map. There is a well-defined* $\mathbb{Z}_p$*–linear map of Lie algebras*

$$d\Gamma_L \colon \mathrm{Lie}(\Gamma_L) \longrightarrow \mathrm{End}_L(\mathscr{M})$$
$$\mathfrak{x} \longmapsto \log\big(\exp_{\Gamma_L}(\mathfrak{x})\big)$$

such that, for every $\mathfrak{x} \in \mathrm{Lie}(\Gamma_L)$,

$$\big(d\Gamma_L(\mathfrak{x})\big)(fm) = \big(d\Gamma_L(\mathfrak{x})\big)(f) \cdot m + f \cdot \big(d\Gamma_L(\mathfrak{x})\big)(m)$$

for all $f \in \mathbf{B}^+_{\mathrm{rig},L}$ *and all* $m \in \mathscr{M}$.

Proof. See lemma 2.1.2 of Kisin–Ren's article [28], whose proof works verbatim for the *relative* Lubin–Tate situation too. □

Definition 96. A module $\mathscr{M} \in \mathrm{Ob}\big((\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}_{\mathbf{B}^+_{\mathrm{rig},L},\mathrm{fr}}\big)$ is called $\mathscr{O}_K$*–analytic* if the $\mathbb{Z}_p$–linear map $d\Gamma_L \colon \mathrm{Lie}(\Gamma_L) \to \mathrm{End}_L(\mathscr{M})$ from lemma 95 is in fact $\mathscr{O}_K$–linear. We write (φ_q, Γ_L)–$\mathrm{Mod}^{\mathrm{an}}_{\mathbf{B}^+_{\mathrm{rig},L},\mathrm{fr}}$ for the full subcategory of $\mathscr{O}_K$–analytic objects in (φ_q, Γ_L)–$\mathrm{Mod}_{\mathbf{B}^+_{\mathrm{rig},L},\mathrm{fr}}$.

Remark. By lemma 3.4.13 and remark 3.4.15 of Berger–Schneider–Xie's article [8], every $\mathscr{M} \in \mathrm{Ob}\big((\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}_{\mathbf{B}^+_{\mathrm{rig},L},\mathrm{fr}}\big)$ is automatically $\mathscr{O}_K$–analytic just because the action of Γ_L is trivial modulo Z. (The proof of this fact is not easy and we will not use it.)

Lemma 97. *Let* $\mathscr{M} \in \mathrm{Ob}\big((\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{\mathrm{an}}_{\mathbf{B}^+_{\mathrm{rig},L},\mathrm{fr}}\big)$.

(1) *For every* $r \in p^{\mathbb{Q}}$ *with* $r < 1$, *the operator*

$$N_\nabla = \frac{\log(\gamma)}{\log(\chi_\phi(\gamma))} \quad \text{on } \mathscr{M}|_{\mathcal{B}[r]}$$

is well-defined (for $\gamma \in \Gamma_L$ *sufficiently close to* 1 *but* $\neq 1$*) and is independent of* γ.

(2) *Gluing these operators for varying* r, *we obtain an* L*–linear differential operator* $N_\nabla \colon \mathscr{M} \to \mathscr{M}$ *that commutes with* φ_q. *In particular,*

$$N_\nabla(fm) = N_\nabla(f) \cdot m + f \cdot N_\nabla(m)$$

for all $f \in \mathbf{B}^+_{\mathrm{rig},L}$ *and all* $m \in \mathscr{M}$.

(3) *There is a singular connection* ∇ *on* $\mathscr{M}$ *with simple poles at the non-zero torsion points of* $\mathfrak{F}_\phi$ *(i.e., the zeros of* $\phi_n(Z)$ *for* $n \geq 1$ *other than* 0*) such that*

$$N_\nabla = \langle \nabla, \log_\phi(Z)\partial_\phi \rangle.$$

Proof. See lemma 2.1.4 of Kisin–Ren's article [28], which works exactly in the same way for the *relative* Lubin–Tate situation. □

7.3.2 Filtered φ_q–modules

Definition 98. A *filtered* φ_q*–module over* L is a finite-dimensional L–vector space D endowed with a φ_q–semilinear bijective map $\varphi_q = \varphi_D \colon D \to D$ and a decreasing, separated and exhaustive filtration, indexed by $\mathbb{Z}$, by L–subspaces. We write $(\mathrm{Fil}, \varphi_q)\text{–}\mathrm{Mod}_L$ for the category of filtered φ_q–modules over L.

Following subsection 2.2 of the article [28] of Kisin and Ren, which in turn adapts the constructions of subsection 1.2 of Kisin's article [27] to the Lubin–Tate situation, we want to exhibit an equivalence between the categories $(\mathrm{Fil}, \varphi_q)\text{–}\mathrm{Mod}_L$ and $(\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{\mathrm{an}}_{\mathbf{B}^+_{\mathrm{rig},L},\mathrm{fr}}$.

7.3.3 The functor $\mathscr{M}$

Fix once and for all a lift of $\varphi_q \in \mathrm{Gal}(L/K)$ to $\varphi_q \in G_K$. Write $t_0 = (z_n)_{n \geq 0}$ and let $v_n = \varphi_q^n(z_n) \in \mathfrak{F}_{\phi,n}$ for all $n \in \mathbb{Z}_{\geq 0}$. Since t_0 is a generator of $\mathrm{T}_\phi \mathfrak{F}_\phi$, for every $n \geq 1$ the element $v_n \in L_n$ is a zero of

$$\frac{\phi_n(Z)}{\phi_{n-1}(Z)} = \varphi_q^{n-1}(Q(Z))$$

and $L_n = L(v_n)$. Kisin and Ren define

$$\lambda(Z) = \prod_{n \geq 0} \varphi_q^n \left(\frac{Q(Z)}{\pi_L} \right) \in \mathbf{B}^+_{\mathrm{rig},L}$$

that, by lemma 56, is nothing else than $\lambda(Z) = \log_\phi(Z)/Z$. In particular, the zeros of $\lambda(Z)$ are the non-zero torsion points of $\mathfrak{F}_\phi$ (cf. corollary 58), namely the Galois conjugates of the v_n for $n \geq 1$. Therefore, given $n \in \mathbb{Z}_{\geq 1}$, the function $\lambda^{\varphi_q^{-n}}(Z)$ has a simple zero at z_n. Also,

$$\varphi_q(\lambda(Z)) = \frac{\pi_L \log_\phi(Z)}{\phi(Z)} = \pi_L \cdot \frac{Z}{\phi(Z)} \cdot \lambda(Z) = \frac{\pi_L}{Q(Z)} \cdot \lambda(Z)$$

by corollary 57.

Let $n \geq 1$ and write x_n for the point of $\mathcal{B}$ corresponding to the Galois conjugacy class of z_n. Let $\mathbf{S}_n$ denote the complete local ring of $\mathcal{B}$ at x_n, which is a complete discrete valuation ring with residue field $L_n = L(z_n)$. The field L_n can be viewed canonically inside $\mathbf{S}_n$ and so we have an obvious uniformizer $Z - z_n$. By the observation at the end of the last paragraph, $\lambda^{\varphi_q^{-n}}(Z)$ is another uniformizer for $\mathbf{S}_n$. We consider $\mathbf{S}_n$ with the natural filtration given by its discrete valuation.

Let $D \in \mathrm{Ob}\big((\mathrm{Fil}, \varphi_q)\text{–}\mathrm{Mod}_L\big)$. For every $n \geq 1$, we define

$$\begin{aligned} \iota_n \colon \mathbf{B}^+_{\mathrm{rig},L}[\lambda^{-1}] \otimes_L D &\longhookrightarrow \mathbf{S}_n[(Z - z_n)^{-1}] \otimes_L D \\ f(Z) \otimes \delta &\longmapsto f^{\varphi_q^{-n}}(Z) \otimes \varphi_q^{-n}(\delta) \end{aligned}$$

(where we use the canonical morphism $\mathbf{B}^+_{\mathrm{rig},L} \to \mathbf{S}_n$ and the fact that $\varphi_q \colon D \to D$ is bijective). Set

$$\mathscr{M}(D) = \Big\{ x \in \mathbf{B}^+_{\mathrm{rig},L}[\lambda^{-1}] \otimes_L D : \iota_n(x) \in \mathrm{Fil}^0\big(\mathbf{S}_n[(Z - z_n)^{-1}] \otimes_L D\big) \text{ for all } n \in \mathbb{Z}_{\geq 1} \Big\},$$

where the filtration on $\mathbf{S}_n[(Z - z_n)^{-1}] \otimes_L D$ is the tensor product of the filtrations on $\mathbf{S}_n[(Z - z_n)^{-1}]$ and on D.

Lemma 99. *Let $D \in \mathrm{Ob}\big((\mathrm{Fil}, \varphi_q)\text{–}\mathrm{Mod}_L\big)$. The operator φ_q on $\mathbf{B}^+_{\mathrm{rig},L}[\lambda^{-1}] \otimes_L D$ induces the structure of a free φ_q–module over $\mathbf{B}^+_{\mathrm{rig},L}$ on $\mathscr{M}(D)$.*

Proof. This is analogous to lemma 1.2.2 of Kisin's article [27]. We reproduce (most of) the proof with the necessary changes here.

Let $r \in \mathbb{Z}_{\geq 0}$ such that $\mathrm{Fil}^{r+1}(D) = 0$. Since $\iota_n(\lambda)$ is the product of $Z - z_n$ and a unit in $\mathbf{S}_n$ and

$$\mathrm{Fil}^0\big(\mathbf{S}_n[(Z - z_n)^{-1}] \otimes_L D\big) = \sum_{j \in \mathbb{Z}} (Z - z_n)^{-j} \mathbf{S}_n \otimes_L \mathrm{Fil}^j(D),$$

we deduce that $\mathscr{M}(D) \subset \lambda^{-r}\mathbf{B}^+_{\mathrm{rig},L} \otimes_L D$. But $\lambda^{-r}\mathbf{B}^+_{\mathrm{rig},L} \otimes_L D$ is finite free over $\mathbf{B}^+_{\mathrm{rig},L}$ and so are its closed submodules (cf. lemma 1.1.5 of Kisin's article [27]). One can prove that $\mathscr{M}(D)$ is a closed submodule using the continuity of the maps ι_n for $n \geq 1$.

To check that

$$\begin{aligned} \varphi_q \colon \lambda^{-r}\mathbf{B}^+_{\mathrm{rig},L} \otimes_L D &\longrightarrow \lambda^{-r}\mathbf{B}^+_{\mathrm{rig},L} \otimes_L D \\ f(Z) \otimes \delta &\longmapsto \varphi_q(f(Z)) \otimes \varphi_q(\delta) = f^{\varphi_q}(\phi(Z)) \otimes \varphi_q(\delta) \end{aligned}$$

induces an isomorphism $\varphi_q^*\big(\mathscr{M}(D)\big)[Q^{-1}] \cong \mathscr{M}(D)[Q^{-1}]$, we can identify $\mathscr{M}(D)$ with its corresponding coherent sheaf on $\mathcal{B}$ and work on points. The result is only unclear at the points of $\mathcal{B}$ where λ is not a unit; that is, at the x'_n corresponding to the Galois conjugacy class of v_n (or equivalently to $\varphi_q^{n-1}(Q(Z))$) for $n \geq 1$.

Let $n \in \mathbb{Z}_{\geq 1}$. Since $z_n = \phi^{\varphi_q^{-n-1}}(z_{n+1})$, we have a well-defined morphism of L_n–algebras $\phi^{\varphi_q^{-n-1}} \colon \mathbf{S}_n \to \mathbf{S}_{n+1}$ defined by $Z \mapsto \phi^{\varphi_q^{-n-1}}(Z)$. In fact, the diagram

$$\begin{array}{ccc} \lambda^{-r}\mathbf{B}^+_{\mathrm{rig},L} \otimes_L D & \xrightarrow{\iota_n} & (Z - z_n)^{-r}\mathbf{S}_n \otimes_L D \\ {\scriptstyle \varphi_q}\big\downarrow & & \big\downarrow{\scriptstyle \phi^{\varphi_q^{-n-1}} \otimes \mathrm{id}_D} \\ \lambda^{-r}\mathbf{B}^+_{\mathrm{rig},L} \otimes_L D & \xrightarrow[\iota_{n+1}]{} & (Z - z_{n+1})^{-r}\mathbf{S}_{n+1} \otimes_L D \end{array}$$

is commutative. In particular, regarding $\mathbf{S}_n$ as a $\mathbf{B}^+_{\mathrm{rig},L}$–module via the morphism $f(Z) \mapsto f^{\varphi_q^{-n}}(Z)$ (and analogously for $\mathbf{S}_{n+1}$), the morphism $\phi^{\varphi_q^{-n-1}} \colon \mathbf{S}_n \to \mathbf{S}_{n+1}$ is

φ_q–semilinear in the sense that the diagram

$$\begin{array}{ccc} \mathbf{S}_n & \xrightarrow{\phi^{\varphi_q^{-n-1}}} & \mathbf{S}_{n+1} \\ \uparrow & & \uparrow \\ \mathbf{B}^+_{\mathrm{rig},L} & \xrightarrow{\varphi_q} & \mathbf{B}^+_{\mathrm{rig},L} \end{array}$$

is commutative. Then, the $\mathbf{B}^+_{\mathrm{rig},L}$–linearization

$$\varphi_q^*\big(\mathbf{S}_n[(Z-z_n)^{-1}]\big) \to \mathbf{S}_{n+1}[(Z-z_{n+1})^{-1}]$$

is an isomorphism taking $\varphi_q^*\big((Z-z_n)^m\mathbf{S}_n\big)$ onto $(Z-z_{n+1})^m\mathbf{S}_{n+1}$ for all $m \in \mathbb{Z}$.

Define

$$\mathscr{M}_n(D) = \Big\{ x \in \mathbf{B}^+_{\mathrm{rig},L}[\lambda^{-1}] \otimes_L D : \iota_n(x) \in \mathrm{Fil}^0\big(\mathbf{S}_n[(Z-z_n)^{-1}] \otimes_L D\big) \Big\}.$$

Clearly $\mathscr{M}(D) \subset \mathscr{M}_n(D)$ and this inclusion becomes an isomorphism at the point x'_n of $\mathcal{B}$. On the other hand, ι_n induces (over the residue fields of x'_n and x_n) a bijection

$$\frac{\mathbf{B}^+_{\mathrm{rig},L} \otimes_L D}{(\varphi_q^{n-1}(Q(Z)))} \underset{\cong}{\xrightarrow{\iota_n}} \frac{\mathbf{S}_n \otimes_L D}{(Z-z_n)} \cong L_n \otimes_L D,$$

which implies that

$$0 \longrightarrow \mathscr{M}_n(D) \longrightarrow \lambda^{-r}\mathbf{B}^+_{\mathrm{rig},L} \otimes_L D \xrightarrow{\iota_n} \frac{(Z-z_n)^{-r}\mathbf{S}_n \otimes_L D}{\mathrm{Fil}^0\big((Z-z_n)^{-r}\mathbf{S}_n \otimes_L D\big)} \longrightarrow 0$$

is a short exact sequence.

Since $\varphi_q \colon \mathbf{B}^+_{\mathrm{rig},L} \to \mathbf{B}^+_{\mathrm{rig},L}$ is flat, we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 \to & \varphi_q^*(\mathscr{M}_n(D)) & \to & \varphi_q^*\big(\lambda^{-r}\mathbf{B}^+_{\mathrm{rig},L} \otimes_L D\big) & \xrightarrow{1\otimes\iota_n} & \varphi_q^*\left(\dfrac{(Z-z_n)^{-r}\mathbf{S}_n \otimes_L D}{\mathrm{Fil}^0}\right) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow \wr\!\!\mid\!\mid & \\ 0 \longrightarrow & \mathscr{M}_{n+1}(D) & \longrightarrow & \lambda^{-r}\mathbf{B}^+_{\mathrm{rig},L} \otimes_L D & \xrightarrow{\iota_{n+1}} & \dfrac{(Z-z_{n+1})^{-r}\mathbf{S}_{n+1} \otimes_L D}{\mathrm{Fil}^0} & \longrightarrow 0 \end{array}$$

with exact rows. Using that

$$\varphi_q(\lambda^{-1}) = \frac{Q(Z)}{\pi_L}\lambda^{-1},$$

we see that the vertical arrow in the middle has image $Q(Z)^r\lambda^{-r}\mathbf{B}^+_{\mathrm{rig},L} \otimes_L D$ and we deduce by the snake lemma that the kernel and cokernel of the left vertical arrow are 0 and $\left(\lambda^{-r}\mathbf{B}^+_{\mathrm{rig},L} \otimes_L D\right) / \left(Q(Z)^r\lambda^{-r}\mathbf{B}^+_{\mathrm{rig},L} \otimes_L D\right)$, respectively. But these are in fact the kernel and cokernel of $\varphi_q^*(\mathcal{M}(D)) \to \mathcal{M}(D)$ at the point x'_{n+1}, as the inclusions $\varphi_q^*(\mathcal{M}(D)) \subset \varphi_q^*(\mathcal{M}_n(D))$ and $\mathcal{M}(D) \subset \mathcal{M}_{n+1}(D)$ become isomorphisms at x'_{n+1}. Since $Q(Z)$ is invertible at x'_{n+1}, we obtain the desired isomorphism at this point.

It remains to prove that $\varphi_q^*(\mathcal{M}(D))[Q^{-1}] \to \mathcal{M}(D)[Q^{-1}]$ becomes an isomorphism at the point x'_1. But x'_1 corresponds to $Q(Z)$ and so inverting Q makes λ into a unit at this point too, which makes the result clear. □

Proposition 100. *Let $D \in \mathrm{Ob}\big((\mathrm{Fil}, \varphi_q)\text{–}\mathrm{Mod}_L\big)$. Then $\mathcal{M}(D)$ is naturally an object of $(\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{\mathrm{an}}_{\mathbf{B}^+_{\mathrm{rig},L},\mathrm{fr}}$.*

Proof. This is analogous to lemma 2.2.1 of Kisin–Ren's article. We adapt it to the relative Lubin–Tate situation here.

Having lemma 99, it remains to show that $\mathcal{M}(D)$ has a natural $\mathcal{O}_K$–analytic action of Γ_L. For $\gamma \in \Gamma_L$,

$$\gamma(\lambda(Z)) = \frac{\log_\phi([\chi_\phi(\gamma)]_\phi(Z))}{[\chi_\phi(\gamma)]_\phi(Z)} = \frac{\chi_\phi(\gamma)\log_\phi(Z)}{[\chi_\phi(\gamma)]_\phi(Z)} = \frac{\chi_\phi(\gamma)\cdot Z}{[\chi_\phi(\gamma)]_\phi(Z)}\cdot\lambda(Z)$$

and so $\lambda(Z)$ and $\gamma(\lambda(Z))$ differ (multiplicatively) by a unit in $\left(\mathbf{A}'^{,+}_L\right)^\times$. Thus, the $\mathcal{O}_K$–analytic action of Γ_L on $\mathbf{B}^+_{\mathrm{rig},L}$ induces an action on $\mathbf{B}^+_{\mathrm{rig},L}[\lambda^{-1}] \otimes_L D$ that is again $\mathcal{O}_K$–analytic.

The same argument shows that Γ_L acts by automorphisms on $\mathbf{S}_n$ for each $n \geq 1$. Since, for every $\gamma \in \Gamma_L$, composition with $[\chi_\phi(\gamma)]_\phi(Z)$ (resp. $[\chi_\phi(\gamma)]_\phi^{\varphi_q^{-n}}(Z)$) preserves the order of vanishing on torsion points of $\mathfrak{F}_\phi$ (resp. $\mathfrak{F}_\phi^{\varphi_q^{-n}}$), we see by the definition of $\mathcal{M}(D) \subset \mathbf{B}^+_{\mathrm{rig},L}[\lambda^{-1}] \otimes_L D$ that $\mathcal{M}(D)$ is stable under the action of Γ_L. □

7.3.4 The functor $\boldsymbol{D}$

To go in the opposite direction, consider $\mathscr{M} \in (\varphi_q, \Gamma_L)\text{–Mod}^{\mathrm{an}}_{\mathbf{B}^+_{\mathrm{rig},L},\mathrm{fr}}$ and set

$$\boldsymbol{D}(\mathscr{M}) = \mathscr{M}/Z\mathscr{M},$$

which is clearly a finite L–vector space and inherits an action of φ_q from $\mathscr{M}$. Observe that, as $\mathscr{M}/Z\mathscr{M} \cong \mathscr{M}[Q^{-1}]/Z\mathscr{M}[Q^{-1}]$, the L–linearization of φ_q is an isomorphism $\varphi_q^*(\boldsymbol{D}(\mathscr{M})) \cong \boldsymbol{D}(\mathscr{M})$. We want to define a filtration on $\boldsymbol{D}(\mathscr{M})$, but for that we need a previous result.

Lemma 101. *Consider $\mathscr{M} \in \mathrm{Ob}\big((\varphi_q, \Gamma_L)\text{–Mod}^{\mathrm{an}}_{\mathbf{B}^+_{\mathrm{rig},L},\mathrm{fr}}\big)$. There exists a unique L–linear φ_q–equivariant map $\xi\colon \boldsymbol{D}(\mathscr{M}) \to \mathscr{M}[\lambda^{-1}]$ whose reduction modulo Z induces $\mathrm{id}_{\boldsymbol{D}(\mathscr{M})}$ and such that the elements of $\mathrm{Im}(\xi)$ are Γ_L–invariant. Furthermore,*

(1) *the morphism ξ induces an isomorphism*

$$1 \otimes \xi\colon \mathbf{B}^+_{\mathrm{rig},L}[\lambda^{-1}] \otimes_L \boldsymbol{D}(\mathscr{M}) \to \mathscr{M}[\lambda^{-1}]$$

and

(2) *the image of $1 \otimes \xi\colon \mathbf{B}^+_{\mathrm{rig},L} \otimes_L \boldsymbol{D}(\mathscr{M}) \to \mathscr{M}[\lambda^{-1}]$ coincides with the image of $(1 \otimes \varphi_q)\colon \varphi_q^*(\mathscr{M}) \to \mathscr{M}[\lambda^{-1}]$ over an admissible open neighbourhood of the point x_1' of $\mathcal{B}$ corresponding to $Q(Z)$.*

Proof. This is analogous to lemma 2.2.2 of Kisin–Ren's article, which in turn adapts lemma 1.2.6 of Kisin's article. We reproduce the proof with the necessary changes here.

Lemma 95 gives a connection ∇ on $\mathscr{M}$. Since $\boldsymbol{D}(\mathscr{M}) = \mathscr{M}/Z\mathscr{M}$ can be viewed as the stalk of $\mathscr{M}$ at the origin, for $r \in p^{\mathbb{Q}}$ small enough there exists a unique parallel (with respect to ∇) section $\xi_r\colon \boldsymbol{D}(\mathscr{M}) \to \mathscr{M}|_{\mathcal{B}[r]}$. Since N_∇ commutes with φ_q and with $\gamma \in \Gamma_L$, we see that the sections $\varphi_q \circ \xi_r \circ \varphi_q^{-1}$ and $\gamma \circ \xi_r \circ \gamma^{-1}$ are also parallel. By uniqueness, ξ_r must be φ_q– and Γ_L–invariant.

To extend ξ_r to the whole $\mathcal{B}$, observe that we can define $\xi_{r^{1/q}}$ by requiring that the diagram

$$\begin{array}{ccc} \varphi_q^*(\boldsymbol{D}(\mathscr{M})) & \xrightarrow{\varphi_q^*(\xi_r)} & \varphi_q^*\big(\mathscr{M}|_{\mathcal{B}[r]}\big)[\lambda^{-1}] \\ \downarrow \wr & & \downarrow \wr \\ \boldsymbol{D}(\mathscr{M}) & \overset{\xi_{r^{1/q}}}{\dashrightarrow} & \big(\mathscr{M}|_{\mathcal{B}[r^{1/q}]}\big)[\lambda^{-1}] \end{array}$$

be commutative. Since r^{1/q^n} approaches 1 as $n \to \infty$, we get $\xi\colon \boldsymbol{D}(\mathscr{M}) \to \mathscr{M}[\lambda^{-1}]$ by repeating this argument inductively using r^{1/q^n} in place of r.

Now the other two claims follow from a variation of the same argument. Namely, the fact that ξ modulo Z is an isomorphism implies that there is some $r \in p^{\mathbb{Q}}$ small enough for which $1 \otimes \xi_r \colon \mathbf{B}^+_{\mathrm{rig},L}[\lambda^{-1}] \otimes_L \boldsymbol{D}(\mathscr{M}) \to \big(\mathscr{M}|_{\mathcal{B}[r]}\big)[\lambda^{-1}]$ is an isomorphism. Next we use the commutativity of the diagram

$$\begin{array}{ccc} \varphi_q^*(\mathbf{B}^+_{\mathrm{rig},L}[\lambda^{-1}] \otimes_L \boldsymbol{D}(\mathscr{M})) & \xrightarrow[\cong]{\varphi_q^*(1\otimes\xi_r)} & \varphi_q^*\big(\big(\mathscr{M}|_{\mathcal{B}[r]}\big)[\lambda^{-1}]\big) \\ \downarrow & & \downarrow \\ \mathbf{B}^+_{\mathrm{rig},L}[\lambda^{-1}] \otimes_L \boldsymbol{D}(\mathscr{M}) & \xrightarrow{1\otimes\xi_{r^{1/q}}} & \big(\mathscr{M}|_{\mathcal{B}[r^{1/q}]}\big)[\lambda^{-1}] \end{array}$$

and the fact that

$$\varphi_q(\lambda^{-1}) = \frac{Q(Z)}{\pi_L}\lambda^{-1},$$

which implies that the two vertical arrows become isomorphisms if we invert $Q(Z)$ (or its multiple $\lambda(Z)$) above. Therefore, the lower horizontal arrow is an isomorphism too and we can repeat the argument inductively with r^{1/q^n} in place of r to obtain claim (1).

For claim (2), choose $r \in p^{\mathbb{Q}}$ such that $p^{-1/e(q-1)} < r < p^{-1/eq(q-1)}$, so that $\mathcal{B}[r^q]$ contains no non-zero torsion points of $\mathfrak{F}_\phi$ but $\mathcal{B}[r]$ contains v_1. Since the morphism $1 \otimes \xi \colon \mathbf{B}^+_{\mathrm{rig},L} \otimes_L \boldsymbol{D}(\mathscr{M}) \to \mathscr{M}[\lambda^{-1}]$ becomes an isomorphism after reduction modulo Z (i.e., over the origin), there exists some $n \in \mathbb{Z}_{\geq 0}$ for which the restriction $1 \otimes \xi_{r^{q^n}} \colon \mathbf{B}^+_{\mathrm{rig},L} \otimes_L \boldsymbol{D}(\mathscr{M}) \to \big(\mathscr{M}|_{\mathcal{B}[r^{q^n}]}\big)[\lambda^{-1}]$ is an isomorphism. If $n > 1$, then λ is invertible over $\mathcal{B}[r^{q^{n-1}}]$ and we obtain a commutative diagram

$$\begin{array}{ccc} \varphi_q^*(\mathbf{B}^+_{\mathrm{rig},L} \otimes_L \boldsymbol{D}(\mathscr{M})) & \xrightarrow[\cong]{\varphi_q^*(1\otimes\xi_{r^{q^n}})} & \varphi_q^*\big(\mathscr{M}|_{\mathcal{B}[r^{q^n}]}\big) \\ \downarrow \wr\wr & & \downarrow \wr\wr \\ \mathbf{B}^+_{\mathrm{rig},L} \otimes_L \boldsymbol{D}(\mathscr{M}) & \xrightarrow{1\otimes\xi_{r^{q^{n-1}}}} & \mathscr{M}|_{\mathcal{B}[r^{q^{n-1}}]} \end{array}$$

showing that $1 \otimes \xi_{r^{q^{n-1}}}$ is also an isomorphism. Finally, for $n = 1$, the commutative diagram

$$\begin{array}{ccc} \varphi_q^*(\mathbf{B}^+_{\mathrm{rig},L} \otimes_L \boldsymbol{D}(\mathscr{M})) & \xrightarrow[\cong]{\varphi_q^*(1\otimes\xi_{r^q})} & \varphi_q^*\big(\mathscr{M}|_{\mathcal{B}[r^q]}\big) \\ \downarrow \wr\wr & & \downarrow \\ \mathbf{B}^+_{\mathrm{rig},L} \otimes_L \boldsymbol{D}(\mathscr{M}) & \xrightarrow{1\otimes\xi_r} & \big(\mathscr{M}|_{\mathcal{B}[r]}\big)[\lambda^{-1}] \end{array}$$

gives the desired result. □

Given $\mathscr{M} \in \mathrm{Ob}\big((\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{\mathrm{an}}_{\mathbf{B}^+_{\mathrm{rig},L},\mathrm{fr}}\big)$, we define a filtration on $\varphi_q^*(\mathscr{M})$ by

$$\mathrm{Fil}^i\big(\varphi_q^*(\mathscr{M})\big) = \big\{\, x \in \varphi_q^*(\mathscr{M}) : (1 \otimes \varphi_q)(x) \in Q^i \mathscr{M} \,\big\} \quad \text{for each } i \in \mathbb{Z}$$

(where $1 \otimes \varphi_q \colon \varphi_q^*(\mathscr{M}) \to \mathscr{M}[Q^{-1}]$ is the $\mathbf{B}^+_{\mathrm{rig},L}$–linearization morphism). This is a decreasing filtration by finite free $\mathbf{B}^+_{\mathrm{rig},L}$–modules. Lemma 101 gives, over a neighbourhood of x_1', an isomorphism $\mathbf{B}^+_{\mathrm{rig},L} \otimes_L \boldsymbol{D}(\mathscr{M}) \cong \varphi_q^*(\mathscr{M})$, where we identified $\varphi_q^*(\mathscr{M})$ with its image inside $\mathscr{M}[Q^{-1}]$. In particular, over x_1' we obtain an isomorphism

$$L_1 \otimes_L \boldsymbol{D}(\mathscr{M}) \cong \varphi_q^*(\mathscr{M}) / Q\varphi_q^*(\mathscr{M}).$$

The filtration on $\varphi_q^*(\mathscr{M})$ now induces a filtration on $L_1 \otimes_L \boldsymbol{D}(\mathscr{M})$ that is stable under the action of Γ_L and descends to a filtration on $\boldsymbol{D}(\mathscr{M})$. It is clear by definition that the filtration is separated and exhaustive. Thus, from now on we view $\boldsymbol{D}(\mathscr{M})$ as an object of $(\mathrm{Fil}, \varphi_q)$–$\mathrm{Mod}_L$.

7.3.5 The equivalence of categories

Lemma 102. *Let $\mathscr{M} \in \mathrm{Ob}\big((\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{\mathrm{an}}_{\mathbf{B}^+_{\mathrm{rig},L},\mathrm{fr}}\big)$. Let $\mathbf{S}_1'$ be the complete local ring of $\mathcal{B}$ at the point x_1' corresponding to $Q(Z)$. The map ξ from lemma 101 induces isomorphisms*

$$\mathrm{Fil}^i\big(\mathbf{S}_1' \otimes_L \boldsymbol{D}(\mathscr{M})\big) = \sum_{j \geq 0} Q^j \mathbf{S}_1' \otimes_L \mathrm{Fil}^{i-j}\big(\boldsymbol{D}(\mathscr{M})\big) \cong \mathbf{S}_1' \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathrm{Fil}^i\big(\varphi_q^*(\mathscr{M})\big)$$

for all $i \in \mathbb{Z}$.

Proof. This is analogous to lemma 2.2.5 of Kisin–Ren's article [28] and to lemma 1.2.12.(4) of Kisin's article [27]. We reproduce it here for the convenience of the reader.

Lemma 101 gives an isomorphism $\mathbf{S}_1' \otimes_L \boldsymbol{D}(\mathscr{M}) \cong \mathbf{S}_1' \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} (1 \otimes \varphi_q)\big(\varphi_q^*(\mathscr{M})\big)$ (over x_1'). Take an index $r \in \mathbb{Z}$ such that $(1 \otimes \varphi_q)\big(\varphi_q^*(\mathscr{M})\big) \subset Q^r \mathscr{M}$ or, equivalently, $\mathrm{Fil}^r\big(\varphi_q^*(\mathscr{M})\big) = \varphi_q^*(\mathscr{M})$. By the definition of the filtration on $\boldsymbol{D}(\mathscr{M})$, we see that $\mathrm{Fil}^r\big(\mathbf{S}_1' \otimes_L \boldsymbol{D}(\mathscr{M})\big) = \mathbf{S}_1' \otimes_L \boldsymbol{D}(\mathscr{M})$. Therefore, it suffices to prove the lemma for $i \geq r$ and we do so by induction. The base case is obvious.

Suppose that we have the result for $i-1$ and we want it for i. The inductive hypothesis shows that

$$Q \cdot \mathrm{Fil}^{i-1}\big(\mathbf{S}_1' \otimes_L \boldsymbol{D}(\mathscr{M})\big) \cong Q\mathbf{S}_1' \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} (1 \otimes \varphi_q)\big(\mathrm{Fil}^{i-1}\, \varphi_q^*(\mathscr{M})\big).$$

But

$$\mathrm{Fil}^i\big(\mathbf{S}_1' \otimes_L \boldsymbol{D}(\mathscr{M})\big) = Q \cdot \mathrm{Fil}^{i-1}\big(\mathbf{S}_1' \otimes_L \boldsymbol{D}(\mathscr{M})\big) + \mathbf{S}_1' \otimes_L \mathrm{Fil}^i\big(\boldsymbol{D}(\mathscr{M})\big).$$

On the other hand, since the filtrations on $\varphi_q^*(\mathscr{M})$ and on $\mathbf{S}_1' \otimes_L \boldsymbol{D}(\mathscr{M})$ both induce the same filtration on their common quotient $L_1 \otimes_L \boldsymbol{D}(\mathscr{M})$, the preimage of $\mathbf{S}_1' \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} (1 \otimes \varphi_q)\big(\mathrm{Fil}^i\, \varphi_q^*(\mathscr{M})\big)$ lies in $\mathrm{Fil}^i\big(\mathbf{S}_1' \otimes_L \boldsymbol{D}(\mathscr{M})\big)$. Thus, it suffices to prove that the image of $\mathrm{Fil}^i\big(\boldsymbol{D}(\mathscr{M})\big)$ lies in $\mathbf{S}_1' \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} (1 \otimes \varphi_q)\big(\mathrm{Fil}^i\, \varphi_q^*(\mathscr{M})\big)$.

Let d be an element in the image of $\mathrm{Fil}^i\big(\boldsymbol{D}(\mathscr{M})\big)$ and decompose it as $d = d_0 + d_1$ with $d_0 \in \mathbf{S}_1' \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} (1 \otimes \varphi_q)\big(\mathrm{Fil}^i\, \varphi_q^*(\mathscr{M})\big)$ and $d_1 \in Q\mathbf{S}_1' \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} (1 \otimes \varphi_q)\big(\varphi_q^*(\mathscr{M})\big)$ (cf. the definition of the filtration on $\boldsymbol{D}(\mathscr{M})$). The operator

$$N_\nabla = \frac{\partial \mathfrak{F}_\phi(X,Y)}{\partial Y}\Big|_{(X,Y)=(Z,0)} \cdot \log_\phi(Z) \cdot \frac{d}{dZ}$$

on $\mathbf{B}^+_{\mathrm{rig},L}$ (cf. lemma 97) extends to $\mathbf{S}_1'$ and then to $\mathbf{S}_1' \otimes_L \boldsymbol{D}(\mathscr{M})$ by acting trivially on $\boldsymbol{D}(\mathscr{M})$. Similarly, the operator N_∇ on $\mathscr{M}$ extends to $\mathbf{S}_1' \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathscr{M}$. By the construction of ξ, the isomorphism $\mathbf{S}_1' \otimes_L \boldsymbol{D}(\mathscr{M}) \cong \mathbf{S}_1' \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} (1 \otimes \varphi_q)\big(\varphi_q^*(\mathscr{M})\big)$ is compatible with N_∇ and so $N_\nabla(d) = 0$. Note also that $N_\nabla(Q) \subset Q\mathbf{B}^+_{\mathrm{rig},L}$ because $\log_\phi(Z)$ is a multiple of $Q(Z)$ and that $(1 \otimes \varphi_q)\big(\varphi_q^*(\mathscr{M})\big)$ is stable under N_∇ by its compatibility with φ_q. Therefore,

$$N_\nabla(d_1) = -N_\nabla(d_0) \in \mathbf{S}_1' \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \big(Q(1 \otimes \varphi_q)\big(\varphi_q^*(\mathscr{M})\big) \cap (1 \otimes \varphi_q)\big(\mathrm{Fil}^i\, \varphi_q^*(\mathscr{M})\big)\big).$$

We claim that N_∇ induces a bijection on

$$M_i = \mathbf{S}_1' \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \big(Q(1 \otimes \varphi_q)(\varphi_q^*(\mathscr{M})) \cap Q^i \mathscr{M}\big)$$

and this claim implies that $d_1 \in M_i \subset \mathbf{S}_1' \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} (1 \otimes \varphi_q)\big(\mathrm{Fil}^i\, \varphi_q^*(\mathscr{M})\big)$, as required.

We prove the last claim by induction on $i \geq r$. The base case follows from the compatibility of the isomorphism $\mathbf{S}_1' \otimes_L \boldsymbol{D}(\mathscr{M}) \cong \mathbf{S}_1' \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} (1 \otimes \varphi_q)\big(\varphi_q^*(\mathscr{M})\big)$ with N_∇ and the fact that N_∇ induces a bijection on $Q\mathbf{S}_1'$. For the inductive step, we use the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_i & \longrightarrow & M_{i-1} & \longrightarrow & M_{i-1}/M_i & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle N_\nabla} & & \wr\!\downarrow{\scriptstyle N_\nabla} & & \downarrow & & \\
0 & \longrightarrow & M_i & \longrightarrow & M_{i-1} & \longrightarrow & M_{i-1}/M_i & \longrightarrow & 0
\end{array}$$

with exact rows and the snake lemma. More precisely, since M_{i-1}/M_i is a finite-dimensional L–vector space, the surjective right vertical arrow must in fact be an isomorphism. Then $N_\nabla\colon M_i \to M_i$ is an isomorphism too. $\square$

Lemma 103. *Let $\mathscr{M} \in \mathrm{Ob}\big((\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{\mathrm{an}}_{\mathbf{B}^+_{\mathrm{rig},L},\mathrm{fr}}\big)$. The map ξ from lemma 101 induces an isomorphism $\boldsymbol{\mathscr{M}}(\boldsymbol{D}(\mathscr{M})) \cong \mathscr{M}$.*

Proof. This is analogous to proposition 1.2.13 of Kisin's article [27]. We adapt it here to our situation.

Write $D_0 = \mathbf{B}^+_{\mathrm{rig},L} \otimes_L \boldsymbol{D}(\mathscr{M})$. By definition, $\mathscr{M}' = \boldsymbol{\mathscr{M}}(\boldsymbol{D}(\mathscr{M}))$ is a submodule of $D_0[\lambda^{-1}] = \mathbf{B}^+_{\mathrm{rig},L}[\lambda^{-1}] \otimes_L \boldsymbol{D}(\mathscr{M})$. On the other hand, by lemma 101 we have an isomorphism $1 \otimes \xi\colon D_0[\lambda^{-1}] \to \mathscr{M}[\lambda^{-1}]$ by means of which we identify $\mathscr{M}$ with a submodule of $D_0[\lambda^{-1}]$ too. Interpreting $D_0[\lambda^{-1}]$ as (the global sections of) a coherent sheaf on $\mathcal{B}$, we have to prove that the two subsheaves $\mathscr{M}$ and $\mathscr{M}'$ coincide at all points.

At the points of $\mathcal{B}$ where λ is a unit, the inclusions $\mathscr{M} \subset D_0[\lambda^{-1}] \supset \mathscr{M}'$ become isomorphisms. Thus, we have to focus on the points x'_n corresponding to $\varphi_q^{n-1}(Q(Z))$ for $n \in \mathbb{Z}_{\geq 1}$ (cf. the proof of lemma 99).

Let $n \geq 1$. At x'_{n+1}, the map $(1 \otimes \varphi_q)\colon \varphi_q^*(D_0[\lambda^{-1}]) \to D_0[\lambda^{-1}]$ becomes an isomorphism because $Q(Z)$ is a unit there. Similarly, the inclusions $\varphi_q^*(\mathscr{M}) \hookrightarrow \mathscr{M}$ and $\varphi_q^*(\mathscr{M}') \hookrightarrow \mathscr{M}'$ are isomorphisms at x'_{n+1}. But, as $\varphi_q(x'_{n+1}) \neq x'_m$ for any $m \geq 1$, the inclusions $\varphi_q^*(\mathscr{M}) \subset \varphi_q^*(D_0[\lambda^{-1}]) \supset \varphi_q^*(\mathscr{M}')$ are isomorphisms at x'_{n+1}. We conclude that $\mathscr{M}$ and $\mathscr{M}'$ coincide at x'_{n+1}.

It remains to study $\mathscr{M}$ and $\mathscr{M}'$ at x'_1. That is, we have to compare $\mathbf{S}'_1 \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathscr{M}$ and $\mathbf{S}'_1 \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathscr{M}'$ inside $\mathbf{S}'_1 \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} D_0[\lambda^{-1}] \cong \mathbf{S}'_1[Q^{-1}] \otimes_L \boldsymbol{D}(\mathscr{M})$. On the one hand, ξ induces an isomorphism

$$\mathbf{S}'_1 \otimes_L \boldsymbol{D}(\mathscr{M}) \cong \mathbf{S}'_1 \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} (1 \otimes \varphi_q)\big(\varphi_q^*(\mathscr{M})\big)$$

by lemma 101. On the other hand, $(1 \otimes \varphi_q)\big(\varphi_q^*(D_0)\big) = D_0$ and at x'_1 the submodules $\varphi_q^*(D_0)$ and $\varphi_q^*(\mathscr{M}')$ become equal, so

$$\mathbf{S}'_1 \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} D_0 = \mathbf{S}'_1 \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} (1 \otimes \varphi_q)\big(\varphi_q^*(D_0)\big) = \mathbf{S}'_1 \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} (1 \otimes \varphi_q)\big(\varphi_q^*(\mathscr{M}')\big).$$

Now we have to check that an element of $\mathbf{S}'_1 \otimes_L \boldsymbol{D}(\mathscr{M})$ lies in $Q^i\mathbf{S}'_1 \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathscr{M}$ if and only if it lies in $Q^i\mathbf{S}'_1 \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathscr{M}'$ (for any $i \in \mathbb{Z}$). But this is equivalent to showing that the filtrations on the two sides of the isomorphism

$$\mathbf{S}'_1 \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} (1 \otimes \varphi_q)\big(\varphi_q^*(\mathscr{M}')\big) = \mathbf{S}'_1 \otimes_L \boldsymbol{D}(\mathscr{M}) \cong \mathbf{S}'_1 \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} (1 \otimes \varphi_q)\big(\varphi_q^*(\mathscr{M})\big)$$

(induced by ξ) coincide, which is the content of lemma 102. □

Theorem 104 (Kisin–Ren). *The functors*

$$(\varphi_q, \Gamma_L)\text{–Mod}^{\mathrm{an}}_{\mathbf{B}^+_{\mathrm{rig},L},\mathrm{fr}} \underset{\mathscr{M}}{\overset{\boldsymbol{D}}{\rightleftarrows}} (\mathrm{Fil}, \varphi_q)\text{–Mod}_L$$

are exact quasi-inverse equivalences of categories that are compatible with tensor products.

Proof. For the non-relative Lubin–Tate case, this is proposition 2.2.6 of Kisin–Ren's article [28], which in turn is analogous to theorem 1.2.15 of Kisin's article [27].

Lemma 103 proves one direction of the fact that $\boldsymbol{D}$ and $\mathscr{M}$ are quasi-inverse. For the other direction, let $D \in \mathrm{Ob}\big((\mathrm{Fil}, \varphi_q)\text{–Mod}_L\big)$. By the definition of $\mathscr{M}(D)$, we have a natural Γ_L–equivariant inclusion $D \subset \mathscr{M}(D)[Q^{-1}]$. This inclusion induces an isomorphism between D and $\boldsymbol{D}(\mathscr{M}(D)) = \mathscr{M}(D)/Z\mathscr{M}(D)$ and so D must be the image of $\xi\colon \boldsymbol{D}(\mathscr{M}(D)) \to \mathscr{M}(D)[\lambda^{-1}]$ (cf. lemma 101). Tracing the definitions and using lemma 102 (modulo Z), one sees that the filtrations on D and on $\boldsymbol{D}(\mathscr{M}(D))$ coincide.

The exactness and the compatibility with tensor products can be proved exactly as in theorem 1.2.15 of Kisin's article [27], working on points of $\mathcal{B}$ and using some of the arguments that appeared in the proof of lemma 99. □

Kisin and Ren restricted this equivalence of categories to certain subcategories that we define next.

7.3.6 An equivalence of subcategories

Definition 105. We say that $\mathscr{M} \in \mathrm{Ob}\big(\varphi_q\text{–Mod}_{\mathbf{B}^+_{\mathrm{rig},L},\mathrm{fr}}\big)$ is *pure of slope* 0 (or *étale*) if its base change $\mathscr{M}_{\mathbf{B}^\dagger_{\mathrm{rig},L}} = \mathbf{B}^\dagger_{\mathrm{rig},L} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathscr{M}$ is pure of slope 0 (in the sense that 0 is the only slope appearing in Kedlaya's slope filtration or, equivalently, that $\mathscr{M}_{\mathbf{B}^\dagger_{\mathrm{rig},L}}$ is étale as in definition 83). We define $(\varphi_q, \Gamma_L)\text{–Mod}^{0,\mathrm{an}}_{\mathbf{B}^+_{\mathrm{rig},L},\mathrm{fr}}$ to be the full subcategory of $(\varphi_q, \Gamma_L)\text{–Mod}^{\mathrm{an}}_{\mathbf{B}^+_{\mathrm{rig},L},\mathrm{fr}}$ of modules that are pure of slope 0 as φ_q–modules.

Remark. For $\mathscr{M} \in \mathrm{Ob}\big(\varphi_q\text{–Mod}_{\mathbf{B}^+_{\mathrm{rig},L},\mathrm{fr}}\big)$, the base change $\mathscr{M}_{\mathbf{B}^\dagger_{\mathrm{rig},L}}$ is indeed an object of $\varphi_q\text{–Mod}_{\mathbf{B}^\dagger_{\mathrm{rig},L},\mathrm{fr}}$ because Q is invertible in $\mathbf{B}^\dagger_{\mathrm{rig},L}$.

Definition 106. Let $D \in \mathrm{Ob}\big((\mathrm{Fil}, \varphi_q)\text{–Mod}_L\big)$.

(1) If D is 1–dimensional over L, choose a basis element $x \in D$ and consider $\alpha \in L$ such that $\varphi_q(x) = \alpha x$. The *Newton number of D* is

$$t_N(D) = \mathrm{v}_{\pi_L}(\alpha)$$

(where v_{π_L} denotes the normalized valuation of L). If D has dimension m over L, the *Newton number of D* is

$$t_N(D) = t_N\Big(\bigwedge^m D\Big).$$

(2) The *Hodge number of D* is

$$t_H(D) = \sum_{i \in \mathbb{Z}} i \cdot \dim_L\big(\mathrm{Gr}^i(D)\big).$$

(3) We say that D is *weakly admissible* if $t_H(D) = t_N(D)$ and $t_H(D') \leq t_N(D')$ for all subobjects D' of D in $(\mathrm{Fil}, \varphi_q)$–$\mathrm{Mod}_L$. We write $(\mathrm{Fil}, \varphi_q)$–$\mathrm{Mod}_L^{\mathrm{wa}}$ for the full subcategory of weakly admissible objects in $(\mathrm{Fil}, \varphi_q)$–$\mathrm{Mod}_L$.

Proposition 107. *An object D of $(\mathrm{Fil}, \varphi_q)$–$\mathrm{Mod}_L$ is weakly admissible if and only if $\mathscr{M}(D)_{\mathbf{B}^\dagger_{\mathrm{rig},L}}$ is pure of slope 0. Therefore, the functors*

$$(\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{0,\mathrm{an}}_{\mathbf{B}^+_{\mathrm{rig},L},\mathrm{fr}} \underset{\mathscr{M}}{\overset{\boldsymbol{D}}{\rightleftarrows}} (\mathrm{Fil}, \varphi_q)\text{–}\mathrm{Mod}_L^{\mathrm{wa}}$$

are quasi-inverse equivalences of categories.

Proof. See proposition 2.3.3 of Kisin–Ren's article [28], which works exactly in the same way for the *relative* Lubin–Tate situation. □

7.4 Filtered φ_q–modules and crystalline representations

We want to apply the theory introduced in the previous sections to study crystalline $\mathscr{O}_K$–linear representations of G_L.

7.4.1 The functors $\mathbf{D}_{\mathrm{crys}}$ and $\mathbf{D}_{\mathrm{crys},K}$

Recall that the ring $\mathbf{B}_{\mathrm{crys}}$ comes equipped with a p–th power Frobenius endomorphism φ_p (and so with a q–th power Frobenius endomorphism φ_q too). Let F (resp. F') denote the maximal absolutely unramified subfield of K (resp. of L). We define $\mathbf{B}_{\mathrm{crys},K} = \mathbf{B}_{\mathrm{crys}} \otimes_F K$ and write again φ_q for the endomorphism $\varphi_q \otimes 1$ on $\mathbf{B}_{\mathrm{crys},K}$. (Note that, since L/K is unramified, we could equivalently use $\mathbf{B}_{\mathrm{crys},L} = \mathbf{B}_{\mathrm{crys}} \otimes_{F'} L$, which is the same as $\mathbf{B}_{\mathrm{crys},K}$.) In addition, $\mathbf{B}_{\mathrm{crys}}$ and $\mathbf{B}_{\mathrm{crys},K}$ inherit filtrations from $\mathbf{B}_{\mathrm{dR}}$ via the natural inclusions $\mathbf{B}_{\mathrm{crys}} \hookrightarrow \mathbf{B}_{\mathrm{crys},K} \hookrightarrow \mathbf{B}_{\mathrm{dR}}$.

Definition 108. Let V be a K–linear representation of G_L (i.e., a finite K–vector space endowed with a continuous action of G_L). We define the *crystalline filtered φ_q–modules*

$$\mathbf{D}_{\text{crys}}(V) = (\mathbf{B}_{\text{crys}} \otimes_{\mathbb{Q}_p} V)^{G_L}$$

and

$$\mathbf{D}_{\text{crys},K}(V) = (\mathbf{B}_{\text{crys},K} \otimes_K V)^{G_L} = (\mathbf{B}_{\text{crys}} \otimes_F V)^{G_L}.$$

We say that V is *crystalline* if $\dim_{F'}\big(\mathbf{D}_{\text{crys}}(V)\big) = \dim_{\mathbb{Q}_p}(V)$. We write $\operatorname{Rep}_K^{\text{crys}}(G_L)$ for the category of K–linear continuous representations of G_L that are crystalline and $\operatorname{Rep}_{\mathscr{O}_K,\text{fr}}^{\text{crys}}(G_L)$ for the category of free $\mathscr{O}_K$–modules T endowed with a continuous action of G_L and such that $K \otimes_{\mathscr{O}_K} T \in \operatorname{Ob}\big(\operatorname{Rep}_K^{\text{crys}}(G_L)\big)$.

Remark. By definition, $\mathbf{D}_{\text{crys},K}(V)$ is a module over $\mathbf{B}_{\text{crys}}^{G_L} \otimes_F K = F' \otimes_F K \cong L$ and so is an object of (Fil, φ_q)–Mod_L. Similarly, $\mathbf{D}_{\text{crys}}(V)$ is a module over $F' \otimes_{\mathbb{Q}_p} K$. We view $\mathbf{D}_{\text{crys},K}(V)$ as the *identity component* of $\mathbf{D}_{\text{crys}}(V)$, in the sense that it corresponds to the identity of $\operatorname{Gal}(F/\mathbb{Q}_p) = \langle \varphi_p \rangle$ in the isomorphism

$$\mathbf{B}_{\text{crys}} \otimes_{\mathbb{Q}_p} V \cong \bigoplus_{\sigma \in \operatorname{Gal}(F/\mathbb{Q}_p)} (\mathbf{B}_{\text{crys}} \otimes_{\sigma,F} V).$$

7.4.2 The functors $\mathbf{V}_{\text{crys}}$ and $\mathbf{V}_{\text{crys},K}$

Definition 109. A *free filtered φ_p–module over $F' \otimes_{\mathbb{Q}_p} K$* is a free $(F' \otimes_{\mathbb{Q}_p} K)$–module $\widetilde{D}$ of finite rank endowed with a $(\varphi_p \otimes \text{id}_K)$–semilinear bijective map $\varphi_p \colon \widetilde{D} \to \widetilde{D}$ and a decreasing, separated and exhaustive filtration on $\widetilde{D}_L = L \otimes_{F'} \widetilde{D}$, indexed by $\mathbb{Z}$, by $(L \otimes_{\mathbb{Q}_p} K)$–modules. We write (Fil, φ_p)–$\text{Mod}_{F' \otimes_{\mathbb{Q}_p} K,\text{fr}}$ for the category of such modules.

Definition 110.

(1) Let $\widetilde{D} \in \operatorname{Ob}\big((\text{Fil}, \varphi_p)\text{–}\text{Mod}_{F' \otimes_{\mathbb{Q}_p} K,\text{fr}}\big)$. We define its *associated representation*

$$\mathbf{V}_{\text{crys}}(\widetilde{D}) = \text{Fil}^0\big(\mathbf{B}_{\text{crys}} \otimes_{F'} \widetilde{D}\big)^{\varphi_p=1} \in \operatorname{Ob}\big(\operatorname{Rep}_K(G_L)\big).$$

(2) Let $D \in \operatorname{Ob}\big((\text{Fil}, \varphi_q)\text{–}\text{Mod}_L\big)$. We define its *associated representation*

$$\mathbf{V}_{\text{crys},K}(D) = \text{Fil}^0\big(\mathbf{B}_{\text{crys},K} \otimes_L D\big)^{\varphi_q=1} \in \operatorname{Ob}\big(\operatorname{Rep}_K(G_L)\big).$$

7.4.3 The equivalence for crystalline representations

One of the most important results of p–adic Hodge theory is that the functors $\mathbf{D}_{\text{crys}}$ and $\mathbf{V}_{\text{crys}}$ are quasi-inverse if we restrict them to appropriate subcategories.

Next we recall that result and later transport it to $\mathbf{D}_{\mathrm{crys},K}$ and $\mathbf{V}_{\mathrm{crys},K}$.

Definition 111. Let $\widetilde{D} \in \mathrm{Ob}\big((\mathrm{Fil}, \varphi_p)\text{–}\mathrm{Mod}_{F'\otimes_{\mathbb{Q}_p}K,\mathrm{fr}}\big)$.

(1) Regard $\widetilde{D}$ as an F'–vector space. Let $m = \dim_{F'}(\widetilde{D})$ and choose a basis element

$$x \in \bigwedge^m \widetilde{D}.$$

Consider $\alpha \in F'$ such that $\varphi_p(x) = \alpha x$. The *Newton number of* $\widetilde{D}$ is

$$t_N(\widetilde{D}) = \mathrm{v}_{\pi_L}(\alpha)$$

(where v_{π_L} denotes the normalized valuation of L).

(2) The *Hodge number of* $\widetilde{D}$ is

$$t_H(\widetilde{D}) = \sum_{i\in\mathbb{Z}} i \cdot \dim_L\big(\mathrm{Gr}^i(\widetilde{D})\big).$$

(3) We say that $\widetilde{D}$ is *weakly admissible* if $t_H(\widetilde{D}) = t_N(\widetilde{D})$ and $t_H(\widetilde{D}') \leq t_N(\widetilde{D}')$ for all subobjects $\widetilde{D}'$ of $\widetilde{D}$ in $(\mathrm{Fil}, \varphi_p)\text{–}\mathrm{Mod}_{F'\otimes_{\mathbb{Q}_p}K,\mathrm{fr}}$. Let $(\mathrm{Fil}, \varphi_p)\text{–}\mathrm{Mod}^{\mathrm{wa}}_{F'\otimes_{\mathbb{Q}_p}K,\mathrm{fr}}$ denote the full subcategory formed of weakly admissible objects inside $(\mathrm{Fil}, \varphi_p)\text{–}\mathrm{Mod}_{F'\otimes_{\mathbb{Q}_p}K,\mathrm{fr}}$.

Theorem 112 (Fontaine, Colmez–Fontaine). *The functors*

$$\mathrm{Rep}_K^{\mathrm{crys}}(G_L) \underset{\mathbf{V}_{\mathrm{crys}}}{\overset{\mathbf{D}_{\mathrm{crys}}}{\rightleftarrows}} (\mathrm{Fil}, \varphi_p)\text{–}\mathrm{Mod}^{\mathrm{wa}}_{F'\otimes_{\mathbb{Q}_p}K,\mathrm{fr}}$$

are exact quasi-inverse equivalences of categories that are compatible with tensor products and duality.

Proof. See proposition 9.1.11 of Brinon–Conrad's notes [11] and theorem 11.19 of Colmez's article [16], for instance. □

7.4.4 Comparison of filtered φ_q– and φ_p–modules

In the remainder of this subsection we explain how one can view the category $(\mathrm{Fil}, \varphi_q)\text{–}\mathrm{Mod}_L$ inside $(\mathrm{Fil}, \varphi_p)\text{–}\mathrm{Mod}_{F'\otimes_{\mathbb{Q}_p}K,\mathrm{fr}}$ and describe the corresponding (full) subcategory of crystalline representations.

Let $r = [F : \mathbb{Q}_p]$, so that $q = p^r$. We extend $\varphi_p^i \colon F' \to F'$ to

$$\varphi_p^i \colon L \cong F' \otimes_F K \xrightarrow{\varphi_p^i \otimes \mathrm{id}_K} F' \otimes_{\varphi_p^i, F} K$$

for every $i \geq 0$. Given $D \in \mathrm{Ob}\big((\mathrm{Fil}, \varphi_q)\text{–}\mathrm{Mod}_L\big)$, we define

$$\widetilde{D} = \bigoplus_{i=0}^{r-1} (\varphi_p^i)^*(D) = \bigoplus_{i=0}^{r-1} \big((F' \otimes_{\varphi_p^i, F} K) \otimes_{\varphi_p^i, L} D\big) \cong F \otimes_{\mathbb{Q}_p} D$$

and regard it as an $(F' \otimes_{\mathbb{Q}_p} K)$–module using the decomposition

$$F' \otimes_{\mathbb{Q}_p} K \cong \bigoplus_{i=0}^{r-1} (F' \otimes_{\varphi_p^i, F} K).$$

Then $\widetilde{D}$ is finite free of the same rank as D.

Next we define $\varphi_p \colon \widetilde{D} \to \widetilde{D}$ as follows. For $0 \leq i < r-1$, we consider the morphism $\varphi_p \colon (\varphi_p^i)^*(D) \to (\varphi_p^{i+1})^*(D)$ naturally induced by $\varphi_p \colon F' \to F'$; for $i = r-1$, we have $\varphi_p \colon (\varphi_p^{r-1})^*(D) \to D$ given by

$$\begin{array}{ccccc} (F' & \otimes_{\varphi_p^{r-1},F} & K) & \otimes_{\varphi_p^{r-1},L} & D \\ & \searrow^{\varphi_p} & \downarrow^{\mathrm{id}_K} & \swarrow^{\varphi_q} & \\ & (F' \otimes_F & K) & \otimes_L D & \end{array}$$

(recall that $q = p^r$, so $\varphi_p \colon F' \to F'$ and $\varphi_q \colon D \to D$ are indeed compatible as shown in the diagram).

Finally, since $L \otimes_{\mathbb{Q}_p} K$ is artinian, we can decompose

$$\widetilde{D}_L = \bigoplus_{\mathfrak{m} \in \mathrm{Spm}(L \otimes_{\mathbb{Q}_p} K)} (\widetilde{D}_L)_{\mathfrak{m}},$$

where the direct sum runs over all maximal ideals of $L \otimes_{\mathbb{Q}_p} K$. In particular, there is one maximal ideal $\mathfrak{m}_0$ corresponding to the natural multiplication morphism $L \otimes_{\mathbb{Q}_p} K \twoheadrightarrow L$ (equivalently, to $\mathrm{id}_K \colon K \hookrightarrow L$ via the decomposition of $L \otimes_{\mathbb{Q}_p} K$ as a direct sum of $L \otimes_{\sigma, K} K$ for $\sigma \in \mathrm{Gal}(K/\mathbb{Q}_p)$) and we call $(\widetilde{D}_L)_{\mathfrak{m}_0} = L \otimes_{L \otimes_{\mathbb{Q}_p} K} \widetilde{D}_L$ the *identity component* of $\widetilde{D}_L$, which is naturally identified with D. We endow $\widetilde{D}_L$ with a filtration that is the direct sum of the filtration on D and the trivial filtration on the other direct summands $(\widetilde{D}_L)_{\mathfrak{m}}$ for $\mathfrak{m} \neq \mathfrak{m}_0$. Thus, we formed an object $\widetilde{D}$ in $(\mathrm{Fil}, \varphi_p)\text{–}\mathrm{Mod}_{F' \otimes_{\mathbb{Q}_p} K, \mathrm{fr}}$.

Remark. We can recover the *identity component* directly from $\widetilde{D}$, without passing to $\widetilde{D}_L$. That is, we can consider an analogous direct sum decomposition of $\widetilde{D}$ using the maximal ideals of $F' \otimes_{\mathbb{Q}_p} K$ and, if $\mathfrak{m}_0$ is the kernel of the multiplication morphism $F' \otimes_{\mathbb{Q}_p} K \twoheadrightarrow L$, then $\widetilde{D}_{\mathfrak{m}_0} = L \otimes_{F' \otimes_{\mathbb{Q}_p} K} \widetilde{D}$ is naturally identified with D.

Definition 113.

(1) We say that $\widetilde{D} \in \mathrm{Ob}\big((\mathrm{Fil}, \varphi_p)\text{–}\mathrm{Mod}_{F'\otimes_{\mathbb{Q}_p}K,\mathrm{fr}}\big)$ is *K–analytic* if the filtration on

$$\widetilde{D}_L = \bigoplus_{\mathfrak{m}\in \mathrm{Spm}(L\otimes_{\mathbb{Q}_p}K)} (\widetilde{D}_L)_{\mathfrak{m}}$$

restricts to the trivial filtration on every direct summand other than the identity component. Let $(\mathrm{Fil}, \varphi_p)\text{–}\mathrm{Mod}^{\mathrm{an}}_{F'\otimes_{\mathbb{Q}_p}K,\mathrm{fr}}$ (resp. $(\mathrm{Fil}, \varphi_p)\text{–}\mathrm{Mod}^{\mathrm{wa,an}}_{F'\otimes_{\mathbb{Q}_p}K,\mathrm{fr}}$) denote the full subcategory of K–analytic objects of $(\mathrm{Fil}, \varphi_p)\text{–}\mathrm{Mod}_{F'\otimes_{\mathbb{Q}_p}K,\mathrm{fr}}$ (resp. of $(\mathrm{Fil}, \varphi_p)\text{–}\mathrm{Mod}^{\mathrm{wa}}_{F'\otimes_{\mathbb{Q}_p}K,\mathrm{fr}}$).

(2) We say that $V \in \mathrm{Ob}\big(\mathrm{Rep}^{\mathrm{crys}}_K(G_L)\big)$ is *K–analytic* if $\mathbf{D}_{\mathrm{crys}}(V)$ is K–analytic. We write $\mathrm{Rep}^{\mathrm{crys,an}}_K(G_L)$ for the full subcategory of such representations.

(3) We say that $T \in \mathrm{Ob}\big(\mathrm{Rep}^{\mathrm{crys}}_{\mathscr{O}_K,\mathrm{fr}}(G_L)\big)$ is *$\mathscr{O}_K$–analytic* if $K \otimes_{\mathscr{O}_K} T$ is K–analytic (i.e., if $\mathbf{D}_{\mathrm{crys}}(T) = (\mathbf{B}_{\mathrm{crys}} \otimes_{\mathbb{Z}_p} T)^{G_L}$ is K–analytic). We write $\mathrm{Rep}^{\mathrm{crys,an}}_{\mathscr{O}_K,\mathrm{fr}}(G_L)$ for the full subcategory of such representations.

Remark. There is a more general notion of K–analytic representation: a K–linear representation V of G_L is called K–analytic if the $\mathbb{C}_p$–semilinear representations $\mathbb{C}_p \otimes_{\sigma,K} V$ are trivial for all the embeddings $\sigma\colon K \hookrightarrow \mathbb{C}_p$ other than the identity. If V is crystalline (and so de Rham and Hodge–Tate), we recover the definition above.

Lemma 114. *The rule $D \mapsto \widetilde{D}$ described above defines a fully faithful functor*

$$(\mathrm{Fil}, \varphi_q)\text{–}\mathrm{Mod}_L \xrightarrow{\;\widetilde{\cdot}\;} (\mathrm{Fil}, \varphi_p)\text{–}\mathrm{Mod}_{F'\otimes_{\mathbb{Q}_p}K,\mathrm{fr}}$$

that is compatible with tensor products. Moreover, the essential image of this functor is $(\mathrm{Fil}, \varphi_p)\text{–}\mathrm{Mod}^{\mathrm{an}}_{F'\otimes_{\mathbb{Q}_p}K,\mathrm{fr}}$.

Proof. This is analogous to lemma 3.3.1 of Kisin–Ren's article [28]. We only sketch the proof here.

It suffices to give a quasi-inverse functor

$$(\mathrm{Fil}, \varphi_p)\text{–}\mathrm{Mod}^{\mathrm{an}}_{F'\otimes_{\mathbb{Q}_p}K,\mathrm{fr}} \to (\mathrm{Fil}, \varphi_q)\text{–}\mathrm{Mod}_L.$$

Given $\widetilde{D} \in \mathrm{Ob}\big((\mathrm{Fil}, \varphi_p)\text{–}\mathrm{Mod}^{\mathrm{an}}_{F'\otimes_{\mathbb{Q}_p}K,\mathrm{fr}}\big)$, we construct $D' = L \otimes_{F'\otimes_{\mathbb{Q}_p}K} \widetilde{D}$. We have a φ_q–semilinear map

$$\varphi_q\colon D' = L \otimes_{F'\otimes_{\mathbb{Q}_p}K} \widetilde{D} \xrightarrow{\varphi_q\otimes\varphi_p^r} L \otimes_{F'\otimes_{\mathbb{Q}_p}K} \widetilde{D} = D'$$

and a filtration on D' coming from that on $\widetilde{D}$. Then the decomposition

$$F' \otimes_{\mathbb{Q}_p} K \cong \bigoplus_{i=0}^{r-1} (F' \otimes_{\varphi_p^i, F} K)$$

allows one to define a canonical isomorphism $\widetilde{D}' \cong \widetilde{D}$ from the equality on identity components. Then one checks easily that the two functors are quasi-inverse to each other and preserve tensor products. ☐

7.4.5 The equivalence for crystalline K–analytic representations

Lemma 115. *Let $D \in \mathrm{Ob}\big((\mathrm{Fil}, \varphi_q)\text{–}\mathrm{Mod}_L\big)$. Then*

$$t_N(D) = t_N(\widetilde{D}) \quad \textit{and} \quad t_H(D) = t_H(\widetilde{D})$$

and D is weakly admissible if and only if $\widetilde{D}$ is weakly admissible.

Proof. See lemma 3.3.2 of Kisin–Ren's article [28], whose proof works verbatim in the *relative* Lubin–Tate situation. ☐

Corollary 116 (Kisin–Ren). *The functors*

$$\mathrm{Rep}_K^{\mathrm{crys,an}}(G_L) \underset{\mathbf{V}_{\mathrm{crys},K}}{\overset{\mathbf{D}_{\mathrm{crys},K}}{\rightleftarrows}} (\mathrm{Fil}, \varphi_q)\text{–}\mathrm{Mod}_L^{\mathrm{wa}}$$

are exact quasi-inverse equivalences of categories that are compatible with tensor products and duality.

Proof. Lemmata 114 and 115 show that we have a commutative diagram of functors

$$\begin{array}{ccc} \mathrm{Rep}_K^{\mathrm{crys}}(G_L) & \underset{\mathbf{V}_{\mathrm{crys}}}{\overset{\mathbf{D}_{\mathrm{crys}}}{\rightleftarrows}} & (\mathrm{Fil}, \varphi_p)\text{–}\mathrm{Mod}_{F'\otimes_{\mathbb{Q}_p}K,\mathrm{fr}}^{\mathrm{wa}} \\ \big\uparrow & & \big\uparrow \wr \\ \mathrm{Rep}_K^{\mathrm{crys,an}}(G_L) & \underset{\mathbf{V}_{\mathrm{crys},K}}{\overset{\mathbf{D}_{\mathrm{crys},K}}{\rightleftarrows}} & (\mathrm{Fil}, \varphi_q)\text{–}\mathrm{Mod}_L^{\mathrm{wa}} \end{array}$$

and so the result is a direct consequence of theorem 112. More precisely, lemma 114 implies that there are canonical functorial isomorphisms

$$\mathbf{D}_{\mathrm{crys}}(V) \cong \mathbf{D}_{\mathrm{crys},K}(V)^{\sim} \quad \text{for all } V \in \mathrm{Ob}\big(\mathrm{Rep}_K^{\mathrm{crys,an}}(G_L)\big)$$

and proposition 3.3.4 of Kisin–Ren's article [28] gives canonical functorial isomorphisms

$$\mathbf{V}_{\mathrm{crys}}(\widetilde{D}) \cong \mathbf{V}_{\mathrm{crys},K}(D) \quad \text{for all } D \in \mathrm{Ob}\big((\mathrm{Fil}, \varphi_q)\text{–}\mathrm{Mod}_L^{\mathrm{wa}}\big)$$

(or rather their duals). □

7.5 Wach modules

In this subsection we combine the results relating (φ_q, Γ_L)–modules with integral (i.e., $\mathscr{O}_K$–linear) representations of G_L with the results for crystalline K–analytic representations. We do so by means of Wach modules.

Definition 117.

(1) A *free φ_q–module over $\mathbf{A}_L^+$* is a finite free $\mathbf{A}_L^+$–module N endowed with a φ_q–semilinear morphism $\varphi_q = \varphi_N \colon N \to N[Q(\omega_\phi)^{-1}]$ with the property that the $\mathbf{A}_L^+$–linearization $1 \otimes \varphi_q \colon \varphi_q^*(N)[Q(\omega_\phi)^{-1}] \to N[Q(\omega_\phi)^{-1}]$ is an isomorphism. We write $\varphi_q\text{–}\mathrm{Mod}_{\mathbf{A}_L^+,\mathrm{fr}}$ for the category of such modules.

(2) A *free (φ_q, Γ_L)–module over $\mathbf{A}_L^+$* is a free φ_q–module N over $\mathbf{A}_L^+$ equipped with an $\mathbf{A}_L^+$–semilinear continuous action of Γ_L commuting with φ_q and such that the induced action on $N/\omega_\phi N$ is trivial. We write $(\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}_{\mathbf{A}_L^+,\mathrm{fr}}$ for the category of such modules.

Recall that we fixed an isomorphism $\mathbf{A}_L^{\prime,+} \cong \mathbf{A}_L^+$ defined by $Z \mapsto \omega_\phi$. Combining this isomorphism with the natural inclusion $\mathbf{A}_L^{\prime,+} \hookrightarrow \mathbf{B}_{\mathrm{rig},L}^+$, we obtain an inclusion $\mathbf{A}_L^+ \hookrightarrow \mathbf{B}_{\mathrm{rig},L}^+$ characterized by $\omega_\phi \mapsto Z$. Then definition 117 is completely analogous to definitions 92 and 93. In fact, base change gives a functor $\mathbf{B}_{\mathrm{rig},L}^+ \otimes_{\mathbf{A}_L^+} \cdot \colon (\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}_{\mathbf{A}_L^+,\mathrm{fr}} \to (\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}_{\mathbf{B}_{\mathrm{rig},L}^+,\mathrm{fr}}$.

Definition 118. A module $N \in \mathrm{Ob}\big((\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}_{\mathbf{A}_L^+,\mathrm{fr}}\big)$ is called *$\mathscr{O}_K$–analytic* if its base change $\mathbf{B}_{\mathrm{rig},L}^+ \otimes_{\mathbf{A}_L^+} N$ is in $(\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}_{\mathbf{B}_{\mathrm{rig},L}^+,\mathrm{fr}}^{\mathrm{an}}$. We write $(\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}_{\mathbf{A}_L^+,\mathrm{fr}}^{\mathrm{an}}$ for the full subcategory of $\mathscr{O}_K$–analytic objects of $(\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}_{\mathbf{A}_L^+,\mathrm{fr}}$.

Similarly, base change by the natural inclusion $\mathbf{A}_L^+ \hookrightarrow \mathbf{A}_L$ induces a functor $\mathbf{A}_L \otimes_{\mathbf{A}_L^+} \cdot \colon (\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}_{\mathbf{A}_L^+,\mathrm{fr}} \to (\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}_{\mathbf{A}_L,\mathrm{fr}}^{\text{ét}}$ because Q is invertible in $\mathbf{A}_L'$. Putting together the results of the previous subsections, we obtain a diagram of functors

$$
\begin{array}{ccccc}
(\varphi_q,\Gamma_L)\text{–}\mathrm{Mod}^{\text{ét}}_{\mathbf{A}_L,\mathrm{fr}} & & \underset{\mathbf{D}}{\overset{\mathbf{V}}{\rightleftarrows}}\ (\cong) & & \mathrm{Rep}_{\mathscr{O}_K,\mathrm{fr}}(G_L) \\
\Big\uparrow {\scriptstyle \mathbf{A}_L\otimes_{\mathbf{A}_L^+}\cdot} & & & & \Big\uparrow \\
(\varphi_q,\Gamma_L)\text{–}\mathrm{Mod}^{\mathrm{an}}_{\mathbf{A}_L^+,\mathrm{fr}} & & \overset{?}{\longleftarrow\dashrightarrow} & & \mathrm{Rep}^{\mathrm{crys,an}}_{\mathscr{O}_K,\mathrm{fr}}(G_L) \\
\Big\downarrow {\scriptstyle \mathbf{B}^+_{\mathrm{rig},L}\otimes_{\mathbf{A}_L^+}\cdot} & & & & \Big\downarrow {\scriptstyle K\otimes_{\mathscr{O}_K}\cdot} \\
(\varphi_q,\Gamma_L)\text{–}\mathrm{Mod}^{\mathrm{an}}_{\mathbf{B}^+_{\mathrm{rig},L},\mathrm{fr}} & \underset{\mathscr{M}}{\overset{D}{\rightleftarrows}}\ (\cong) & (\mathrm{Fil},\varphi_q)\text{–}\mathrm{Mod}_L & & \\
\Big\uparrow & & \Big\uparrow & & \\
(\varphi_q,\Gamma_L)\text{–}\mathrm{Mod}^{0,\mathrm{an}}_{\mathbf{B}^+_{\mathrm{rig},L},\mathrm{fr}} & \underset{\mathscr{M}}{\overset{D}{\rightleftarrows}}\ (\cong) & (\mathrm{Fil},\varphi_q)\text{–}\mathrm{Mod}^{\mathrm{wa}}_L & \underset{\mathbf{D}_{\mathrm{crys},K}}{\overset{\mathbf{V}_{\mathrm{crys},K}}{\rightleftarrows}}\ (\cong) & \mathrm{Rep}^{\mathrm{crys,an}}_K(G_L) \\
 & & \Big\downarrow {\scriptstyle \sim} & & \Big\downarrow \\
 & & (\mathrm{Fil},\varphi_p)\text{–}\mathrm{Mod}^{\mathrm{wa}}_{F'\otimes_{\mathbb{Q}_p}K,\mathrm{fr}} & \underset{\mathbf{D}_{\mathrm{crys}}}{\overset{\mathbf{V}_{\mathrm{crys}}}{\rightleftarrows}}\ (\cong) & \mathrm{Rep}^{\mathrm{crys}}_K(G_L)
\end{array}
$$

in which the horizontal pairs of arrows are quasi-inverse equivalences of categories and the small squares are commutative. Our goal in this subsection is to fill in the dashed arrow with an equivalence of categories that makes the whole diagram commutative.

7.5.1 The functors $\mathbf{V}^*$

As the upper half of the previous diagram shows, in order to define the functor $(\varphi_q,\Gamma_L)\text{–}\mathrm{Mod}^{\mathrm{an}}_{\mathbf{A}_L^+,\mathrm{fr}} \to \mathrm{Rep}^{\mathrm{crys,an}}_{\mathscr{O}_K,\mathrm{fr}}$ we just need to check that the composition of $\mathbf{A}_L\otimes_{\mathbf{A}_L^+}\cdot$ with $\mathbf{V}$ lands in the right subcategory. To study this composition, however, it is be more convenient to work with the *dual* (or contravariant version) of the functor $\mathbf{V}$: for $M\in\mathrm{Ob}\big(\varphi_q\text{–}\mathrm{Mod}^{\text{ét}}_{\mathbf{A}_L,\mathrm{fr}}\big)$, we define

$$\mathbf{V}^*(M)=\mathrm{Hom}_{\mathbf{A}_L,\varphi_q}(M,\mathbf{A})\cong\big(\mathbf{A}\otimes_{\mathbf{A}_L}\mathrm{Hom}_{\mathbf{A}_L}(M,\mathbf{A}_L)\big)^{\varphi_q=1}.$$

Analogously, and by an abuse of notation that will be justified by lemma 119, for $N\in\mathrm{Ob}\big(\varphi_q\text{–}\mathrm{Mod}_{\mathbf{A}_L^+,\mathrm{fr}}\big)$ we set

$$\mathbf{V}^*(N)=\mathrm{Hom}_{\mathbf{A}_L^+,\varphi_q}(N,\mathbf{A}^+[\omega_\phi^{-1}]).$$

Lemma 119. *Let* $N\in\mathrm{Ob}\big(\varphi_q\text{–}\mathrm{Mod}_{\mathbf{A}_L^+,\mathrm{fr}}\big)$.

(1) *If the map* φ_q *on* N *induces a morphism* $\varphi_q^*(N)\to N$ *(without inverting* $Q(\omega_\phi)$*),*

then

$$\mathbf{V}^*(N) = \mathrm{Hom}_{\mathbf{A}_L^+,\varphi_q}(N, \mathbf{A}^+).$$

(2) *The natural map*

$$\mathbf{V}^*(N) \to \mathbf{V}^*(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} N)$$

is an isomorphism and both sides are free $\mathscr{O}_K$–modules of the same rank as N.

Proof. See lemma 3.2.1 of Kisin–Ren's article [28], whose proof works exactly in the same way for the *relative* Lubin–Tate situation. □

In order to prove that, for every $N \in \mathrm{Ob}\big((\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{\mathrm{an}}_{\mathbf{A}_L^+,\mathrm{fr}}\big)$, the representation $\mathbf{V}^*(N)$ is crystalline and $\mathscr{O}_K$–analytic, we need to study more carefully the lower half of the previous large diagram.

7.5.2 Isogeny categories

Observe that the functor $K \otimes_{\mathscr{O}_K} \cdot : \mathrm{Rep}^{\mathrm{crys,an}}_{\mathscr{O}_K,\mathrm{fr}} \to \mathrm{Rep}^{\mathrm{crys,an}}_K$ factors through the isogeny category $\mathrm{Rep}^{\mathrm{crys,an}}_{\mathscr{O}_K,\mathrm{fr}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ (i.e., the category obtained by applying $\cdot \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ to the Hom modules of $\mathrm{Rep}^{\mathrm{crys,an}}_{\mathscr{O}_K,\mathrm{fr}}$). As a matter of fact, we have an equivalence of categories $K \otimes_{\mathscr{O}_K} \cdot : \mathrm{Rep}^{\mathrm{crys,an}}_{\mathscr{O}_K,\mathrm{fr}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong \mathrm{Rep}^{\mathrm{crys,an}}_K$. There is an analogous result for (φ_q, Γ_L)–modules:

Proposition 120. *There is an equivalence of categories*

$$\varphi_q\text{–}\mathrm{Mod}_{\mathbf{A}_L^+,\mathrm{fr}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong \varphi_q\text{–}\mathrm{Mod}^0_{\mathbf{B}^+_{\mathrm{rig},L},\mathrm{fr}}$$

given by $N \mapsto \mathbf{B}^+_{\mathrm{rig},L} \otimes_{\mathbf{A}_L^+} N$ that is exact and compatible with tensor products, where

- *$\varphi_q\text{–}\mathrm{Mod}_{\mathbf{A}_L^+,\mathrm{fr}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is the isogeny category obtained by applying $\cdot \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ to the* Hom *modules of $\varphi_q\text{–}\mathrm{Mod}_{\mathbf{A}_L^+,\mathrm{fr}}$ and*
- *$\varphi_q\text{–}\mathrm{Mod}^0_{\mathbf{B}^+_{\mathrm{rig},L},\mathrm{fr}}$ is the full subcategory of $\varphi_q\text{–}\mathrm{Mod}_{\mathbf{B}^+_{\mathrm{rig},L},\mathrm{fr}}$ of modules that are pure of slope* 0*.*

Consequently, we obtain an equivalence of categories

$$(\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{\mathrm{an}}_{\mathbf{A}_L^+,\mathrm{fr}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong (\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{0,\mathrm{an}}_{\mathbf{B}^+_{\mathrm{rig},L},\mathrm{fr}}$$

defined in the same way.

Proof. The characterization of $\mathscr{M} \in \mathrm{Ob}\big(\varphi_q\text{–}\mathrm{Mod}_{\mathbf{B}^+_{\mathrm{rig},L},\mathrm{fr}}\big)$ being pure of slope 0 in terms of φ_q–stable lattices of $\mathscr{M}_{\mathbf{B}^\dagger_{\mathrm{rig},L}}$ implies that the objects obtained by base change from $\varphi_q\text{–}\mathrm{Mod}_{\mathbf{A}_L^+,\mathrm{fr}}$ are in $\varphi_q\text{–}\mathrm{Mod}^0_{\mathbf{B}^+_{\mathrm{rig},L},\mathrm{fr}}$.

As for the equivalence of categories, the proof of lemma 1.3.13 of Kisin's article [27] works exactly in the same way (ignoring the monodromy). □

Proposition 121. *Let* $N \in \mathrm{Ob}\big((\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{\mathrm{an}}_{\mathbf{A}_L^+,\mathrm{fr}}\big)$ *and let* $D = \boldsymbol{D}\big(\mathbf{B}^+_{\mathrm{rig},L} \otimes_{\mathbf{A}_L^+} N\big)$. *There is a canonical* G_L*–equivariant bijection*

$$K \otimes_{\mathscr{O}_K} \mathbf{V}^*(N) \longrightarrow \mathbf{V}^*_{\mathrm{crys},K}(D) = \mathrm{Hom}_{K,\mathrm{Fil},\varphi_q}(D, \mathbf{B}_{\mathrm{crys},K})$$

that is compatible with tensor products. In particular, both sides have the same dimension over K.

Proof. This is proposition 3.2.3 of Kisin–Ren's article [28]. We reproduce it here for the convenience of the reader.

Recall that ω_ϕ and t_ϕ are units in $\mathbf{B}_{\mathrm{crys},K}$ and so $\lambda(\omega_\phi) = t_\phi/\omega_\phi$ and $Q(\omega_\phi)$ are invertible in $\mathbf{B}_{\mathrm{crys},K}$ too.

Write $\mathscr{M} = \mathbf{B}^+_{\mathrm{rig},L} \otimes_{\mathbf{A}_L^+} N$ and consider the natural maps

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{A}_L^+,\varphi_q}(N, \mathbf{A}^+[\omega_\phi^{-1}]) & \longrightarrow & \mathrm{Hom}_{\mathbf{B}^+_{\mathrm{rig},L},\varphi_q}(\mathscr{M}, \mathbf{B}_{\mathrm{crys},K}) \\ & \searrow & \downarrow \\ & & \mathrm{Hom}_{\mathbf{B}^+_{\mathrm{rig},L},\varphi_q}(\varphi_q^*(\mathscr{M}), \mathbf{B}_{\mathrm{crys},K}) \end{array}$$

(the vertical arrow is well-defined because $Q(\omega_\phi)$ is invertible in $\mathbf{B}_{\mathrm{crys},K}$). We claim that the image of the composition consists of morphisms that respect the filtrations.

An $\mathbf{A}_L^+$–linear map $f\colon N \to \mathbf{A}^+$ induces $f_1\colon \mathscr{M} \to \mathbf{B}_{\mathrm{crys},K}$ by composition with $\mathbf{A}^+ \hookrightarrow \mathbf{B}^+_{\mathrm{crys},K} \hookrightarrow \mathbf{B}_{\mathrm{crys},K}$ and base change by $\mathbf{A}_L^+ \hookrightarrow \mathbf{B}^+_{\mathrm{rig},L}$. Localization at Q yields $f_2\colon \mathscr{M}[Q^{-1}] \to \mathbf{B}_{\mathrm{crys},K}$ and precomposition with $(1 \otimes \varphi_q)\colon \varphi^*(\mathscr{M}) \to \mathscr{M}[Q^{-1}]$ gives $f_3\colon \varphi_q^*(\mathscr{M}) \to \mathbf{B}_{\mathrm{crys},K}$. Consider $x \in \mathrm{Fil}^i\big(\varphi_q^*(\mathscr{M})\big)$ for some $i \in \mathbb{Z}$, which means that $(1 \otimes \varphi_q)(x) \in Q^i\mathscr{M}$. Then $f_3(x) \in Q(\omega_\phi)^i \cdot \mathbf{B}^+_{\mathrm{crys},K} \subset \mathrm{Fil}^i(\mathbf{B}_{\mathrm{dR}})$ because $Q(\omega_\phi) \in \mathrm{Fil}^1(\mathbf{B}_{\mathrm{dR}})$ by proposition 2.1.19 of Schneider's book [31]. Therefore, $f_3 \in \mathrm{Hom}_{\mathbf{B}^+_{\mathrm{rig},L},\mathrm{Fil},\varphi_q}(\varphi_q^*(\mathscr{M}), \mathbf{B}_{\mathrm{crys},K})$ as claimed.

Next, consider the inclusions

$$\varphi_q^*(\mathscr{M})[Q^{-1}] \subseteq \varphi_q^*(\mathscr{M})[\lambda^{-1}] \supseteq \varphi_q^*\big(\mathscr{M}[\lambda^{-1}]\big).$$

Since these modules differ by elements of $\mathbf{B}^\times_{\mathrm{crys},K}$, having f_3 is equivalent to having $f_4\colon \varphi_q^*\big(\mathscr{M}[\lambda^{-1}]\big) \to \mathbf{B}_{\mathrm{crys},K}$. But lemma 101 gives a map $\xi\colon D \to \mathscr{M}[\lambda^{-1}]$ and finally we can compose f_4 with $1 \otimes \xi$ to obtain the desired $f_5\colon D \to \mathbf{B}_{\mathrm{crys},K}$. By the definition of the filtration on $D = \boldsymbol{D}(\mathscr{M})$ in terms of ξ, we conclude that $f_5 \in \mathrm{Hom}_{K,\mathrm{Fil},\varphi_q}(D, \mathbf{B}_{\mathrm{crys},K})$.

In this way, we have constructed a canonical G_L–equivariant map

$$K \otimes_{\mathscr{O}_K} \mathrm{Hom}_{\mathbf{A}_L^+,\varphi_q}(N, \mathbf{A}^+[\omega_\phi^{-1}]) \to \mathrm{Hom}_{K,\mathrm{Fil},\varphi_q}(D, \mathbf{B}_{\mathrm{crys},K})$$

given by $f \mapsto f_5$. Tracing the steps of the construction, one checks that this map is compatible with tensor products and injective (the only non-clear step is $f_4 \mapsto f_5$, but that follows from lemma 101). Finally, comparing dimensions we obtain the bijectivity. □

7.5.3 The equivalence of categories

Theorem 122 (Kisin–Ren). *The functor $N \mapsto \mathbf{V}(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} N)$ induces an exact equivalence of categories*

$$\mathbf{V}\colon (\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{\mathrm{an}}_{\mathbf{A}_L^+,\mathrm{fr}} \xrightarrow{\cong} \mathrm{Rep}^{\mathrm{crys,an}}_{\mathscr{O}_K,\mathrm{fr}}(G_L)$$

that is compatible with tensor products and duals.

Proof. This is corollary 3.3.8 of Kisin–Ren's article [28]. We reproduce its proof here for the convenience of the reader.

Combining lemma 119 and proposition 121, we see that $N \mapsto \mathbf{V}(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} N)$ induces a fully faithful functor $(\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{\mathrm{an}}_{\mathbf{A}_L^+,\mathrm{fr}} \to \mathrm{Rep}^{\mathrm{crys,an}}_{\mathscr{O}_K,\mathrm{fr}}$ that is exact and compatible with tensor products and duals.

It remains to prove that the functor is essentially surjective. To that aim, take $T \in \mathrm{Ob}\big(\mathrm{Rep}^{\mathrm{crys,an}}_{\mathscr{O}_K,\mathrm{fr}}(G_L)\big)$ and consider $M = \mathbf{D}(T) \in \mathrm{Ob}\big((\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{\text{ét}}_{\mathbf{A}_L,\mathrm{fr}}\big)$ and $\mathscr{M} = \mathscr{M}(\mathbf{D}_{\mathrm{crys},K}(K \otimes_{\mathscr{O}_K} T)) \in \mathrm{Ob}\big((\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{0,\mathrm{an}}_{\mathbf{B}^+_{\mathrm{rig},L},\mathrm{fr}}\big)$. By proposition 120, we find $N' \in \mathrm{Ob}\big((\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{\mathrm{an}}_{\mathbf{A}_L^+,\mathrm{fr}}\big)$ such that $\mathscr{M} = \mathbf{B}^+_{\mathrm{rig},L} \otimes_{\mathbf{A}_L^+} N'$. Therefore, lemma 119 and proposition 121 imply that the representation $\mathbf{V}(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} N')$ is isomorphic to a G_L–stable $\mathscr{O}_K$–lattice T' of $K \otimes_{\mathscr{O}_K} T$.

Now we can use the equivalence of theorem 74, which provides isomorphisms

$$\mathbf{B}_L \otimes_{\mathbf{A}_L^+} N' \cong \mathbf{D}(K \otimes_{\mathscr{O}_K} T') \cong \mathbf{D}(K \otimes_{\mathscr{O}_K} T) \cong \mathbf{B}_L \otimes_{\mathbf{A}_L} M$$

that we regard as an identification. We define

$$N = M \cap N'[p^{-1}] \subset M[p^{-1}] = \mathbf{B}_L \otimes_{\mathbf{A}_L} M.$$

Since $\mathbf{A}_L \cap \mathbf{A}_L^+[p^{-1}] = \mathbf{A}_L^+$ (inside $\mathbf{B}_L$), we see that N is a finite (φ_q, Γ_L)–module over $\mathbf{A}_L^+$. Using the structure theorem of finite modules over $\mathbf{A}_L^+$ (equivalently, over $\mathbf{A}_L^{\prime,+} = \mathscr{O}_L[\![Z]\!]$), one can check that N is free over $\mathbf{A}_L^+$ (cf. the argument at the end of lemma 1.3.13 of Kisin's article [27]). All in all, $N \in \mathrm{Ob}\big((\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{\mathrm{an}}_{\mathbf{A}_L^+,\mathrm{fr}}\big)$ and $\mathbf{A}_L \otimes_{\mathbf{A}_L^+} N \cong M$. We obtain isomorphisms $\mathbf{V}(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} N) \cong \mathbf{V}(M) \cong T$. $\square$

The proof of theorem 122 shows that, for every $T \in \mathrm{Ob}\big(\mathrm{Rep}^{\mathrm{crys,an}}_{\mathscr{O}_K,\mathrm{fr}}(G_L)\big)$, there exists an $\mathbf{A}_L^+$–submodule N of $\mathbf{D}(T)$ that inherits actions of φ_q and Γ_L making it an object of $(\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{\mathrm{an}}_{\mathbf{A}_L^+,\mathrm{fr}}$ and with the property that $\mathbf{A}_L \otimes_{\mathbf{A}_L^+} N = \mathbf{D}(T)$. We fix once and for all such an N for every T forming a quasi-inverse

$$\mathbf{N}\colon\ \mathrm{Rep}^{\mathrm{crys,an}}_{\mathscr{O}_K,\mathrm{fr}}(G_L) \to (\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{\mathrm{an}}_{\mathbf{A}_L^+,\mathrm{fr}}$$

to the functor $\mathbf{V}$ of theorem 122. The results of this section can be summarized in the commutative (up to natural isomorphisms) diagram of functors

$$\begin{array}{ccc}
(\varphi_q,\Gamma_L)\text{–}\mathrm{Mod}^{\text{ét}}_{\mathbf{A}_L,\mathrm{fr}} & \underset{\mathbf{D}}{\overset{\mathbf{V}}{\rightleftarrows}}\ (\cong\ (1)) & \mathrm{Rep}_{\mathscr{O}_K,\mathrm{fr}}(G_L) \\
\uparrow {\scriptstyle \mathbf{A}_L \otimes_{\mathbf{A}_L^+} \cdot} & & \uparrow \\
(\varphi_q,\Gamma_L)\text{–}\mathrm{Mod}^{\mathrm{an}}_{\mathbf{A}_L^+,\mathrm{fr}} & \underset{\mathbf{N}}{\overset{\mathbf{V}}{\rightleftarrows}}\ (\cong\ (2)) & \mathrm{Rep}^{\mathrm{crys,an}}_{\mathscr{O}_K,\mathrm{fr}}(G_L) \\
\downarrow & & \downarrow \\
(\varphi_q,\Gamma_L)\text{–}\mathrm{Mod}^{\mathrm{an}}_{\mathbf{A}_L^+,\mathrm{fr}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p & & \mathrm{Rep}^{\mathrm{crys,an}}_{\mathscr{O}_K,\mathrm{fr}}(G_L) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \\
\downarrow {\scriptstyle \mathbf{B}^+_{\mathrm{rig},L} \otimes_{\mathbf{A}_L^+} \cdot}\ \cong (3) & & \downarrow {\scriptstyle K \otimes_{\mathscr{O}_K} \cdot}\ \cong \\
(\varphi_q,\Gamma_L)\text{–}\mathrm{Mod}^{0,\mathrm{an}}_{\mathbf{B}^+_{\mathrm{rig},L},\mathrm{fr}} \ \underset{\mathscr{M}}{\overset{D}{\rightleftarrows}}\ (\cong\ (4)) & (\mathrm{Fil}, \varphi_q)\text{–}\mathrm{Mod}^{\mathrm{wa}}_L & \underset{\mathbf{D}_{\mathrm{crys},K}}{\overset{\mathbf{V}_{\mathrm{crys},K}}{\rightleftarrows}}\ (\cong\ (5))\ \mathrm{Rep}^{\mathrm{crys,an}}_K(G_L)
\end{array}$$

where the equivalences of categories are:

(1) theorem 74 (section 7.1);
(2) theorem 122 (section 7.5);
(3) proposition 120 (section 7.5);
(4) theorem 104 and proposition 107 (section 7.3), and
(5) theorem 112 and corollary 116 (section 7.4).

7.5.4 Behaviour with respect to Hodge–Tate weights

Next we want to study, for $T \in \mathrm{Ob}\big(\mathrm{Rep}^{\mathrm{crys,an}}_{\mathscr{O}_K,\mathrm{fr}}(G_L)\big)$, the relations between the Wach module $\mathbf{N}(T)$ and certain submodules defined in terms of the operators φ_q and ψ_q depending on the Hodge–Tate weights of $V = K \otimes_{\mathscr{O}_K} T$.

By the definition of K–analyticity, most of those Hodge–Tate weights must be 0. More precisely, if $m = \dim_K(V)$, then there are $m \cdot [K : \mathbb{Q}_p]$ weights, from which only the m corresponding to the identity component of $\mathbb{C}_p \otimes_{\mathbb{Q}_p} V$ might be non-zero. We call the other $m\big([K : \mathbb{Q}_p] - 1\big)$ weights the *trivial weights*. We also adopt the convention that the cyclotomic character χ_{cyc} has weight 1.

If the symbol ? is an interval $[a, b]$ (with $a, b \in \mathbb{Z}$) or ≤ 0 or ≥ 0 (for the intervals $(-\infty, 0]$ or $[0, \infty)$, respectively), we write $\mathrm{Rep}_K^{\mathrm{crys,an,?}}(G_L)$ for the full subcategory of representations in $\mathrm{Rep}_K^{\mathrm{crys,an}}(G_L)$ whose Hodge–Tate weights lie in the corresponding interval. Similarly, we write $\mathrm{Rep}_{\mathscr{O}_K,\mathrm{fr}}^{\mathrm{crys,an,?}}(G_L)$ for the full subcategory of $T \in \mathrm{Ob}\big(\mathrm{Rep}_{\mathscr{O}_K,\mathrm{fr}}^{\mathrm{crys,an}}(G_L)\big)$ such that $K \otimes_{\mathscr{O}_K} T \in \mathrm{Ob}\big(\mathrm{Rep}_K^{\mathrm{crys,an,?}}(G_L)\big)$.

Lemma 123. *Let $T \in \mathrm{Ob}\big(\mathrm{Rep}_{\mathscr{O}_K,\mathrm{fr}}^{\mathrm{crys,an,}[a,b]}(G_L)\big)$ for some interval $[a, b]$. If we write $N = \mathbf{N}(T)$, then*

$$Q(\omega_\phi)^{-a} N \subseteq \mathbf{A}_L^+ \cdot \varphi_q(N) \subseteq Q(\omega_\phi)^{-b} N.$$

Proof. The analogous result for $\mathscr{M} = \mathbf{B}_{\mathrm{rig},L}^+ \otimes_{\mathbf{A}_L^+} N$ is corollary 3.37.i of Berger–Schneider–Xie's article [8] (where they use the notation I_{Z_0} for the ideal generated by Q and normalize the weights in such a way that our interval $[a, b]$ is their $[-b, -a]$).

The statement of the lemma follows with the same argument as in remark 1.6 of Schneider–Venjakob's preprint [35], checking that $\mathbf{A}_L^{\prime,+}[Q^{-1}] \cap \mathbf{B}_{\mathrm{rig},L}^+ = \mathbf{A}_L^{\prime,+}$. □

Lemma 124. *Let $T \in \mathrm{Ob}\big(\mathrm{Rep}_{\mathscr{O}_K,\mathrm{fr}}^{\mathrm{crys,an}}(G_L)\big)$.*

(1) *The Wach module $\mathbf{N}(T)$ is the unique $\mathbf{A}_L^+$–submodule of $\mathbf{D}(T)$ which inherits actions of φ_q and Γ_L making it an object in (φ_q, Γ_L)–$\mathrm{Mod}_{\mathbf{A}_L^+,\mathrm{fr}}^{\mathrm{an}}$ and such that $\mathbf{A}_L \otimes_{\mathbf{A}_L^+} \mathbf{N}(T) = \mathbf{D}(T)$.*

(2) *For every $r \in \mathbb{Z}$, $\mathbf{N}(T(\chi_\phi^r)) = \omega_\phi^{-r} \mathbf{N}(T) \otimes t_0^r$, where t_0 is a generator of $\mathscr{O}_K(\chi_\phi)$.*

Proof. See proposition 1.10 and lemma 1.12 of Schneider–Venjakob's preprint [35], whose proofs work verbatim in the *relative* Lubin–Tate situation too. □

Since the Lubin–Tate character χ_ϕ has (non-trivial) Hodge–Tate weight 1, up to twisting we may always work with representations whose Hodge–Tate weights are ≥ 0, which is convenient for some proofs.

7.5.5 Comparison between Wach modules and $\mathbf{D}_{\mathrm{crys},K}$

Lemma 125. *Let $T \in \mathrm{Ob}\big(\mathrm{Rep}_{\mathscr{O}_K,\mathrm{fr}}^{\mathrm{crys,an,}[-r,0]}(G_L)\big)$ with $r \in \mathbb{Z}_{\geq 0}$ and set $V = K \otimes_{\mathscr{O}_K} T$, $D = \mathbf{D}_{\mathrm{crys},K}(V)$ and $\mathscr{M} = \mathbf{B}_{\mathrm{rig},L}^+ \otimes_{\mathbf{A}_L^+} \mathbf{N}(T)$. Identifying D with $\mathbf{D}(\mathscr{M})$, the map ξ*

from lemma 101 induces inclusions

$$\lambda^r \mathscr{M} \subseteq (1 \otimes \xi)\big(\mathbf{B}^+_{\mathrm{rig},L} \otimes_L D\big) \subseteq \mathscr{M}.$$

Proof. The definition of the functor $\mathscr{M}$ (see section 7.3.3) gives inclusions

$$\mathbf{B}^+_{\mathrm{rig},L} \otimes_L D \subseteq \mathscr{M}(D) \subseteq \lambda^{-r}\mathbf{B}^+_{\mathrm{rig},L} \otimes_L D.$$

Applying $1 \otimes \xi$ and using lemma 103 we obtain

$$(1 \otimes \xi)(\mathbf{B}^+_{\mathrm{rig},L} \otimes_L D) \subseteq \mathscr{M} \subseteq (1 \otimes \xi)(\lambda^{-r}\mathbf{B}^+_{\mathrm{rig},L} \otimes_L D),$$

whence the lemma follows. □

Definition 126. Let $V \in \mathrm{Ob}\big(\mathrm{Rep}^{\mathrm{crys,an}}_K(G_L)\big)$ and choose a G_L–stable $\mathscr{O}_K$–lattice T of V. We define

$$\mathbf{N}(V) = \mathbf{N}(T)[p^{-1}] \quad \text{and} \quad \mathbf{N}^{(\varphi_q)}(V) = \mathbf{A}^+_L \cdot \varphi_q(\mathbf{N}(V)) \subset \mathbf{N}(V)[Q(\omega_\phi)^{-1}].$$

Remark. Since $Q(\omega_\phi)$ is already invertible in $\mathbf{A}_L$, we can view $\mathbf{N}^{(\varphi_q)}(V) \subset \mathbf{D}(V)$. The operator ψ_q on $\mathbf{D}(V)$ restricts to

$$\psi_q \colon \mathbf{N}^{(\varphi_q)}(V) \to \mathbf{N}(V).$$

Proposition 127. *Let $T \in \mathrm{Rep}^{\mathrm{crys,an},\geq 0}_{\mathscr{O}_K,\mathrm{fr}}(G_L)$ and set $V = K \otimes_{\mathscr{O}_K} T$. Consider also $\mathscr{M} = \mathbf{B}^+_{\mathrm{rig},L} \otimes_{\mathbf{A}^+_L} \mathbf{N}(T)$. Identifying $\boldsymbol{D}(\mathscr{M})$ with $\mathbf{D}_{\mathrm{crys},K}(V)$, the map ξ from lemma 101 induces inclusions*

1. $\mathbf{N}(V) \subseteq (1 \otimes \xi)\big(\mathbf{B}^+_{\mathrm{rig},L} \otimes_L \mathbf{D}_{\mathrm{crys},K}(V)\big)$,
2. $\mathbf{N}^{(\varphi_q)}(V) \subseteq (1 \otimes \xi)\big(\mathbf{B}^+_{\mathrm{rig},L} \otimes_L \mathbf{D}_{\mathrm{crys},K}(V)\big)$ *and*
3. $\big(\mathbf{N}^{(\varphi_q)}(V)\big)^{\psi_q=0} \subseteq (1 \otimes \xi)\big((\mathbf{B}^+_{\mathrm{rig},L})^{\psi_q=0} \otimes_L \mathbf{D}_{\mathrm{crys},K}(V)\big)$.

Proof. This is corollary 1.14 of Schneider–Venjakob's preprint [35]. We reproduce it here for the convenience of the reader.

Applying lemma 125 to $T(\chi_\phi^{-r})$ and using lemma 124, we get that

$$\begin{aligned} Z^r\lambda^r \mathbf{B}^+_{\mathrm{rig},L} \otimes_{\mathbf{A}^+_L} \mathbf{N}(T) \otimes t_0^{-r} &\subseteq (1 \otimes \xi)\big(\mathbf{B}^+_{\mathrm{rig},L} \otimes_L \mathbf{D}_{\mathrm{crys},K}(K \otimes_{\mathscr{O}_K} T(\chi_\phi^{-r}))\big) \\ &= \mathbf{B}^+_{\mathrm{rig},L} \otimes_L t_\phi^r \mathbf{D}_{\mathrm{crys},K}(V) \otimes t_0^{-r}. \end{aligned}$$

Since $Z\lambda(Z) = \log_\phi(Z)$ and $t_\phi = \log_\phi(\omega_\phi)$, we deduce the first inclusion. The second inclusion follows from the first and the compatibility of $1 \otimes \xi$ with φ_q.

To prove the last inclusion, take

$$x = \sum_{i=1}^{n} f_i \varphi_q(x_i) \in \mathbf{N}^{(\varphi_q)}(V)$$

such that

$$\psi_q(x) = \sum_{i=1}^{n} \psi_q(f_i) x_i = 0.$$

Let $e_1, \ldots, e_m$ be an L–basis of $\mathbf{D}_{\mathrm{crys},K}(V)$ and express

$$x_i = \sum_{j=1}^{m} g_{ij} \otimes \xi(e_j) \quad \text{with } g_{ij} \in \mathbf{B}^{+}_{\mathrm{rig},L}$$

(which is possible by the first inclusion that we already proved). By the linear independence of the e_j, the condition $\psi_q(x) = 0$ means that

$$0 = \sum_{i=1}^{n} \psi_q(f_i) g_{ij} = \psi_q\Big(\sum_{i=1}^{n} f_i \varphi_q(g_{ij})\Big)$$

for each j. Therefore,

$$x = (1 \otimes \xi)\Big(\sum_{i=1}^{n} f_i \varphi_q(x_i)\Big) = \sum_{j=1}^{m} \Big[\sum_{i=1}^{n} f_i \varphi_q(g_{ij})\Big] \otimes \xi(\varphi_q(e_j))$$

lies in $(1 \otimes \xi)\big((\mathbf{B}^{+}_{\mathrm{rig},L})^{\psi_q=0} \otimes \mathbf{D}_{\mathrm{crys},K}(V)\big)$. □

7.5.6 Comparison between $\mathbf{N}(T)^{\psi_q=1}$ and $\mathbf{D}(T)^{\psi_q=1}$

Lemma 128. *For every $m \in \mathbb{Z}_{\geq 1}$,*

$$\psi_q\Big(\frac{1}{\omega_\phi^m}\Big) \in \frac{\varphi_q^{-1}(\pi_L)^{m-1}}{\omega_\phi^m} + \frac{1}{\omega_\phi^{m-1}} \mathbf{A}_L^{+}.$$

Proof. The proof is analogous to that of lemma 1.25 of Schneider–Venjakob's preprint [35]. We adapt it to the *relative* Lubin–Tate situation here for the convenience of the reader.

Let

$$h(Z) = \psi_q(Q(Z)^m) = \psi_q\Big(\frac{\phi(Z)^m}{Z^m}\Big) \in \mathbf{A}_L^{\prime,+}.$$ □

We can write $h(Z) = \psi_q(\varphi_q(Z^m) Z^{-m}) = Z^m \psi_q(Z^{-m})$, so we need to compute $h(Z)/Z^m$. As a matter of fact, since $h(Z) - h(0) \in Z\mathbf{A}_L^{\prime,+}$, we just need to check that $h(0) = \varphi_q^{-1}(\pi_L)^{m-1}$. Indeed,

$$\begin{aligned}\varphi_q(h(0)) &= \varphi_q(h)(0) = (\varphi_q \circ \psi_q)\Big(\frac{\phi(Z)^m}{Z^m}\Big)\Big|_{Z=0} \\ &= \frac{1}{\pi_L}\sum_{v_1 \in \mathfrak{F}_{\phi,1}} \frac{\phi(\mathfrak{F}_\phi(v_1, Z))^m}{\mathfrak{F}_\phi(v_1, Z)^m}\Big|_{Z=0} = \frac{1}{\pi_L}\sum_{v_1 \in \mathfrak{F}_{\phi,1}} \frac{\phi(Z)^m}{\mathfrak{F}_\phi(v_1, Z)^m}\Big|_{Z=0} \\ &= \frac{1}{\pi_L} Q(0)^m = \pi_L^{m-1}.\end{aligned}$$

Lemma 129. *Let $T \in \mathrm{Ob}\big(\mathrm{Rep}_{\mathscr{O}_K,\mathrm{fr}}^{\mathrm{crys,an},\geq 0}(G_L)\big)$ and let $k \in \mathbb{Z}_{\geq 1}$.*

(1) *We have an inclusion*

$$\psi_q\big(\pi_L \mathbf{D}(T) + \omega_\phi^{-k}\mathbf{N}(T)\big) \subseteq \begin{cases} \pi_L \mathbf{D}(T) + \omega_\phi^{-1}\mathbf{N}(T) & \text{if } k = 1, \\ \pi_L \mathbf{D}(T) + \omega_\phi^{-k+1}\mathbf{N}(T) & \text{if } k > 1. \end{cases}$$

(2) *If $x \in \mathbf{D}(T)$ satisfies that $(\psi_q - 1)(x) = \psi_q(x) - x \in \pi_L \mathbf{D}(T) + \omega_\phi^{-k}\mathbf{N}(T)$, then $x \in \pi_L \mathbf{D}(T) + \omega_\phi^{-k}\mathbf{N}(T)$.*

Proof. The proof is completely analogous to those of lemmata 1.26 and 1.27 of Schneider–Venjakob's preprint [35], taking into account that π_L and $\varphi_q^{\pm 1}(\pi_L)$ differ (multiplicatively) by a unit and that, even if ψ_q is not $\mathscr{O}_L$–linear, it is still φ_q^{-1}–semilinear. □

Proposition 130. *For every $T \in \mathrm{Ob}\big(\mathrm{Rep}_{\mathscr{O}_K,\mathrm{fr}}^{\mathrm{crys,an},\geq 0}(G_L)\big)$,*

$$\mathbf{D}(T)^{\psi_q = 1} \subseteq \omega_\phi^{-1}\mathbf{N}(T).$$

Proof. This is analogous to lemma 1.28 of Schneider–Venjakob's preprint [35]. We adapt it here to the *relative* Lubin–Tate situation for the convenience of the reader.

We prove by induction on $k \in \mathbb{Z}_{\geq 1}$ that $\mathbf{D}(T)^{\psi_q=1} \subseteq \pi_L^k \mathbf{D}(T) + \omega_\phi^{-1}\mathbf{N}(T)$. Then taking $k \to \infty$ gives, for each $x \in \mathbf{D}(T)^{\psi_q=1}$, a sequence in $\omega_\phi^{-1}\mathbf{N}(T)$ converging to x.

The base case $k = 1$ follows directly from the second part of lemma 129 applied to $x \in \mathbf{D}(T)^{\psi_q=1}$. For the inductive step, write $x = \pi_L^k y + n$ with $y \in \mathbf{D}(T)$ and $n \in \omega_\phi^{-1}\mathbf{N}(T)$. Since $(\psi_q - 1)(x) = 0$, we get that

$$\psi_q(n) - n = -\pi_L^k\Big(\Big(\frac{\varphi_q^{-1}(\pi_L)}{\pi_L}\Big)^k \psi_q(y) - y\Big).$$

But the first part of lemma 129 shows that $\psi_q(n) - n \in \omega_\phi^{-1}\mathbf{N}(T)$. We deduce that

$$\Big(\frac{\varphi_q^{-1}(\pi_L)}{\pi_L}\Big)^k \psi_q(y) - y \in \omega_\phi^{-1}\mathbf{N}(T)$$

because $\pi_L^k\mathbf{D}(T) \cap \omega_\phi^{-1}\mathbf{N}(T) = \pi_L^k\omega_\phi^{-1}\mathbf{N}(T)$.

We claim that $y \in \pi_L\mathbf{D}(T) + \omega_\phi^{-1}\mathbf{N}(T)$. To prove it, take $l \in \mathbb{Z}_{\geq 0}$ such that $y \in \pi_L\mathbf{D}(T) + \omega_\phi^{-l}\mathbf{N}(T)$ but $y \notin \pi_L\mathbf{D}(T) + \omega_\phi^{-l+1}\mathbf{N}(T)$. If $l > 1$, then we would have $\psi_q(y) \in \pi_L\mathbf{D}(T) + \omega_\phi^{-l+1}\mathbf{N}(T)$ by the first part of lemma 129 and so

$$y = \Big(\frac{\varphi_q^{-1}(\pi_L)}{\pi_L}\Big)^k \psi_q(y) - \bigg(\Big(\frac{\varphi_q^{-1}(\pi_L)}{\pi_L}\Big)^k \psi_q(y) - y\bigg) \in \pi_L\mathbf{D}(T) + \omega_\phi^{-l+1}\mathbf{N}(T),$$

thus contradicting the choice of l. Therefore, $l \leq 1$.

The previous claim allows us to write $y = \pi_L y' + n'$ with $y' \in \mathbf{D}(T)$ and $n' \in \omega_\phi^{-1}\mathbf{N}(T)$. In conclusion, we can express $x = \pi_L^{k+1}y' + (\pi_L^k n' + n)$, which completes the proof of the inductive step. □

Lemma 131. *Let $V \in \mathrm{Ob}\big(\mathrm{Rep}_K^{\mathrm{crys,an},\geq 0}(G_L)\big)$. If $\mathbf{D}_{\mathrm{crys},K}(V)^{\varphi_q=1} \neq 0$, then V has the trivial representation K as a quotient.*

Proof. This is lemma 1.29 of Schneider–Venjakob's preprint [35]. □

Proposition 132. *Let $T \in \mathrm{Ob}\big(\mathrm{Rep}_{\mathscr{O}_K,\mathrm{fr}}^{\mathrm{crys,an},\geq 0}(G_L)\big)$. If $V = K \otimes_{\mathscr{O}_K} T$ has no quotient isomorphic to the trivial representation K (and so $\mathbf{D}_{\mathrm{crys},K}(V)^{\varphi_q=1} = 0$), then*

$$\mathbf{D}(T)^{\psi_q=1} = \mathbf{N}(T)^{\psi_q=1}.$$

Proof. See lemma 1.30 of Schneider–Venjakob's preprint [35], whose proof works in the same way for the *relative* Lubin–Tate situation too. We reproduce it here for the convenience of the reader.

By proposition 130, it suffices to prove that $(\omega_\phi^{-1}\mathbf{N}(T))^{\psi_q=1} \subseteq \mathbf{N}(T)$. Let $e_1, \ldots, e_m$ be an $\mathbf{A}_L^+$–basis of $\mathbf{N}(T)$. By lemma 123, we have $\mathbf{N}(T) \subseteq \mathbf{A}_L^+ \cdot \varphi_q(\mathbf{N}(T))$ and so we can express

$$e_i = \sum_{j=1}^m g_{ij}\varphi_q(e_j) \quad \text{with } g_{ij} \in \mathbf{A}_L^+.$$

Take

$$\omega_\phi^{-1}n = \sum_{i=1}^m f_i e_i \in (\omega_\phi^{-1}\mathbf{N}(T))^{\psi_q=1} \quad (\text{with } f_i \in \omega_\phi^{-1}\mathbf{A}_L^+).$$

Comparing the coefficients of each e_j, the condition $\psi_q(\omega_\phi^{-1}n) = \omega_\phi^{-1}n$ implies that

$$f_j = \sum_{i=1}^{m} \psi_q(f_i g_{ij}) \quad \text{for every } j.$$

Now, writing $f_{i,-1}$ (resp. $g_{ij,0}$) for the coefficient of ω_ϕ^{-1} (resp. the constant coefficient) in $f_i(\omega_\phi)$ (resp. in $g_{ij}(\omega_\phi)$),

$$\begin{aligned}\psi_q(f_i g_{ij}) \in \psi_q(f_{i,-1} \cdot g_{ij,0} \cdot \omega_\phi^{-1} + \mathbf{A}_L^+) &\subseteq \varphi_q^{-1}(f_{i,-1} g_{ij,0})\psi_q(\omega_\phi^{-1}) + \mathbf{A}_L^+ \\ &\subseteq \varphi_q^{-1}(f_{i,-1} g_{ij,0})\omega_\phi^{-1} + \mathbf{A}_L^+,\end{aligned}$$

whence

$$\varphi_q(\omega_\phi f_j) = \varphi_q\left(\sum_{i=1}^{m} \omega_\phi \psi_q(f_i g_{ij})\right) \in \sum_{i=1}^{m} f_{i,-1} g_{ij,0} + \omega_\phi \mathbf{A}_L^+$$

Combining these congruences modulo ω_ϕ with the definitions of the f_i and the g_{ij}, we conclude that

$$\begin{aligned}\varphi_q(n) = \sum_{j=1}^{m} \varphi_q(\omega_\phi f_j e_j) &\equiv \sum_{i=1}^{m}\sum_{j=1}^{m} f_{i,-1} g_{ij,0} \varphi_q(e_j) \\ &\equiv \sum_{i=1}^{m} f_{i,-1} e_i \equiv n \mod \omega_\phi \mathbf{N}(T).\end{aligned}$$

That is, the operator φ_q on $\mathbf{D}_{\mathrm{crys},K}(V) \cong \big(\mathbf{N}(T)/\omega_\phi \mathbf{N}(T)\big)[p^{-1}]$ acts as the identity on $n \mod \omega_\phi \mathbf{N}(T)$, which means that $n \equiv 0 \mod \omega_\phi \mathbf{N}(T)$ by hypothesis. □

8 Rigid character varieties and distributions

Schneider–Teitelbaum's article [33] constructed a *character variety*, whose points parametrize characters on the ring of integers of a finite extension of $\mathbb{Q}_p$, and related its rigid analytic functions to locally analytic distributions via what they called p–adic Fourier theory. Furthermore, they used a Lubin–Tate formal group (plus the theory of p–divisible groups) to show that such character variety is a *twisted version* of the open unit disc.

In this section, we recall the constructions of Schneider and Teitelbaum and adapt certain technical parts to obtain the analogous results using a *relative* Lubin–Tate formal group to give the module structure on the open unit disc. After that, we use the identification between distributions and functions on the character variety to study spaces of distributions and even some larger spaces, following the preprint [35] of Schneider and Venjakob.

8.1 The construction of the character variety

To begin with, we summarize the first two sections of Schneider–Teitelbaum's article [33], where they construct the character variety. This part does not use Lubin–Tate theory and so there are no changes, but beware that we use the symbol K where they use L (and their K has a different meaning).

8.1.1 Locally analytic characters

Let H denote the p–adic Lie group $\mathscr{O}_K$, regarded as a locally analytic manifold over K, and let H_0 be its restriction of scalars to $\mathbb{Q}_p$. We want to describe a rigid variety $\mathcal{X}$ over K whose points parametrize locally K–analytic characters of $\mathscr{O}_K$. Such $\mathcal{X}$ will be a closed subvariety of a larger $\mathcal{X}_0$ whose points parametrize locally $\mathbb{Q}_p$–analytic characters of $\mathscr{O}_K$. (The symbols H and H_0 are introduced to avoid confusions between the two notions of analyticity, depending on the base field one wants to consider.)

Let $C^{\mathrm{an}}(H, \mathbb{C}_p)$ (resp. $C^{\mathrm{an}}(H_0, \mathbb{C}_p)$) denote the locally convex $\mathbb{C}_p$–vector space of locally analytic functions on H (resp. on H_0) with values in $\mathbb{C}_p$. Consider the subsets $\widehat{H}(\mathbb{C}_p) \subset C^{\mathrm{an}}(H, \mathbb{C}_p)$ and $\widehat{H}_0(\mathbb{C}_p) \subset C^{\mathrm{an}}(H_0, \mathbb{C}_p)$ of locally analytic characters. We define the space of locally analytic distributions $D(H, \mathbb{C}_p)$ (resp. $D(H_0, \mathbb{C}_p)$) to be the continuous dual of $C^{\mathrm{an}}(H, \mathbb{C}_p)$ (resp. $C^{\mathrm{an}}(H_0, \mathbb{C}_p)$), endowed with the strong dual topology; it is in fact a Fréchet algebra over $\mathbb{C}_p$ (with the convolution product).

Lemma 133. *Forgetting the K–analyticity identifies $C^{\mathrm{an}}(H,\mathbb{C}_p)$ with a closed subspace of $C^{\mathrm{an}}(H_0,\mathbb{C}_p)$, in the sense that $C^{\mathrm{an}}(H,\mathbb{C}_p) \hookrightarrow C^{\mathrm{an}}(H_0,\mathbb{C}_p)$ is a topological embedding with closed image.*

Proof. See lemma 1.2 of Schneider–Teitelbaum's article [33]. Lemma 1.1 of ibid. even gives an explicit characterization of the image in terms of the action of the Lie algebra $\mathfrak{h} = \mathrm{Lie}(H_0) = \mathrm{Lie}(H) = K$ (by means of an exponential map). □

Remark. Thanks to lemma 133, we can apply Hahn–Banach's theorem to obtain a surjective continuous morphism of Fréchet algebras

$$D(H_0,\mathbb{C}_p) \twoheadrightarrow D(H,\mathbb{C}_p)$$

that has to be then a quotient map.

Using the action of the Lie algebra $\mathfrak{h} = \mathrm{Lie}(H_0) = K$ on $C^{\mathrm{an}}(H_0,\mathbb{C}_p)$, we can define a group morphism

$$d\colon \widehat{H}_0 \to \mathrm{Hom}_{\mathbb{Q}_p}(K,\mathbb{C}_p)$$

as follows: given $\chi \in \widehat{H}_0$,

$$d\chi(\mathfrak{x}) = \left[\frac{d}{dt}\chi(t\mathfrak{x})\right]\bigg|_{t=0} \qquad \text{for all } \mathfrak{x} \in K.$$

Lemma 134. *Via the embedding $C^{\mathrm{an}}(H,\mathbb{C}_p) \hookrightarrow C^{\mathrm{an}}(H_0,\mathbb{C}_p)$, the set $\widehat{H}(\mathbb{C}_p)$ is identified with the subset of $\chi \in \widehat{H}_0(\mathbb{C}_p)$ such that $d\chi$ is K–linear (and not just $\mathbb{Q}_p$–linear).*

Proof. See lemma 1.3 of Schneider–Teitelbaum's article [33]. □

Remark. The diagram

$$\begin{array}{ccc} \widehat{H}(\mathbb{C}_p) & \xrightarrow{d} & \mathrm{Hom}_K(K,\mathbb{C}_p) \\ \big\downarrow & \ulcorner & \big\downarrow \\ \widehat{H}_0(\mathbb{C}_p) & \xrightarrow{d} & \mathrm{Hom}_{\mathbb{Q}_p}(K,\mathbb{C}_p) \end{array}$$

is cartesian.

8.1.2 The rigid variety $\mathcal{X}_0$

Next, we recall Schneider–Teitelbaum's construction of the rigid variety $\mathcal{X}_0$, which is a polydisc of dimension $[K:\mathbb{Q}_p]$ with $\mathcal{X}_0(\mathbb{C}_p) = \widehat{H}_0(\mathbb{C}_p)$. To that aim, let $\mathcal{B}_1$

denote the rigid analytic open disc of radius 1 centred at 1 over K. Thus, its set of points $\mathcal{B}_1(\mathbb{C}_p) = 1 + \mathfrak{m}_{\mathbb{C}_p}$ has the structure of a $\mathbb{Z}_p$–module, where the addition is multiplication in $(1 + \mathfrak{m}_{\mathbb{C}_p})^\times$ and the multiplication by scalars is raising elements of $(1 + \mathfrak{m}_{\mathbb{C}_p})^\times$ to powers in $\mathbb{Z}_p$. (Note that this is precisely the module structure given by the multiplicative formal group $\widehat{\mathbb{G}}_{\mathrm{m}}$. Later we will use the $\mathscr{O}_K$–module structure given by a relative Lubin–Tate formal group as in section 5.)

Lemma 135. *The set of locally $\mathbb{Q}_p$–analytic characters of $\mathscr{O}_K$ with values in $\mathbb{C}_p$ can be described as*

$$\begin{aligned}\widehat{H}_0(\mathbb{C}_p) = \mathrm{Hom}^{\mathrm{cont}}_{\mathbb{Z}}(\mathscr{O}_K, \mathbb{C}_p^\times) &= \mathrm{Hom}_{\mathbb{Z}_p}(\mathscr{O}_K, \mathcal{B}_1(\mathbb{C}_p)) \\ &\cong \mathcal{B}_1(\mathbb{C}_p) \otimes_{\mathbb{Z}_p} \mathrm{Hom}_{\mathbb{Z}_p}(\mathscr{O}_K, \mathbb{Z}_p).\end{aligned}$$

In particular, $z \otimes \beta \in \mathcal{B}_1(\mathbb{C}_p) \otimes_{\mathbb{Z}_p} \mathrm{Hom}_{\mathbb{Z}_p}(\mathscr{O}_K, \mathbb{Z}_p)$ corresponds to the character $\chi_{z\otimes\beta} \in \widehat{H}_0(\mathbb{C}_p)$ defined by

$$\chi_{z\otimes\beta}(a) = z^{\beta(a)} \quad \text{for all } a \in H_0 = \mathscr{O}_K.$$

Proof. Observing that $\mathscr{O}_K$ is a finite free $\mathbb{Z}_p$–module, this follows from standard results:

- Continuity of characters forces the images to lie in $1 + \mathfrak{m}_{\mathbb{C}_p}$.
- A $\mathbb{Z}_p$–linear map $f : \mathbb{Z}_p \to \mathcal{B}_1(\mathbb{C}_p)$ is of the form $f(a) = z^a$ (for $z = f(1)$) and that is automatically locally $\mathbb{Q}_p$–analytic.

TODO: find references (maybe in Colmez's article on one-variable p–adic analytic functions, maybe in Amice's transform original article?) □

Since $\mathrm{Hom}_{\mathbb{Z}_p}(\mathscr{O}_K, \mathbb{Z}_p)$ is a free $\mathbb{Z}_p$–module of finite rank $[K : \mathbb{Q}_p]$, we can form a rigid analytic variety $\mathcal{X}_0 = \mathcal{B}_1 \otimes_{\mathbb{Z}_p} \mathrm{Hom}_{\mathbb{Z}_p}(\mathscr{O}_K, \mathbb{Z}_p)$ (this "tensor product" is just notation) over K whose $\mathbb{C}_p$–points are $\mathcal{B}_1(\mathbb{C}_p) \otimes_{\mathbb{Z}_p} \mathrm{Hom}_{\mathbb{Z}_p}(\mathscr{O}_K, \mathbb{Z}_p)$: once we choose a $\mathbb{Z}_p$–basis of $\mathrm{Hom}_{\mathbb{Z}_p}(\mathscr{O}_K, \mathbb{Z}_p)$, $\mathcal{X}_0$ is (non-canonically) isomorphic to the open polydisc $\mathcal{B}_1^{[K:\mathbb{Q}_p]}$. Lemma 135 shows that $\mathcal{X}_0(\mathbb{C}_p) \cong \widehat{H}_0(\mathbb{C}_p)$, as desired.

8.1.3 The rigid variety $\mathcal{X}$

Finally, to define the character variety $\mathcal{X}$ as a closed subvariety of $\mathcal{X}_0$, we need to identify "rigid analytic equations" characterizing $\widehat{H}(\mathbb{C}_p)$ inside $\widehat{H}_0(\mathbb{C}_p)$. But lemma 134 gives a condition for $\chi \in \widehat{H}_0(\mathbb{C}_p)$ to belong to $\widehat{H}(\mathbb{C}_p)$ in terms of $d\chi$. A simple computation shows that, for $z \in \mathcal{B}_1(\mathbb{C}_p)$ and $\beta \in \mathrm{Hom}_{\mathbb{Z}_p}(\mathscr{O}_K, \mathbb{Z}_p)$,

$$d\chi_{z\otimes\beta} = \log(z) \cdot \beta$$

(where we view $\beta \in \mathrm{Hom}_{\mathbb{Q}_p}(K, \mathbb{Q}_p)$ in the obvious way). Now lemma 134 can be reformulated as saying that $\mathcal{X}(\mathbb{C}_p)$ should be defined inside $\mathcal{X}_0(\mathbb{C}_p)$ by the equations

$$\big(\beta(t) - t \cdot \beta(1)\big) \log(z) = 0 \quad \text{for all } t \in K.$$

In fact, it suffices to consider the $[K : \mathbb{Q}_p]$ equations corresponding to a basis of K over $\mathbb{Q}_p$.

Let $\mathscr{O}(\mathcal{X}_0/\mathbb{C}_p)$ denote the Fréchet algebra of global analytic functions on the base change of $\mathcal{X}_0$ to $\mathbb{C}_p$. Schneider and Teitelbaum extend previous work of Amice and exhibit for each $t \in K$ a rigid analytic function $F_t \in \mathscr{O}(\mathcal{X}_0/\mathbb{C}_p)$ such that $F_t(\chi_{z\otimes\beta}) = \big(\beta(t) - t \cdot \beta(1)\big) \log(z)$ for all $z \otimes \beta \in \mathcal{B}_1(\mathbb{C}_p) \otimes_{\mathbb{Z}_p} \mathrm{Hom}_{\mathbb{Z}_p}(\mathscr{O}_K, \mathbb{Z}_p)$. They explain that the ideal sheaf $\mathscr{J}$ in $\mathscr{O}_{\mathcal{X}_0}$ generated by these functions is a coherent sheaf defining a reduced closed K–analytic subvariety $\mathcal{X}$ of $\mathcal{X}_0$ with the desired property that the canonical isomorphism $\mathcal{X}_0(\mathbb{C}_p) \cong \widehat{H}_0(\mathbb{C}_p)$ restricts to $\mathcal{X}(\mathbb{C}_p) \cong \widehat{H}(\mathbb{C}_p)$.

Definition 136. The *Fourier transform* of a distribution $\mu \in D(H, \mathbb{C}_p)$ is the function $F_\mu \colon \widehat{H}(\mathbb{C}_p) \to \mathbb{C}_p$ defined by

$$F_\mu(\chi) = \mu(\chi) = \int_H \chi(a)\, \mu(a).$$

Theorem 137. *The Fourier transform defines an isomorphism $D(H, \mathbb{C}_p) \cong \mathscr{O}(\mathcal{X}/\mathbb{C}_p)$ of Fréchet algebras over $\mathbb{C}_p$. More generally, for every subextension $\mathbb{C}_p/K'/K$ with K' complete, we obtain an isomorphism $D(H, K') \cong \mathscr{O}(\mathcal{X}/K')$.*

Proof. This is theorem 2.3 of Schneider–Teitelbaum's article [33]. □

8.2 Twisted unit discs: the isomorphism at the level of points

The construction of the character variety $\mathcal{X}$ in section 8.1 relies on the open unit disc with the $\mathbb{Z}_p$–module structure given by the formal multiplicative group $\widehat{\mathbb{G}}_{\mathrm{m}}$. In section 3 of their article [33], Schneider and Teitelbaum study the open unit disc endowed with the $\mathscr{O}_K$–module structure given by a Lubin–Tate formal group and find that, after base change to $\mathbb{C}_p$, it is isomorphic to $\mathcal{X}$ in a very explicit way. Here we adapt the theory of loc. cit. to the case of the relative Lubin–Tate formal group $\mathfrak{F}_\phi$ (keeping the notation from section 5).

8.2.1 Formal modules and p–divisible groups

Let $\mathcal{B}$ denote the rigid analytic open unit disc centred at the origin over L, which we regard as an $\mathscr{O}_K$–module object by means of the relative Lubin–Tate structure. Namely, $\mathcal{B}(\mathbb{C}_p)$ is an $\mathscr{O}_K$–module with the addition given by $\mathfrak{F}_\phi(\,\cdot\,,\,\cdot\,)$ and the multiplication by scalars given by $[\,\cdot\,]_\phi(\,\cdot\,)$. As explained in section 2.2 of Tate's article [39], the formal group $\mathfrak{F}_\phi$ corresponds to a connected p–divisible group $G = (G_n)_{n\geq 0}$ over $\mathscr{O}_L$ of dimension 1 and height $[K : \mathbb{Q}_p]$. More precisely, we can write $G_n = \operatorname{Spec}(A_n)$ for $A_n = \mathscr{O}_L[\![Z]\!]/([p^n]_\phi(Z))$ and the comultiplication morphism $A_n \to A_n \otimes_{\mathscr{O}_L} A_n$ is defined by $Z \mapsto \mathfrak{F}_\phi$.

Let $G' = (G'_n)_{n\geq 0}$ be the Cartier dual of G. Thus, G' is an étale p–divisible group of dimension $[K : \mathbb{Q}_p] - 1$ and height $[K : \mathbb{Q}_p]$. By Cartier duality, there are canonical isomorphisms $G'_n(\mathscr{O}_{\mathbb{C}_p}) \cong \operatorname{Hom}(G_{n,\mathscr{O}_{\mathbb{C}_p}}, \mathbb{G}_{\mathrm{m},\mathscr{O}_{\mathbb{C}_p}})$ for all $n \geq 0$ (where the symbol Hom means morphisms of finite flat group schemes over $\mathscr{O}_{\mathbb{C}_p}$). Let $\mathrm{T}_p\, G'$ be the p–adic Tate module of G'. If $\mu_{\mathscr{O}_{\mathbb{C}_p}}$ denotes the p–divisible group of roots of unity over $\mathscr{O}_{\mathbb{C}_p}$, then $\mathrm{T}_p\, G' \cong \operatorname{Hom}(G_{\mathscr{O}_{\mathbb{C}_p}}, \mu_{\mathscr{O}_{\mathbb{C}_p}})$ (where the symbol Hom means morphisms of p–divisible groups over $\mathscr{O}_{\mathbb{C}_p}$). But, by the equivalence between connected p–divisible groups and divisible commutative formal Lie groups given in proposition 1 of Tate's article [39], $\operatorname{Hom}(G_{\mathscr{O}_{\mathbb{C}_p}}, \mu_{\mathscr{O}_{\mathbb{C}_p}}) \cong \operatorname{Hom}_{\mathscr{O}_{\mathbb{C}_p}}(\mathfrak{F}_\phi, \widehat{\mathbb{G}}_{\mathrm{m}})$.

Thus, every $t' \in \mathrm{T}_p\, G'$ determines a morphism of formal groups $F_{t'} \colon \mathfrak{F}_\phi \to \widehat{\mathbb{G}}_{\mathrm{m}}$ over $\mathscr{O}_{\mathbb{C}_p}$. Such $F_{t'}$ is a power series in $Z\mathscr{O}_{\mathbb{C}_p}[\![Z]\!]$ and we call its leading coefficient $\Omega_{t'} = F'_{t'}(0)$ the *Lubin–Tate period* associated with t'. Alternatively, using the isomorphisms to the additive formal group $\widehat{\mathbb{G}}_{\mathrm{a}}$ given by the formal logarithms of $\mathfrak{F}_\phi$ and $\widehat{\mathbb{G}}_{\mathrm{m}}$, the power series $F_{t'}(Z)$ is characterized by

$$1 + F_{t'}(Z) = \exp\big(\Omega_{t'} \log_\phi(Z)\big).$$

There is a canonical isomorphism $\mathrm{T}_p\, G' \cong \operatorname{Hom}_{\mathbb{Z}_p}(\mathrm{T}_p\, G, \mathrm{T}_p\, \mu)$ as G_L–modules (see step 1 of the proof of proposition 11 in Tate's article [39]) and so $\mathrm{T}_p\, G' \cong \mathscr{O}_K(\tau)$, where $\tau = \chi_{\mathrm{cyc}} \cdot \chi_\phi^{-1}$. The structure of $\mathscr{O}_K$–module is given as follows:

$$F_{at'}(Z) = F_{t'}\big([a]_\phi(Z)\big) \quad \text{for all } t' \in \mathrm{T}_p\, G' \text{ and all } a \in \mathscr{O}_K.$$

From the isomorphism $\mathrm{T}_p\, G' \cong \operatorname{Hom}_{\mathscr{O}_{\mathbb{C}_p}}(\mathfrak{F}_\phi, \widehat{\mathbb{G}}_{\mathrm{m}})$ we obtain (on points) a natural G_L–equivariant, $\mathbb{Z}_p$–bilinear and $\mathscr{O}_K$–invariant pairing

$$\langle\,\cdot\,,\,\cdot\,\rangle \colon\ \mathrm{T}_p\, G' \otimes_{\mathscr{O}_K} \mathcal{B}(\mathbb{C}_p) \to \mathcal{B}_1(\mathbb{C}_p)$$

given by

$$\langle t', z \rangle = 1 + F_{t'}(z) \quad \text{for all } t' \in \mathrm{T}_p\, G' \text{ and all } z \in \mathcal{B}(\mathbb{C}_p).$$

Lemma 138. *The maps*

$$\begin{aligned} \alpha \colon \mathcal{B}(\mathbb{C}_p) &\longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{T}_p\, G', \mathcal{B}_1(\mathbb{C}_p)) \\ z &\longmapsto \langle\, \cdot\,, z \rangle \end{aligned} \quad \textit{and} \quad \begin{aligned} d\alpha \colon \mathbb{C}_p &\longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{T}_p\, G', \mathbb{C}_p) \\ \mathfrak{x} &\longmapsto \big(t' \mapsto \Omega_{t'}\mathfrak{x}\big) \end{aligned}$$

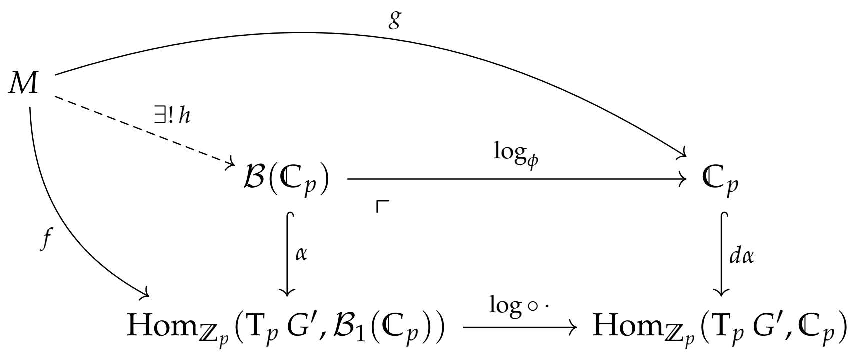
are injective and

$$\begin{array}{ccc} \mathcal{B}(\mathbb{C}_p) & \xrightarrow{\ \log_\phi\ } & \mathbb{C}_p \\ {\scriptstyle \alpha}\big\downarrow \ulcorner & & \big\downarrow{\scriptstyle d\alpha} \\ \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{T}_p\, G', \mathcal{B}_1(\mathbb{C}_p)) & \xrightarrow{\ \log\circ\,\cdot\ } & \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{T}_p\, G', \mathbb{C}_p) \end{array}$$

is a cartesian square.

Proof. The isomorphism $\mathrm{T}_p\, G' \cong \mathrm{Hom}_{\mathcal{O}_{\mathbb{C}_p}}(\mathfrak{F}_\phi, \widehat{\mathbb{G}}_{\mathrm{m}})$ induces α on points and $d\alpha$ on tangent spaces. Thus the commutativity of the square is clear. The injectivity of α and $d\alpha$ is part of proposition 11 of Tate's article [39].

We consider a commutative diagram of $\mathbb{Z}_p$–modules



and we have to check that there exists a unique h fitting in it. By the injectivity of α and $d\alpha$, it suffices to prove that $\mathrm{Im}(f) \subset \mathrm{Im}(\alpha)$. But proposition 11 of Tate's article [39] implies that the lower horizontal arrow induces an isomorphism $\mathrm{Coker}(\alpha) \cong \mathrm{Coker}(d\alpha)$. Therefore, the image of f inside $\mathrm{Coker}(\alpha)$ corresponds to the image of $d\alpha \circ g$ inside $\mathrm{Coker}(d\alpha)$, which is trivial. $\square$

8.2.2 The isomorphism on points

Proposition 139. *The map*

$$\begin{aligned} \mathcal{B}(\mathbb{C}_p) \otimes_{\mathscr{O}_K} \mathrm{T}_p\, G' &\longrightarrow \widehat{H}(\mathbb{C}_p) \\ z \otimes t' &\longmapsto \kappa_{z\otimes t'} \end{aligned}$$

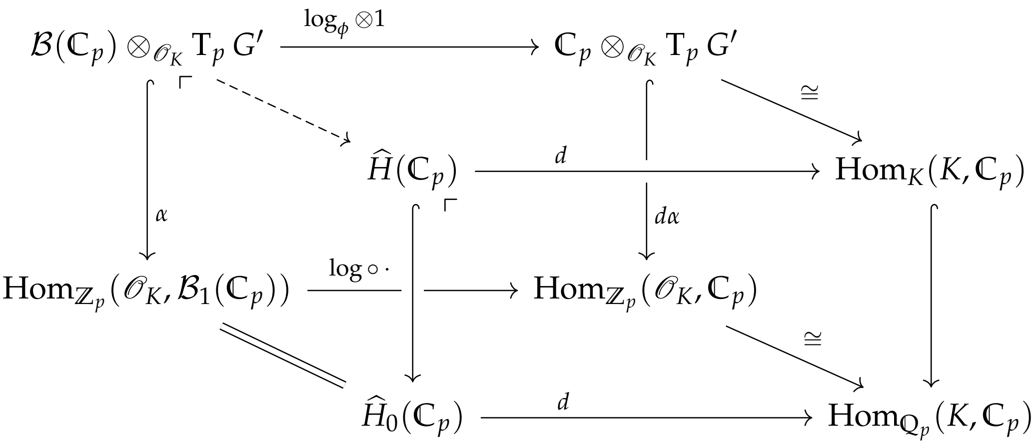
defined by

$$\kappa_{z\otimes t'}(a) = \langle t', [a]_\phi(z)\rangle$$

is a well-defined isomorphism of $\mathbb{Z}$-modules.

Proof. For a non-relative Lubin–Tate formal group, this is proposition 3.1 of Schneider–Teitelbaum's article [33]. The proof in loc. cit. uses the general results of Tate's article [39] and so works exactly the same for the relative Lubin–Tate situation. We repeat it here for the convenience of the reader.

The map of the statement will be defined by commutativity of the "cube"



that we explain next.

- The front face is the cartesian square from the remark after lemma 134.
- In the bottom face, the equality on the left is part of lemma 135 and the isomorphism on the right is the obvious one. The commutativity of this square comes from the computation

$$d\chi_{z\otimes\beta}(\,\cdot\,) = \log(z)\cdot\beta(\,\cdot\,) = \log\bigl(z^{\beta(\,\cdot\,)}\bigr) = \log\circ\chi_{z\otimes\beta}(\,\cdot\,).$$

- The rear face comes from the cartesian diagram of lemma 138 after tensoring with $\mathrm{T}_p\, G'$ over $\mathscr{O}_K$ and using the isomorphisms

$$\mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{T}_p\, G', \,\cdot\,) \otimes_{\mathscr{O}_K} \mathrm{T}_p\, G' \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(\mathscr{O}_K, \,\cdot\,)$$

$$f \otimes t' \longmapsto \big(a \mapsto f(at')\big)$$

in the lower row.

- In the right face, the upper isomorphism is defined to make the square commutative. This is possible because any map in the image of $d\alpha$ is clearly $\mathscr{O}_K$–linear:
$$d\alpha(\mathfrak{x} \otimes t')(a) = \Omega_{at'}\mathfrak{x} = a\Omega_{t'}\mathfrak{x}.$$
By definition, this morphism $\mathbb{C}_p \otimes_{\mathscr{O}_K} \mathrm{T}_p\, G' \to \mathrm{Hom}_K(K, \mathbb{C}_p)$ is obviously injective. Counting dimensions over $\mathbb{C}_p$, one sees that it is also surjective.

Now the universal property of the front face implies the existence of a unique morphism $\mathcal{B}(\mathbb{C}_p) \otimes_{\mathscr{O}_K} \mathrm{T}_p\, G' \to \widehat{H}(\mathbb{C}_p)$ making the whole diagram commutative. This dashed arrow must be the map in the statement of the proposition by the commutativity of the left face and the definition of α. Finally, the same argument with the universal property of the rear face gives an inverse. □

Fix once and for all a generator t'_0 of the $\mathscr{O}_K$–module $\mathrm{T}_p\, G'$. From now on we write $\Omega = \Omega_{t'_0}$ and $\kappa_z = \kappa_{z\otimes t'_0}$ to simplify the notation. By proposition 139, we have an isomorphism

$$\kappa(\mathbb{C}_p)\colon \mathcal{B}(\mathbb{C}_p) \longrightarrow \mathcal{X}(\mathbb{C}_p)$$
$$z \longmapsto \kappa_z$$

of $\mathbb{Z}$–modules. The next goal is to prove that this isomorphism on points comes from an isomorphism $\kappa\colon \mathcal{B}_{\mathbb{C}_p} \to \mathcal{X}_{\mathbb{C}_p}$ of rigid varieties over $\mathbb{C}_p$. More precisely, we are going to construct compatible admissible coverings by affinoid open subsets giving both $\mathcal{B}$ and $\mathcal{X}$ the structure of a quasi-Stein space.

8.3 The isomorphism of rigid analytic varieties

8.3.1 Covering the disc $\mathcal{B}$ by affinoids

For each $r \in p^{\mathbb{Q}}$, let $\mathcal{B}[r]$ denote the closed disc of radius r over L. If $r < 1$, we regard $\mathcal{B}[r]$ as an affinoid subdomain of $\mathcal{B}$.

Lemma 140. *Let e be the ramification index of $K/\mathbb{Q}_p$. Let π_K be a uniformizer of $\mathscr{O}_K$ and take $r \in p^{\mathbb{Q}}$ such that $p^{-q/e(q-1)} \leq r < 1$. Then*

$$[\pi_K]_\phi^{-1}(\mathcal{B}[r]) = \mathcal{B}[r^{1/q}] \quad \textit{and} \quad [p]_\phi^{-1}(\mathcal{B}[r]) = \mathcal{B}[r^{1/q^e}]$$

and the maps $[p^n]_\phi\colon \mathcal{B}[r^{1/q^{en}}] \to \mathcal{B}[r]$ are finite étale for all $n \in \mathbb{Z}_{\geq 0}$.

Proof. This is analogous to lemma 3.2 of Schneider–Teitelbaum's article [33]. We repeat the proof in the relative case for the convenience of the reader.

Since p and π_K^e differ by a unit $u \in \mathscr{O}_K^\times$ and $|[u]_\phi(z)|_p = |z|_p$, it suffices to prove the statements for π_K. But multiplication by π_K is given by a power series

$$[\pi_K]_\phi(Z) = \sum_{n \geq 1} a_n Z^n \in \mathscr{O}_L[\![Z]\!]$$

with $a_1 = \pi_K$ and whose first unit coefficient is that of Z^q, as $\mathfrak{F}_\phi$ is a formal group of height $[K : \mathbb{Q}_p]$. Therefore,

$$|a_i z^i| \leq \begin{cases} |\pi_K z|_p = p^{-1/e}|z|_p & \text{if } 1 \leq n < q, \\ |z^q|_p = |z|_p^q & \text{if } n \geq q. \end{cases}$$

We deduce that $|[\pi_K]_\phi(z)|_p \leq \max\{\, p^{-1/e}|z|_p, |z|_p^q \,\}$ and this maximum is $|z|_p^q$ precisely when $|z|_p \geq p^{-1/e(q-1)}$. This completes the proof of the first part because $r^{1/q} \geq p^{-1/e(q-1)}$.

The finiteness and the étaleness of $[\pi_K]_\phi$ follows from the form of the power series $[\pi_K]_\phi(Z)$ and Weierstrass's preparation theorem, which allows us to reduce it to a distinguished polynomial. □

8.3.2 Covering $\mathcal{X}$ by affinoids

Similarly, for each $r \in p^{\mathbb{Q}}$ with $r < 1$, let $\mathcal{B}_1[r]$ denote the closed disc of radius r centred at 1 over K, regarded as an affinoid subdomain of $\mathcal{B}_1$. Then we define the affinoid subdomains $\mathcal{X}_0[r] = \mathcal{B}_1[r] \otimes_{\mathbb{Z}_p} \mathrm{Hom}_{\mathbb{Z}_p(\mathscr{O}_K, \mathbb{Z}_p)}$ of $\mathcal{X}_0$ and $\mathcal{X}[r] = \mathcal{X}_0[r] \cap \mathcal{X}$ of $\mathcal{X}$.

Lemma 141. *Let $r \in p^{\mathbb{Q}}$ such that $p^{-p/(p-1)} \leq r < 1$. Then*

$$[p]^{-1}(\mathcal{X}[r]) = \mathcal{X}[r^{1/p}]$$

and the maps $[p^n] \colon \mathcal{X}[r^{1/p^n}] \to \mathcal{X}[r]$ are finite étale for all $n \in \mathbb{Z}_{\geq 0}$.

Proof. This is a special case of lemma 140 using the multiplicative Lubin–Tate formal group $\widehat{\mathbb{G}}_\mathrm{m}$ over $\mathbb{Q}_p$ (in place of $\mathfrak{F}_\phi$ relative to L/K). □

8.3.3 The valuation of the period Ω

Lemma 142. *If $z \in \mathcal{B}(\mathbb{C}_p)$ is a zero of $F_{t_0'}$, then z must be a torsion point of $\mathfrak{F}_\phi$.*

Proof. This is lemma 6.2 of de Shalit's article [38] and a claim inside the proof of lemma 3.4.c of Schneider–Teitelbaum's article. We reproduce the proof here for the convenience of the reader.

Since $F_{t'_0}$ defines an element of $\mathrm{Hom}_{\mathscr{O}_{\mathbb{C}_p}}(\mathfrak{F}_\phi, \widehat{\mathbb{G}}_m)$, the hypothesis that $F_{t'_0}(z) = 0$ implies that $F_{t'_0}([a]_\phi(z)) = \left(1 + F_{t'_0}(z)\right)^a - 1 = 0$ for all $a \in \mathbb{Z}_p^\times$. If z were not a torsion point, then $F_{t'_0}$ would have infinitely many zeros of p–adic absolute value $r = |z|_p$. But this is impossible because a non-zero function on the affinoid $\mathcal{B}[r]$ can have at most finitely many zeros. □

Lemma 143. *Let* $f(Z) = bZ(1 + b_1 Z + b_2 Z^2 + \cdots) \in \mathbb{C}_p[\![Z]\!]$ *which converges for* $|Z|_p \leq 1$ *and has no zeros in* $0 < |Z|_p < 1$.

1. *The coefficients* b_n *for* $n \in \mathbb{Z}_{\geq 1}$ *are all in* $\mathscr{O}_{\mathbb{C}_p}$ *and tend to* 0 *as* $n \to \infty$.
2. *The function* f *has only finitely many zeros on* $|Z|_p \leq 1$.
3. *Let* $z_0 \in \mathscr{O}_{\mathbb{C}_p}^\times$ *(i.e.,* $|z_0|_p = 1$*). If* f *has a zero on the residue disc* $|Z - z_0|_p < 1$*, then* $|f(Z)|_p < |b|_p$ *throughout that disc; otherwise,* $|f(Z)|_p = |b|_p$ *for* $|Z - z_0|_p < 1$.

Proof. This is lemma 6.3 of de Shalit's article [38]. TODO: find a better reference (de Shalit only says it is "well-known" and cites a book where I couldn't find the result). □

Proposition 144. *The p–adic valuation of the Lubin–Tate period associated with* t'_0 *is*

$$\mathrm{v}_p(\Omega) = \frac{1}{p-1} - \frac{1}{e(q-1)}.$$

Proof. This result is proved (for non-relative Lubin–Tate groups) in the appendix of Schneider–Teitelbaum's article [33] using complicated constructions in p–adic Hodge theory from Fontaine's article [20]. Here we adapt the more elementary proof from proposition 6.1 of de Shalit's article [38].

Fix $\rho \in \mathscr{O}_{\mathbb{C}_p}$ with $|\rho|_p = p^{-1/e(q-1)}$ and consider

$$f(Z) = F_{t'_0}(\rho Z) = \rho\Omega Z(1 + b_1 Z + b_2 Z^2 + \cdots) \in \mathbb{C}_p[\![Z]\!].$$

This power series converges for $|Z|_p \leq 1$ because $F_{t'_0}(Z)$ does for $|Z|_p < 1$. We claim that $f(Z)$ has no zeros in $0 < |Z|_p < 1$ and so we can apply lemma 143. Indeed, if $z_0 \in \mathcal{B}(\mathbb{C}_p)$ satisfies that $f(z_0) = 0$, then $z = \rho z_0$ is a zero of $F_{t'_0}$ and must be a torsion point of $\mathfrak{F}_\phi$ by lemma 142. But the non-zero torsion points of $\mathfrak{F}_\phi$ have p–adic valuations of the form $1/eq^{n-1}(q-1)$ with $n \geq 1$ and all these numbers are strictly larger than $\mathrm{v}_p(z)$. Therefore, $z_0 = 0$.

Consider the torsion points in $\mathfrak{F}_{\phi,1}$. If π_K is a uniformizer of $\mathscr{O}_K$, we can write $\mathfrak{F}_{\phi,1} = \mathrm{Ker}([\pi_K]_\phi)$. Since p differs (multiplicatively) from π_K^e by a unit and $F_{t_0'}$ is a morphism of formal groups, we deduce that $F_{t_0'}(\mathfrak{F}_{\phi,1}) \subset \{\, \zeta_p - 1 : \zeta_p \in \mu_p \,\}$. We can find $z \in \mathfrak{F}_{\phi,1}$ such that $F_{t_0'}(z) = \zeta_p - 1 \neq 0$: otherwise, $F_{t_0'}$ would factor through $[\pi_K]_\phi$, but this is impossible because t_0' is a generator of $\mathrm{T}_p\, G' \cong \mathscr{O}_K(\tau)$. Write $z = \rho z_0$. Comparing p–adic valuations, we see that $|z_0|_p = 1$. Now lemma 142 implies that $f(Z)$ cannot have any zeros in $|Z - z_0|_p < 1$, as z is the only torsion point of $\mathfrak{F}_\phi$ in the corresponding disc. The last part of lemma 143 says that

$$|\rho\Omega|_p = |f(z_0)|_p = |\zeta_p - 1|_p = p^{-1/(p-1)}$$

and this conclude the proof. □

8.3.4 The isomorphism of varieties over $\mathbb{C}_p$

Proposition 145. *Let $r \in p^{\mathbb{Q}}$ with $r < p^{-1/e(q-1)}$. The power series $F_{t_0'}(Z)$ defines an analytic isomorphism between $\mathcal{B}[r]_{\mathbb{C}_p}$ and $\mathcal{B}[r|\Omega|_p]_{\mathbb{C}_p}$.*

Proof. This is lemma 3.4.c of Schneider–Teitelbaum's article [33].

Write $F_{t_0'}(Z) = \Omega Z(1 + b_1 Z + b_2 Z^2 + \cdots) \in \mathbb{C}_p[\![Z]\!]$. To prove that this power series defines an isomorphism from $\mathcal{B}[r]_{\mathbb{C}_p}$ to $\mathcal{B}[r|\Omega|_p]_{\mathbb{C}_p}$, it suffices to check that $|b_n|_p \leq p^{n/e(q-1)}$ for all $n \in \mathbb{Z}_{\geq 1}$, by the hypothesis on r. If there were some $n \geq 1$ for which $|b_n|_p > p^{n/e(q-1)}$, then the Newton polygon of $1 + b_1 Z + b_2 Z^2 + \cdots$ would tell us that this series has a zero z with $|z|_p < p^{-1/e(q-1)}$. But lemma 142 implies that the only zero of $F_{t_0'}(Z)$ with absolute value $< p^{-1/e(q-1)}$ is 0, so such z cannot exist. □

We finally have all the ingredients to prove that the isomorphism on points $\kappa(\mathbb{C}_p)\colon \mathcal{B}(\mathbb{C}_p) \to \mathcal{X}(\mathbb{C}_p)$ is induced by an isomorphism of rigid analytic varieties $\kappa\colon \mathcal{B}_{\mathbb{C}_p} \to \mathcal{X}_{\mathbb{C}_p}$ compatible with certain quasi-Stein coverings. That is, we see κ as an isomorphism of two families of affinoids exhausting $\mathcal{B}_{\mathbb{C}_p}$ and $\mathcal{X}_{\mathbb{C}_p}$.

To define κ as a rigid analytic morphism (on affinoids), we choose a $\mathbb{Z}_p$–basis $e_1, \ldots, e_{[K:\mathbb{Q}_p]}$ of $\mathscr{O}_K$ and let $e_1^*, \ldots, e_{[K:\mathbb{Q}_p]}^*$ be the dual basis of $\mathrm{Hom}_{\mathbb{Z}_p}(\mathscr{O}_K, \mathbb{Z}_p)$. This allows us to identify $\mathcal{X}_0$ with the polydisc $\mathcal{B}_1^{[K:\mathbb{Q}_p]}$ and then $\kappa\colon \mathcal{B}_{\mathbb{C}_p} \to (\mathcal{X}_0)_{\mathbb{C}_p}$ is given in coordinates by

$$\kappa_z = \sum_{i=1}^{[K:\mathbb{Q}_p]} \big(1 + F_{e_i t_0'}(z)\big) \otimes e_i^*.$$

The expression in terms of power series shows that κ is rigid analytic over $\mathbb{C}_p$ and we have seen (on points) that it factors through the closed subvariety $\mathcal{X}_{\mathbb{C}_p}$ of $(\mathcal{X}_0)_{\mathbb{C}_p}$ (cf. proposition 139).

Theorem 146 (Schneider–Teitelbaum). *Let $r = p^{-q/e(q-1)}$. For every $n \in \mathbb{Z}_{\geq 0}$,*

$$\kappa \colon \mathcal{B}\big[r^{1/q^{en}}\big]_{\mathbb{C}_p} \to \mathcal{X}\big[(r|\Omega|_p)^{1/p^n}\big]_{\mathbb{C}_p}$$

is an isomorphism of affinoids over $\mathbb{C}_p$. Consequently, $\kappa \colon \mathcal{B}_{\mathbb{C}_p} \to \mathcal{X}_{\mathbb{C}_p}$ is an isomorphism of rigid varieties over $\mathbb{C}_p$.

Proof. For a non-relative Lubin–Tate formal group, this is theorem 3.6 of Schneider–Teitelbaum's article [33]. The proof in for the relative case is exactly the same using the results that we have previously stated (especially proposition 144). We reproduce it here for the convenience of the reader.

First we prove the case $n = 0$. By the expression of κ in terms of power series just before the theorem and proposition 145, we have a well-defined rigid analytic morphism $\kappa \colon \mathcal{B}[r]_{\mathbb{C}_p} \to \mathcal{X}[r|\Omega|_p]_{\mathbb{C}_p}$. Since $r < p^{-1/e(q-1)}$, the formal logarithm $\log_\phi$ defines a rigid isomorphism $\log_\phi \colon \mathcal{B}[r] \to \mathcal{B}[r]$. The same argument applied to the formal group $\widehat{\mathbb{G}}_{\mathrm{m}}$ shows that $\log \colon \mathcal{B}_1[r|\Omega|_p] \to \mathcal{B}[r|\Omega|_p]$ is also a rigid isomorphism, as $r|\Omega|_p < p^{-1/(p-1)}$. Therefore, we obtain a commutative diagram

$$\begin{array}{ccc}
\mathcal{B}[r]_{\mathbb{C}_p} & \xrightarrow[\log_\phi]{\cong} & \mathcal{B}[r]_{\mathbb{C}_p} \\
{\scriptstyle \kappa}\big\downarrow & & \big\downarrow{\scriptstyle \cdot\Omega}\ \wr\| \\
\mathcal{X}[r|\Omega|_p]_{\mathbb{C}_p} & \xrightarrow[\log(\,\cdot\,(1))]{} & \mathcal{B}[r|\Omega|_p]_{\mathbb{C}_p}
\end{array}$$

of rigid analytic morphisms. (Here, the notation $\cdot\,(1)$ means the following: if the $\mathbb{Z}_p$–basis $e_1, \ldots, e_{[K:\mathbb{Q}_p]}$ of $\mathcal{O}_K$ whose dual $e_1^*, \ldots, e_{[K:\mathbb{Q}_p]}^*$ gives the identification $\mathcal{X}_0 \cong \mathcal{B}_1^{[K:\mathbb{Q}_p]}$ has $e_1 = 1$, we restrict to the intersection of $\mathcal{X}$ and the copy of $\mathcal{B}_1$ corresponding to e_1^*.) At the level of points, this commutative diagram corresponds to the top of the cube appearing in the proof of proposition 139 once we identify $\mathrm{Hom}_K(K, \mathbb{C}_p) \cong \mathbb{C}_p$ via evaluation at 1.

We claim that the lower horizontal arrow is injective on $\mathbb{C}_p$–points. Indeed, if $\log(\kappa_z(1)) = 0$ or, equivalently, $\kappa_z(1) = 1$, then by local K–analyticity κ_z must be locally constant and so of finite order. But $\mathcal{B}_1[r|\Omega|_p](\mathbb{C}_p)$ has no non-trivial torsion. Thus κ_z must be the trivial character.

Looking at the diagram on points, we see that the top and right arrows are isomorphisms and the left and bottom arrows are at least injective. Therefore, all arrows must be isomorphisms on $\mathbb{C}_p$–points. This implies that all arrows are rigid isomorphisms because all the affinoids in the diagram are reduced.

Next, we prove the theorem for $n > 0$. We use the commutative diagram

$$\begin{array}{ccc} \mathcal{B}[r^{1/q^{en}}]_{\mathbb{C}_p} & \xrightarrow{\kappa} & \mathcal{X}\big[(r|\Omega|_p)^{1/p^n}\big]_{\mathbb{C}_p} \\ {\scriptstyle [p^n]_\phi}\big\downarrow & & \big\downarrow{\scriptstyle [p^n]} \\ \mathcal{B}[r]_{\mathbb{C}_p} & \xrightarrow[\kappa]{} & \mathcal{X}[r|\Omega|_p]_{\mathbb{C}_p} \end{array}$$

and lemmata 140 and 141 to reduce to the previous case. (Observe that we can apply lemma 141 because $r|\Omega| = p^{-1/(p-1)-1/e} \geq p^{-p/(p-1)}$.)

First, we consider the diagram at the level of $\mathbb{C}_p$–points. We know that $\kappa\colon \mathcal{B}(\mathbb{C}_p) \to \mathcal{X}(\mathbb{C}_p)$ is an isomorphism of groups by proposition 139. Thus, since the lower arrow $\kappa\colon \mathcal{B}[r](\mathbb{C}_p) \to \mathcal{X}[r|\Omega|_p](\mathbb{C}_p)$ is an isomorphism and lemmata 140 and 141 tell us that the upper arrow is obtained from the lower one by taking preimages under $[p^n]$, we deduce that the upper arrow is an isomorphism on $\mathbb{C}_p$–points.

Again by lemmata 140 and 141, the two vertical arrows are finite étale affinoid morphisms of degree $q^{ne} = p^{n[K:\mathbb{Q}_p]}$. Thus the problem is reduced to a general result on reduced affinoids (cf. the end of the proof of theorem 3.6 in Schneider–Teitelbaum's article [33]). □

8.3.5 Global functions

The isomorphism of rigid varieties $\mathcal{B}_{\mathbb{C}_p} \cong \mathcal{X}_{\mathbb{C}_p}$ from theorem 146 induces an isomorphism on rigid functions $\mathscr{O}(\mathcal{X}/\mathbb{C}_p) \cong \mathscr{O}(\mathcal{B}/\mathbb{C}_p) = \mathbf{B}^+_{\mathrm{rig},\mathbb{C}_p}$. We can now translate the natural action of G_L on $\mathscr{O}(\mathcal{X}/\mathbb{C}_p) = \mathscr{O}(\mathcal{X}/K) \mathbin{\widehat{\otimes}}_K \mathbb{C}_p$ to $\mathscr{O}(\mathcal{B}/\mathbb{C}_p)$ as follows. For $z \in \mathcal{B}(\mathbb{C}_p)$ and $\sigma \in G_L$, we define $\sigma * z \in \mathcal{B}(\mathbb{C}_p)$ as the unique element satisfying that $\kappa_{\sigma * z} = \sigma \circ \kappa_z$. More precisely,

$$\begin{aligned} \sigma \circ \kappa_z(a) &= \sigma\big(\langle t'_0, [a]_\phi(z)\rangle\big) = \langle \sigma(t'_0), \sigma \circ [a]_\phi(z)\rangle = \langle \tau(\sigma)t'_0, [a]_\phi(\sigma(z))\rangle \\ &= \langle t'_0, [\tau(\sigma)a]_\phi(\sigma(z))\rangle = \big\langle t'_0, [a]_\phi\big([\tau(\sigma)]_\phi(\sigma(z))\big)\big\rangle = \kappa_{[\tau(\sigma)]_\phi(\sigma(z))}(a) \end{aligned}$$

for all $a \in \mathscr{O}_K$ and so $\sigma * z = [\tau(\sigma)]_\phi(\sigma(z))$. Then, for $F \in \mathscr{O}(\mathcal{B}/\mathbb{C}_p)$ and $\sigma \in G_L$, the action of σ "coefficientwise on $\mathscr{O}(\mathcal{X}/\mathbb{C}_p)$" gives

$$(\sigma * F)(z) = \sigma\big(F(\sigma^{-1} * z)\big) = F^\sigma\big(\sigma(\sigma^{-1} * z)\big) = F^\sigma \circ [\tau(\sigma^{-1})]_\phi(z)$$

for all $z \in \mathcal{B}(\mathbb{C}_p)$. That is, $\sigma * F = F^\sigma \circ [\tau(\sigma^{-1})]_\phi$.

Corollary 147. *There is an $\mathscr{O}_K$–equivariant isomorphism of Fréchet algebras over $\mathbb{C}_p$*

$$\begin{aligned} D(H, \mathbb{C}_p) \cong \mathscr{O}(\mathcal{X}/\mathbb{C}_p) &\cong \mathscr{O}(\mathcal{B}/\mathbb{C}_p) \\ \mu \longmapsto & \; A_\mu \end{aligned}$$

characterized by

$$A_\mu(z) = \int_{\mathscr{O}_K} \kappa_z(a)\, \mu(a) \quad \textit{for all } z \in \mathcal{B}(\mathbb{C}_p).$$

More generally, for every subextension $\mathbb{C}_p/L'/L$ with L' complete, we obtain an isomorphism $D(H, L') \cong \mathscr{O}(\mathcal{B}/\mathbb{C}_p)^{G_{L'}}$ (where the Galois action is the one described above).

Proof. This is an immediate consequence of theorem 137 and theorem 146. □

8.4 Distributions on Γ_L

(Relative) Lubin–Tate theory provides us with an isomorphism $\chi_{\xi_K} \colon \Gamma_L \to \mathscr{O}_K^\times$. In particular, Γ_L is a locally analytic manifold over K, isomorphic to q copies of $\mathscr{O}_K$ via χ_{ξ_K} and the canonical decomposition $\mathscr{O}_K^\times \cong \mathbb{F}_q \times (1 + \mathfrak{m}_K)^\times$ given by the Teichmüller character. Our objective in this subsection is to describe the algebra $D(\Gamma_L, \mathbb{C}_p)$ of $\mathbb{C}_p$–valued locally analytic distributions on Γ_L in terms of character varieties. We follow section 2.1 of Schneider–Venjakob's preprint [35].

Choosing a uniformizer π_K of K (later we fix a particular one), we obtain an isomorphism $(1 + \pi_K \mathscr{O}_K)^\times \cong \mathscr{O}_K$ mapping $1 + \pi_K$ to 1. Now, identifying $\mathscr{O}_K^\times$ with $\mathbb{F}_q \times \mathscr{O}_K$, we can construct a character variety $\mathcal{X}^\times$ over K as the product of q copies (indexed by $\mathbb{F}_q$) of $\mathcal{X}$. The $\mathbb{C}_p$–points of $\mathcal{X}^\times$ are the locally K–analytic characters of the p–adic Lie group $\mathscr{O}_K^\times$ with values in $\mathbb{C}_p$. The Fourier transform defines an isomorphism $D(\mathscr{O}_K^\times, \mathbb{C}_p) \cong \mathscr{O}(\mathcal{X}^\times/\mathbb{C}_p)$ (cf. theorem 137).

It is more convenient to view $D(\mathscr{O}_K^\times, \mathbb{C}_p)$ inside $D(\mathscr{O}_K, \mathbb{C}_p)$ and describe these distributions as a subset of $\mathscr{O}(\mathcal{B}/\mathbb{C}_p)$ by means of corollary 147. To that aim, we first need to translate the extra structure on $\mathscr{O}(\mathcal{B}/\mathbb{C}_p)$ provided by Lubin–Tate theory.

8.4.1 The action of $\mathscr{O}_K$ on distributions

We have an "action" of the multiplicative monoid $\mathscr{O}_K \setminus \{ 0 \}$ on $\mathcal{B}$ given by $[\,\cdot\,]_\phi$: in terms of $\mathbb{C}_p$–points,

$$a * z = [a]_\phi(z) \quad \text{for all } a \in \mathscr{O}_K \text{ and } z \in \mathcal{B}(\mathbb{C}_p).$$

A straight-forward computation shows how to define the analogous action on $\mathcal{X}$ and later on $D(\mathscr{O}_K, \mathbb{C}_p)$. Namely, for $z \in \mathcal{B}(\mathbb{C}_p)$ and $a, b \in \mathscr{O}_K$,

$$\kappa_{a*z}(b) = 1 + F_{t_0'}\big([b]_\phi(a * z)\big) = 1 + F_{t_0'}\big([ab]_\phi(z)\big) = \kappa_z(ab).$$

We define the action of $\mathscr{O}_K \setminus \{0\}$ on $C^{\mathrm{an}}(\mathscr{O}_K, \mathbb{C}_p)$ and on $D(\mathscr{O}_K, \mathbb{C}_p)$ as follows: for $a \in \mathscr{O}_K$, $f \in C^{\mathrm{an}}(\mathscr{O}_K, \mathbb{C}_p)$ and $\mu \in D(\mathscr{O}_K, \mathbb{C}_p)$,

$$a^*(f) = f(a\,\cdot\,) \quad \text{and} \quad a_*(\mu) = \mu \circ a^*.$$

The restriction of this action to characters (i.e., to $\mathcal{X}(\mathbb{C}_p)$) comes in fact from the analogous action on the rigid variety $\mathcal{X}_0 = \mathcal{B}_1 \otimes_{\mathbb{Z}_p} \operatorname{Hom}_{\mathbb{Z}_p}(\mathscr{O}_K, \mathbb{Z}_p)$ (or rather on the second factor of this "tensor product").

Using the notation of corollary 147, we see by construction that

$$A_{a_*(\mu)}(Z) = A_\mu([a]_\phi(Z)) \quad \text{for all } a \in \mathscr{O}_K \text{ and } \mu \in D(\mathscr{O}_K, \mathbb{C}_p).$$

8.4.2 The action of φ_q on distributions

Next we want to define an action of φ_q. Consider $G(Z) \in \mathscr{O}(\mathcal{B}/\mathbb{C}_p)$ corresponding to $F \in \mathscr{O}(\mathcal{X}/\mathbb{C}_p)$ via the isomorphism $\mathcal{B}_{\mathbb{C}_p} \cong \mathcal{X}_{\mathbb{C}_p}$. By definition,

$$\begin{aligned} \varphi_q(G)(z) &= G^{\varphi_q}(\phi(z)) = \varphi_q\big(G(\varphi_q^{-1} \circ \phi(z))\big) \\ &= \varphi_q\big(F\big(\kappa_{\varphi_q^{-1}(\phi(z))}\big)\big) = F^{\varphi_q}\big(\varphi_q \circ \kappa_{\varphi_q^{-1}(\phi(z))}\big). \end{aligned}$$

We have to understand the argument of F^{φ_q} above in terms of κ_z. Using the explicit description of $\kappa(\mathbb{C}_p)$ in proposition 139, we can compute

$$\begin{aligned} \varphi_q \circ \kappa_{\varphi_q^{-1}(\phi(z))}(a) &= \varphi_q\big(1 + F_{t_0'}\big([a]_\phi(\varphi_q^{-1} \circ \phi(z))\big)\big) \\ &= \varphi_q\big(\exp\big(\Omega \log_\phi\big([a]_\phi(\varphi_q^{-1} \circ \phi(z))\big)\big)\big) \\ &= \exp\big(\varphi_q(\Omega) \log_\phi^{\varphi_q}\big([a]_\phi^{\varphi_q}(\phi(z))\big)\big) \\ &= \exp\big(\varphi_q(\Omega)\pi_L \log_\phi\big([a]_\phi(z)\big)\big). \end{aligned}$$

Since $\exp\big(\varphi_q(\Omega)\pi_L \log_\phi(\,\cdot\,)\big)$ defines an element of $\operatorname{Hom}_{\mathscr{O}_{\mathbb{C}_p}}(\mathfrak{F}_\phi, \widehat{\mathbb{G}}_{\mathrm{m}})$, it must be of the form $1 + F_{t'}(\,\cdot\,)$ for some $t' \in \mathrm{T}_p\, G'$. But $\mathrm{T}_p\, G'$ is free of rank 1 over $\mathscr{O}_K$ and, comparing the valuations of Ω and $\varphi_q(\Omega)\pi_L$, we conclude that $t' = \pi_K \cdot t_0'$ for some uniformizer π_K of K. From now on, π_K always denotes this particular uniformizer (which depends on many choices). Then

$$\varphi_q \circ \kappa_{\varphi_q^{-1}(\phi(z))}(a) = 1 + F_{t'}([a]_\phi(z)) = 1 + F_{t'_0}([\pi_K a]_\phi(z)) = \kappa_z(\pi_K a)$$

and so

$$\varphi_q(G)(z) = F^{\varphi_q}(\pi_K^*(\kappa_z)).$$

Therefore, we define

$$\varphi_q(F) = F^{\varphi_q} \circ \pi_K^* \quad \text{for all } F \in \mathscr{O}(\mathcal{X}/\mathbb{C}_p).$$

Let $\mu \in D(\mathscr{O}_K, \mathbb{C}_p)$ and consider its Fourier transform $F_\mu \in \mathscr{O}(\mathcal{X}/\mathbb{C}_p)$. For every $\chi \in \mathcal{X}(\mathbb{C}_p)$, we can compute

$$\begin{aligned}\varphi_q(F_\mu)(\chi) &= F_\mu^{\varphi_q}(\pi_K^*(\chi)) = \varphi_q\big(F_\mu(\varphi_q^{-1}(\pi_K^*(\chi)))\big) \\ &= \varphi_q\Big(\int_{\mathscr{O}_K} \varphi_q^{-1}(\pi_K^*(\chi)(a))\,\mu(a)\Big) = \varphi_q\Big(\int_{\mathscr{O}_K} \varphi_q^{-1}(\chi(\pi_K a))\,\mu(a)\Big).\end{aligned}$$

We define $\varphi_q(\mu) \in D(\mathscr{O}_K, \mathbb{C}_p)$ by $\varphi_q(\mu)(f) = \varphi_q\big(\mu(\varphi_q^{-1} \circ \pi_K^*(f))\big)$ or, equivalently,

$$\int_{\mathscr{O}_K} f\,\varphi_q(\mu) = \varphi_q\Big(\int_{\mathscr{O}_K} \big[\varphi_q^{-1} \circ \pi_K^*(f)\big]\,\mu\Big) \quad \text{for all } f \in C^{\mathrm{an}}(\mathscr{O}_K, \mathbb{C}_p).$$

Lemma 148. *The endomorphism φ_q makes $\mathscr{O}(\mathcal{B}/\mathbb{C}_p)$ (resp. $\mathscr{O}(\mathcal{X}/\mathbb{C}_p)$, $D(\mathscr{O}_K, \mathbb{C}_p)$) into a free module over itself of rank q.*

Proof. Since the endomorphisms φ_q are defined to be compatible with the isomorphisms $\mathscr{O}(\mathcal{B}/\mathbb{C}_p) \cong \mathscr{O}(\mathcal{X}/\mathbb{C}_p) \cong D(\mathscr{O}_K, \mathbb{C}_p)$, it suffices to prove the claim for $D(\mathscr{O}_K, \mathbb{C}_p)$. But $\mu \mapsto (\pi_K)_*(\mu)$ defines an isomorphism between $D(\mathscr{O}_K, \mathbb{C}_p)$ and $D(\pi_K\mathscr{O}_K, \mathbb{C}_p)$, while $\mu \mapsto \varphi_q \circ \mu(\varphi_q^{-1} \circ \cdot\,)$ is an automorphism that preserves $D(\pi_K\mathscr{O}_K, \mathbb{C}_p)$ inside $D(\mathscr{O}_K, \mathbb{C}_p)$. The lemma follows from these observations because the Dirac distributions δ_a, where a runs over a set of representatives of $\mathscr{O}_K/\pi_K\mathscr{O}_K$, form a basis of $D(\mathscr{O}_K, \mathbb{C}_p)$ over $D(\pi_K\mathscr{O}_K, \mathbb{C}_p)$. □

Lemma 149. *For every $G(Z) \in \mathbf{B}^+_{\mathrm{rig},\mathbb{C}_p}$,*

$$G([\pi_K]_\phi(Z)) \in \varphi_q\big(\mathbf{B}^+_{\mathrm{rig},\mathbb{C}_p}\big).$$

Furthermore, the morphism $\mathbf{B}^+_{\mathrm{rig},\mathbb{C}_p} \to \varphi_q\big(\mathbf{B}^+_{\mathrm{rig},\mathbb{C}_p}\big)$ given by $Z \mapsto [\pi_K]_\phi(Z)$ is bijective.

Proof. By corollary 147, we can view $G = A_\mu$ for some $\mu \in D(\mathscr{O}_K, \mathbb{C}_p)$. Consider $\widetilde{\mu} \in D(\mathscr{O}_K, \mathbb{C}_p)$ defined by

$$\int_{\mathscr{O}_K} f(a)\, \widetilde{\mu}(a) = \varphi_q^{-1}\Big(\int_{\mathscr{O}_K} (\varphi_q \circ f)(a)\, \mu(a)\Big) \quad \text{for all } f \in C^{\mathrm{an}}(\mathscr{O}_K, \mathbb{C}_p),$$

so that $\varphi_q(\widetilde{\mu}) = (\pi_K)_*(\mu)$. Then

$$G([\pi_K]_\phi(Z)) = A_{(\pi_K)_*(\mu)}(Z) = A_{\varphi_q(\widetilde{\mu})}(Z) = \varphi_q(A_{\widetilde{\mu}}(Z)).$$

The assignation $\mu \mapsto \widetilde{\mu}$ is clearly bijective and we can reverse the construction. □

8.4.3 The action of ψ_q on distributions

Definition 150. We define the operator ψ_q on $D(\mathscr{O}_K, \mathbb{C}_p)$ to be the unique additive endomorphism of $D(\mathscr{O}_K, \mathbb{C}_p)$ satisfying that

$$\varphi_q \circ \psi_q = \frac{1}{\pi_L} \operatorname{Tr}_{D(\mathscr{O}_K, \mathbb{C}_p)/D(\pi_K \mathscr{O}_K, \mathbb{C}_p)} .$$

The operator ψ_q can be defined on $\mathscr{O}(\mathcal{B}/\mathbb{C}_p)$ and $\mathscr{O}(\mathcal{X}/\mathbb{C}_p)$ analogously.

Remark. In section 2.1.1 of their preprint [35], Schneider and Venjakob give a more explicit definition of ψ_q (in fact, without the factor π_L). Namely, they define $(\pi_K)_!\colon C^{\mathrm{an}}(\mathscr{O}_K, \mathbb{C}_p) \to C^{\mathrm{an}}(\mathscr{O}_K, \mathbb{C}_p)$ by

$$\big((\pi_K)_!(f)\big)(a) = \begin{cases} f(\pi_K^{-1} a) & \text{if } a \in \pi_K \mathscr{O}_K, \\ 0 & \text{otherwise,} \end{cases}$$

and consider its dual $(\pi_K)^!\colon D(\mathscr{O}_K, \mathbb{C}_p) \to D(\mathscr{O}_K, \mathbb{C}_p)$. Then

$$\psi_q(\mu) = \varphi_q^{-1} \circ \frac{1}{\pi_L}\big((\pi_K)^!(\mu)\big)(\varphi_q \circ \cdot\,) \quad \text{for all } \mu \in D(\mathscr{O}_K, \mathbb{C}_p).$$

8.4.4 The Mellin transform

By definition,

$$\psi_q \circ \varphi_q = \frac{q}{\varphi_q^{-1}(\pi_L)}.$$

Therefore, we get a decomposition

$$D(\mathscr{O}_K, \mathbb{C}_p) = \varphi_q\big(D(\mathscr{O}_K, \mathbb{C}_p)\big) \oplus D(\mathscr{O}_K, \mathbb{C}_p)^{\psi_q = 0}$$

$$\mu = \frac{\pi_L}{q}\varphi_q(\psi_q(\mu)) + \Big(\mu - \frac{\pi_L}{q}\varphi_q(\psi_q(\mu))\Big)$$

which allows us to identify $D(\mathscr{O}_K^\times, \mathbb{C}_p)$ with the second direct summand. Indeed, the decomposition $\mathscr{O}_K = \pi_K \mathscr{O}_K \sqcup \mathscr{O}_K^\times$ induces a decomposition

$$D(\mathscr{O}_K, \mathbb{C}_p) = D(\pi_K \mathscr{O}_K, \mathbb{C}_p) \oplus D(\mathscr{O}_K^\times, \mathbb{C}_p)$$

and we have seen in the proof of lemma 148 that $\varphi_q(D(\mathscr{O}_K, \mathbb{C}_p)) = D(\pi_K \mathscr{O}_K, \mathbb{C}_p)$.

Definition 151. The *Mellin transform* is the isomorphism

$$\mathfrak{M} \colon D(\Gamma_L, \mathbb{C}_p) \to \big(\mathbf{B}^+_{\mathrm{rig},\mathbb{C}_p}\big)^{\psi_q = 0}$$

obtained as the composition of the isomorphisms

$$D(\Gamma_L, \mathbb{C}_p) \cong D(\mathscr{O}_K^\times, \mathbb{C}_p) \cong D(\mathscr{O}_K, \mathbb{C}_p)^{\psi_q=0} \cong \mathscr{O}(\mathcal{X}/\mathbb{C}_p)^{\psi_q=0} \cong \mathscr{O}(\mathcal{B}/\mathbb{C}_p)^{\psi_q=0}$$

described above.

Lemma 152. *If $\mu \in D(\mathscr{O}_K, \mathbb{C}_p)$ corresponds to $A_\mu \in \mathbf{B}^+_{\mathrm{rig},\mathbb{C}_p}$ via the isomorphism of corollary 147, then the distribution*

$$f \longmapsto \Big(\int_{\mathscr{O}_K} a f(a)\, \mu(a)\Big)$$

corresponds to $\Omega^{-1}\partial_\phi(A_\mu) \in \mathbf{B}^+_{\mathrm{rig},\mathbb{C}_p}$.

Proof. First we claim that, given $z \in \mathcal{B}(\mathbb{C}_p)$, the distribution μ_z given by

$$\int_{\mathscr{O}_K} f(a)\, \mu_z(a) = \int_{\mathscr{O}_K} \kappa_z(a) f(a)\, \mu(a)$$

corresponds to $A_{\mu_z} = A_\mu(\mathfrak{F}_\phi(z, \cdot\,))$. Since the characters are dense in $C^{\mathrm{an}}(\mathscr{O}_K, \mathbb{C}_p)$ and we have the isomorphism $\mathcal{B}(\mathbb{C}_p) \cong \mathcal{X}(\mathbb{C}_p)$ from proposition 139, it suffices to check that $A_{\mu_z}(z') = A_\mu(\mathfrak{F}_\phi(z, z'))$ for all $z' \in \mathcal{B}(\mathbb{C}_p)$. Indeed,

$$A_{\mu_z}(z') = \int_{\mathscr{O}_K} \kappa_z(a)\kappa_{z'}(a)\, \mu(a) = \int_{\mathscr{O}_K} \kappa_{\mathfrak{F}_\phi(z,z')}(a)\, \mu(a) = A_\mu(\mathfrak{F}_\phi(z, z')).$$

Now, to prove the assertion of the lemma, we observe that

$$\partial_\phi(A_\mu)(Z) = \lim_{\varepsilon \to 0} \frac{A_\mu\big(\exp_\phi(\log_\phi(Z) + \varepsilon)\big) - A_\mu(Z)}{\varepsilon}$$

$$= \lim_{\varepsilon\to 0} \frac{A_\mu(Z +_{\mathfrak{F}_\phi} \exp_\phi(\varepsilon)) - A_\mu(Z)}{\varepsilon}$$

and use the claim with $z = \exp_\phi(\varepsilon)$. Thus, writing μ' for the distribution corresponding to $\partial_\phi(A_\mu)(Z)$,

$$\begin{aligned} \int_{\mathscr{O}_K} f(a)\,\mu'(a) &= \lim_{\varepsilon\to 0}\int_{\mathscr{O}_K}\left[\frac{\kappa_{\exp_\phi(\varepsilon)}(a)f(a) - f(a)}{\varepsilon}\right]\mu(a) \\ &= \lim_{\varepsilon\to 0}\int_{\mathscr{O}_K}\frac{F_{t_0'}([a]_\phi(\exp_\phi(\varepsilon)))}{\varepsilon}f(a)\,\mu(a) = \int_{\mathscr{O}_K}\Omega a f(a)\,\mu(a), \end{aligned}$$

where in the last equality we used that

$$\exp_\phi(Z) = Z + \cdots, \quad [a]_\phi(Z) = aZ + \cdots \quad \text{and} \quad F_{t_0'}(Z) = \Omega Z + \cdots. \qquad \square$$

8.5 Robba rings over $\mathbb{C}_p$

In the same way as we defined the rings $\mathbf{B}^+_{\mathrm{rig},L}$, $\mathbf{B}^\dagger_L$ and $\mathbf{B}^\dagger_{\mathrm{rig},L}$ using the rigid analytic variety $\mathcal{B}$ over L, we define rings $\mathbf{B}^+_{\mathrm{rig},\mathbb{C}_p}$, $\mathbf{B}^\dagger_{\mathbb{C}_p}$ and $\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}$ using the base change $\mathcal{B}_{\mathbb{C}_p}$. More precisely, all the definitions appearing in section 6.3 extend to $\mathbb{C}_p$ in the obvious way. (Note however that the structure of the Robba ring is more complicated over $\mathbb{C}_p$ than over a discretely valued field.) Similarly, definitions 78, 79 and 89 and proposition 80 work in the same way replacing $\mathbf{B}^\dagger_{\mathrm{rig},L}$ with $\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}$.

8.5.1 Explicit bases over the image of φ_q

As in section 1.1.3 of Colmez's article [17], we set

$$\eta(a, Z) = 1 + F_{at_0'}(Z) = \exp\big(a\Omega\log_\phi(Z)\big) \quad \text{for every } a \in \mathscr{O}_K.$$

Lemma 153. *Let $a, b \in \mathscr{O}_K$.*

(1) $\eta(a+b, Z) = \eta(a, Z)\eta(b, Z)$.

(2) $\eta(a, \mathfrak{F}_\phi(X, Y)) = \eta(a, X)\eta(a, Y)$.

(3) $\gamma(\eta(a, Z)) = \eta(\chi_\phi(\gamma)a, Z)$ *for every* $\gamma \in \Gamma_L$.

(4) $\varphi_q(\eta(a, Z)) = \eta(a\pi_K, Z) = \eta(a, [\pi_K]_\phi(Z))$.

(5) $\psi_q(\eta(a, Z)) = \begin{cases} \dfrac{q}{\varphi_q^{-1}(\pi_L)}\eta\left(\dfrac{a}{\pi_K}, Z\right) & \text{if } a \in \pi_K\mathscr{O}_K, \\ 0 & \text{otherwise.} \end{cases}$

Proof. The first three properties are clear from the definition of η and the properties of $\log_\phi$. The fourth property is just a rewriting of the definition of π_K given in section 8.4.2 and the last property follows from the relation $\psi_q \circ \varphi_q = q/\varphi_q^{-1}(\pi_L)$. □

Proposition 154. *Let $f \in \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}$. For every $n \in \mathbb{Z}_{\geq 1}$,*

$$f = \frac{\varphi_q^{n-1}(\pi_L)\cdots\varphi_q(\pi_L)\pi_L}{q^n} \sum_{a \in \mathscr{O}_K/\pi_K^n\mathscr{O}_K} (\varphi_q^n \circ \psi_q^n)\big(\eta(-a,Z)f\big)\,\eta(a,Z),$$

where the sum runs over any system of representatives of $\mathscr{O}_K/\pi_K^n\mathscr{O}_K$.

Proof. We prove the formula by induction on n using the expression of $\varphi_q \circ \psi_q$ in terms of Tr_{φ_q}.

For the base case $n = 1$, we write for each $a \in \mathscr{O}_K$

$$\begin{aligned}(\varphi_q \circ \psi_q)\big(\eta(-a,Z)f(Z)\big) &= \frac{1}{\pi_L} \sum_{v_1 \in \mathfrak{F}_{\phi,1}} \eta(-a,\mathfrak{F}_\phi(v_1,Z))f(\mathfrak{F}_\phi(v_1,Z)) \\ &= \frac{1}{\pi_L} \sum_{v_1 \in \mathfrak{F}_{\phi,1}} \eta(-a,v_1)\eta(-a,Z)f(\mathfrak{F}_\phi(v_1,Z)).\end{aligned}$$

But, given $v_1 \in \mathfrak{F}_{\phi,1}$, $\eta(\,\cdot\,,v_1)$ defines a finite character of $\mathbb{F}_q = \mathscr{O}_K/\pi_K\mathscr{O}_K$ which is trivial if and only if $v_1 = 0$. In particular,

$$\sum_{a \in \mathscr{O}_K/\pi_K\mathscr{O}_K} \eta(-a,v_1) = \begin{cases} q & \text{if } v_1 = 0, \\ 0 & \text{if } v_1 \neq 0. \end{cases}$$

Therefore, using that $\eta(-a,Z)\eta(a,Z) = 1$ and summing first over $a \in \mathscr{O}_K/\pi_K\mathscr{O}_K$ and then over $v_1 \in \mathfrak{F}_{\phi,1}$, we conclude that

$$\sum_{a \in \mathscr{O}_K/\pi_K\mathscr{O}_K} (\varphi_q \circ \psi_q)\big(\eta(-a,Z)f(Z)\big)\eta(a,Z) = \frac{q}{\pi_L} f(Z)$$

as desired.

Now suppose that we have the formula for n and let us prove it for $n+1$. We can pick a system of representatives of $\mathscr{O}_K/\pi_K^{n+1}\mathscr{O}_K$ of the form $c = a + \pi_K^n b$ with $a \in \mathscr{O}_K/\pi_K^n\mathscr{O}_K$ and $b \in \mathscr{O}_K/\pi_K\mathscr{O}_K$. Then

$$\eta(c,Z) = \eta(a+\pi_K^n b,Z) = \eta(a,Z)\eta(\pi_K^n b,Z) = \eta(a,Z)\varphi_q^n(\eta(b,Z))$$

and by the projection formula

$$
\begin{aligned}
&(\varphi_q^{n+1} \circ \psi_q^{n+1})\big(\eta(-c,Z)f(Z)\big)\eta(c,Z) \\
&\quad = \big(\varphi_q^n \circ (\varphi_q \circ \psi_q) \circ \psi_q^n\big)\big(\varphi_q^n(\eta(-b,Z))\eta(-a,Z)f(Z)\big)\varphi_q^n(\eta(b,Z))\eta(a,Z) \\
&\quad = \varphi_q^n\Big((\varphi_q \circ \psi_q)\big(\eta(-b,Z)\psi_q^n(\eta(-a,Z)f(Z))\big)\eta(b,Z)\Big)\eta(a,Z).
\end{aligned}
$$

Summing over $b \in \mathcal{O}_K/\pi_K\mathcal{O}_K$ and using the base case we obtain that

$$
\begin{aligned}
\sum_{b \in \mathcal{O}_K/\pi_K\mathcal{O}_K} &(\varphi_q \circ \psi_q)\big(\eta(-b,Z)\psi_q^n(\eta(-a,Z)f(Z))\big)\eta(b,Z) \\
&= \frac{q}{\pi_L}\psi_q^n(\eta(-a,Z)f(Z)).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\sum_{c \in \mathcal{O}_K/\pi_K^{n+1}\mathcal{O}_K} &(\varphi_q^n \circ \psi_q^{n+1})\big(\eta(-c,Z)f(Z)\big)\eta(c,Z) \\
&= \frac{q}{\varphi_q^n(\pi_L)} \sum_{a \in \mathcal{O}_K/\pi_K^n\mathcal{O}_K} (\varphi_q^n \circ \psi_q^n)\big(\eta(-a,Z)f(Z)\big)\eta(a,Z) \\
&= \frac{q^{n+1}}{\varphi_q^n(\pi_L)\cdots\varphi_q(\pi_L)\pi_L} f(Z)
\end{aligned}
$$

by the induction hypothesis. □

Corollary 155. *Let* $\mathscr{M} \in \mathrm{Ob}\big(\varphi_q\text{–}\mathrm{Mod}_{\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p},\mathrm{fr}}\big)$ *(resp.* $\mathscr{M} \in \mathrm{Ob}\big(\varphi_q\text{–}\mathrm{Mod}_{\mathbf{B}^{+}_{\mathrm{rig},\mathbb{C}_p},\mathrm{fr}}\big)$*). For every* $n \in \mathbb{Z}_{\geq 1}$ *and every system of representatives of* $\mathcal{O}_K/\pi_K^n\mathcal{O}_K$*, we have a decomposition of LF (resp. Fréchet) spaces*

$$
\mathscr{M} = \bigoplus_{a \in \mathcal{O}_K/\pi_K^n\mathcal{O}_K} \eta(a,Z)\varphi_q^n(\mathscr{M})
$$

and also decompositions of Banach spaces

$$
\mathscr{M}|_{\mathcal{B}_{\mathbb{C}_p}[r,s]} = \bigoplus_{a \in \mathcal{O}_K/\pi_K^n\mathcal{O}_K} \eta(a,Z)\varphi_q^n\big(\mathscr{M}|_{\mathcal{B}_{\mathbb{C}_p}[r^{q^n},s^{q^n}]}\big)
$$

for all $r,s \in p^{\mathbb{Q}}$ *with* $0 \ll r \leq s < 1$ *(resp.* $r \leq s < 1$*).*

Proof. These decompositions follow from the formula in proposition 154 and (iterates of) the isomorphism $(1 \otimes \varphi_q)\colon \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p} \otimes_{\varphi_q,\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}} \mathscr{M} \to \mathscr{M}$. □

Corollary 156. *Let* $\mathscr{M} \in \mathrm{Ob}\big(\varphi_q\text{–}\mathrm{Mod}_{\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p},\mathrm{fr}}\big)$ *or* $\mathscr{M} \in \mathrm{Ob}\big(\varphi_q\text{–}\mathrm{Mod}_{\mathbf{B}^{+}_{\mathrm{rig},\mathbb{C}_p},\mathrm{fr}}\big)$*. The decomposition in corollary 155 (for* $n = 1$*) restricts to*

$$\mathscr{M}^{\psi_q=0} = \bigoplus_{a\in(\mathscr{O}_K/\pi_K\mathscr{O}_K)^\times} \eta(a,Z)\varphi_q(\mathscr{M}).$$

In particular, if we apply corollary 156 to $\mathbf{B}^{+}_{\mathrm{rig},\mathbb{C}_p}$ and use the isomorphism $D(\mathscr{O}_K,\mathbb{C}_p) \cong \mathbf{B}^{+}_{\mathrm{rig},\mathbb{C}_p}$ from corollary 147, we recover the decomposition explained in the beginning of section 8.4.4.

8.5.2 The Robba ring of Γ_L

Next we recall the construction of the ring $\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L)$ following section 2.2.2 of Schneider–Venjakob's preprint [35].

The isomorphism $\mathbf{B}^{+}_{\mathrm{rig},\mathbb{C}_p} \cong D(\mathscr{O}_K,\mathbb{C}_p)$ from corollary 147 sends Z to some distribution μ_Z. Then, if the symbol ? means either a subinterval of $(0,1)$ as the ones appearing in section 6.3 or nothing, we can define a ring $\mathbf{B}^{\dagger,?}_{\mathrm{rig},\mathbb{C}_p}(\mathscr{O}_K)$ by formally replacing Z with μ_Z in $\mathbf{B}^{\dagger,?}_{\mathrm{rig},\mathbb{C}_p}$. Our objective is to extend this construction to define a ring $\mathbf{B}^{\dagger,?}_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L)$.

Recall that the Lubin–Tate character χ_ϕ defines isomorphisms $\Gamma_L \cong \mathscr{O}_K^\times$ and $\Gamma_{L_n} = \mathrm{Gal}(L_\infty/L_n) \cong (1+\pi_K^n\mathscr{O}_K)^\times$ for all $n \in \mathbb{Z}_{\geq 1}$. Fix $n_0 \in \mathbb{Z}_{\geq 1}$ such that log and exp define isomorphisms between $(1+\pi_K^{n_0}\mathscr{O}_K)^\times$ and $\pi_K^{n_0}\mathscr{O}_K$. Consider the isomorphisms $\ell_n\colon \Gamma_{L_n} \to \mathscr{O}_K$ defined by

$$\ell_n(\gamma) = \frac{1}{\pi_K^n}\log(\chi_\phi(\gamma))$$

for all $n \geq n_0$. We get isomorphisms $\ell_{n,*}\colon D(\Gamma_{L_n},\mathbb{C}_p) \to D(\mathscr{O}_K,\mathbb{C}_p)$ and set $\mu_{Z,n} = \ell_{n,*}^{-1}(\mu_Z)$. We define rings $\mathbf{B}^{\dagger,?}_{\mathrm{rig},\mathbb{C}_p}(\Gamma_{L_n})$ by formally replacing Z with $\mu_{Z,n}$ in $\mathbf{B}^{\dagger,?}_{\mathrm{rig},\mathbb{C}_p}$.

Consider $m,n \in \mathbb{Z}$ with $m \geq n \geq n_0$. The natural inclusion $\Gamma_{L_m} \subseteq \Gamma_{L_n}$ induces a morphism $\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}(\Gamma_{L_m}) \to \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}(\Gamma_{L_n})$ that we want to study. Observe that we have a commutative diagram

$$\begin{array}{ccc}
\Gamma_{L_m} & \xrightarrow{\ \ell_m\ } & \mathscr{O}_K \\
\big\downarrow & & \big\downarrow{\scriptstyle \cdot\pi_K^{m-n}} \\
\Gamma_{L_n} & \xrightarrow{\ \ell_n\ } & \mathscr{O}_K
\end{array}$$

and that the action of π_K on distributions is "almost" φ_q. More precisely, the computations in section 8.4.2 show that φ_q^{m-n} acts on $D(\mathscr{O}_K, \mathbb{C}_p)$ as the composition of the automorphism $\mu \mapsto \varphi_q^{m-n} \circ \mu(\varphi_q^{m-n} \circ \cdot)$ and $(\pi_K^{m-n})_*$. All in all, the inclusion $\Gamma_{L_m} \subseteq \Gamma_{L_n}$ induces a commutative diagram

$$\begin{array}{ccc} \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_{L_m}) & \xrightarrow[\cong]{\ell_{m,*}} & \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\mathscr{O}_K) \cong \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p} \\ \downarrow & & \downarrow{\scriptstyle Z \mapsto [\pi_K^{m-n}]_\phi(Z)} \\ \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_{L_n}) & \xrightarrow[\cong]{\ell_{n,*}} & \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\mathscr{O}_K) \cong \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p} \end{array}$$

where the dashed arrow defines an isomorphism from $\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}$ to $\varphi_q^{m-n}(\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p})$ by lemma 149.

On the other hand, corollary 155 shows that

$$\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p} = \bigoplus_{a \in \mathscr{O}_K/\pi_K^{m-n}\mathscr{O}_K} \eta(a, Z)\varphi_q^{m-n}(\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}).$$

Equivalently, we may view $\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}$ as a module over $\pi_K^{m-n}\mathscr{O}_K$ via $\eta(\,\cdot\,/\pi_K^{m-n}, Z)$ and then

$$\begin{aligned} \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p} \rtimes_{\pi_K^{m-n}\mathscr{O}_K} \mathscr{O}_K &\longrightarrow \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p} \\ f(Z) \otimes [a] &\longmapsto f([\pi_K^{m-n}]_\phi(Z))\eta(a, Z) \end{aligned}$$

is an isomorphism. Therefore, the natural map $\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_{L_m}) \hookrightarrow \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_{L_n})$ induces an isomorphism

$$\begin{aligned} \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_{L_m}) \rtimes_{\Gamma_{L_m}} \Gamma_{L_n} &\longrightarrow \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_{L_n}) \\ f(\mu_{Z,m}) \otimes [\gamma] &\longmapsto f([\pi_K^{m-n}]_\phi(\mu_{Z,n}))\delta_\gamma \end{aligned}$$

making the diagram

$$\begin{array}{ccc} \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_{L_m}) \rtimes_{\Gamma_{L_m}} \Gamma_{L_n} & \xrightarrow{\cong} & \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_{L_n}) \\ {\scriptstyle \ell_{m,*}}\downarrow{\scriptstyle \cong} \quad {\scriptstyle \cong}\downarrow{\scriptstyle \ell_{n,*}} & & {\scriptstyle \cong}\downarrow{\scriptstyle \ell_{n,*}} \\ \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p} \rtimes_{\pi_K^{m-n}\mathscr{O}_K} \mathscr{O}_K & \xrightarrow[\cong]{} & \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p} \end{array}$$

commutative.

Definition 157. The *Robba ring of* Γ_L is

$$\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L) = \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_{L_{n_0}}) \rtimes_{\Gamma_{L_{n_0}}} \Gamma_L.$$

Definition 158. We define $\iota\colon \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L) \to \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L)$ to be the involution induced by the automorphism $\gamma \mapsto \gamma^{-1}$ of Γ_L. More precisely, ι acts on $\mathbb{Z}[\Gamma_L]$ by $[\gamma] \mapsto [\gamma^{-1}]$ and on $\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}(\Gamma_{L_{n_0}})$ by requiring that the diagram

$$\begin{array}{ccc} \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}(\Gamma_{L_{n_0}}) & \xrightarrow[\cong]{\ell_{n_0,*}} & \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p} \\ \downarrow{\scriptstyle\iota} & & \downarrow{\scriptstyle\gamma_{-1}} \\ \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}(\Gamma_{L_{n_0}}) & \xrightarrow[\cong]{\ell_{n_0,*}} & \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p} \end{array}$$

be commutative, where $\gamma_{-1} \in \Gamma_L$ is characterized by $\chi_\phi(\gamma_{-1}) = -1 \in \mathscr{O}_K^\times$ (i.e., the right vertical arrow is defined by $Z \mapsto [-1]_\phi(Z)$).

8.5.3 The action of $\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L)$ on $\mathrm{Ker}(\psi_q)$

Let $\mathscr{M} \in \mathrm{Ob}\big((\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{\mathrm{an}}_{\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p},\mathrm{fr}}\big)$. One might wonder if the action of Γ_L on $\mathscr{M}$ extends to a continuous action of $\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L)$ (where we identify each $\gamma \in \Gamma_L$ with the corresponding Dirac distribution $\delta_\gamma \in D(\Gamma_L, \mathbb{C}_p) \subset \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L)$). In their preprint [35], Schneider and Venjakob prove that this is true at least for $\mathscr{M}^{\psi_q=0}$.

By corollary 156, we obtain an isomorphism

$$\begin{aligned} \eta(1,Z)\varphi_q(\mathscr{M}) \rtimes_{\Gamma_{L_1}} \Gamma_L &\longrightarrow \bigoplus_{a\in(\mathscr{O}_K/\pi_K\mathscr{O}_K)^\times} \eta(a,Z)\varphi_q(\mathscr{M}) = \mathscr{M}^{\psi_q=0} \\ \eta(1,Z)\varphi_q(m) \otimes [\gamma] &\longmapsto \gamma\big(\eta(1,Z)\varphi_q(m)\big) = \eta(\chi_\phi(\gamma), Z)\varphi_q(\gamma(m)) \end{aligned}$$

(well-defined by the identity (4) of lemma 153) thanks to which it suffices to prove that the action of Γ_{L_1} induces a continuous action of $\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}(\Gamma_{L_1})$ on $\eta(1,Z)\varphi_q(\mathscr{M})$. This is done in sections 2.2.4 and 2.2.5 of Schneider–Venjakob's preprint [35], where they use proposition 80 to work over $\mathscr{M}|_{\mathcal{B}_{\mathbb{C}_p}[r,s]}$ for certain closed intervals $[r,s] \subset (0,1)$ and then reduce the problem to the analysis of an analogous action of $\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}(\Gamma_{L_n})$ for $n \gg 0$.

Theorem 159 (Schneider–Venjakob). *Let $\mathscr{M} \in \mathrm{Ob}\big((\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{\mathrm{an}}_{\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p},\mathrm{fr}}\big)$. The action of Γ_L on $\mathscr{M}$ induces a unique continuous action of $\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L)$ on $\mathscr{M}^{\psi_q=0}$ which makes $\mathscr{M}^{\psi_q=0}$ into a free $\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L)$–module of the same rank as $\mathscr{M}$: if $e_1, \dots, e_r$ is a $\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}$–basis of $\mathscr{M}$, then $\eta(1,Z)\varphi_q(e_1), \dots, \eta(1,Z)\varphi_q(e_r)$ is a $\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L)$–basis of $\mathscr{M}^{\psi_q=0}$.*

Proof. See theorem 2.31 of Schneider–Venjakob's preprint [35], whose proof works exactly in the same way for the *relative* Lubin–Tate situation. □

8.5.4 The Mellin transform

In section 8.4.4 we introduced an isomorphism $\mathfrak{M}\colon D(\Gamma_L, \mathbb{C}_p) \to \left(\mathbf{B}^{+}_{\mathrm{rig},\mathbb{C}_p}\right)^{\psi_q=0}$ that we can now extend using the constructions of sections 8.5.2 and 8.5.3.

By corollary 147, for every $\mu \in D(\mathscr{O}_K, \mathbb{C}_p)$ we have

$$A_\mu(Z) = \int_{\mathscr{O}_K} \eta(a, Z)\, \mu(a).$$

In particular, if δ_1 is the Dirac distribution supported on 1, then $A_{\delta_1}(Z) = \eta(1, Z)$. But δ_1 is the unit element of $D(\mathscr{O}_K^\times, \mathbb{C}_p)$ with respect to the convolution product, which means that every $\lambda \in D(\mathscr{O}_K^\times, \mathbb{C}_p)$ can be expressed as $\lambda \cdot \delta_1$. These observations combined with theorem 159 motivate the following definition.

Definition 160. The *Mellin transform* is the isomorphism

$$\begin{aligned} \mathfrak{M}\colon \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L) &\longrightarrow \left(\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}\right)^{\psi_q=0} \\ \lambda &\longmapsto \lambda\big(\eta(1, Z)\big) \end{aligned}$$

induced by the action of Γ_L on $\left(\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}\right)^{\psi_q=0}$ (see theorem 159).

Remark. The fact that this definition does extend definition 151 follows from lemmata 2.2 and 2.5 of Schneider–Venjakob's preprint [35].

Since we always identify Γ_L with $\mathscr{O}_K^\times$ via χ_ϕ, lemma 152 motivates the next definition.

Definition 161. We define the *twist by χ_ϕ* to be the unique isomorphism Tw_{χ_ϕ} that makes the diagram

$$\begin{array}{ccc} \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L) & \xrightarrow{\mathrm{Tw}_{\chi_\phi}} & \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L) \\ {\scriptstyle\mathfrak{M}}\downarrow & & \downarrow{\scriptstyle\mathfrak{M}} \\ \left(\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}\right)^{\psi_q=0} & \xrightarrow{\Omega^{-1}\partial_\phi} & \left(\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}\right)^{\psi_q=0} \end{array}$$

commutative. More generally, for $i \in \mathbb{Z}_{\geq 0}$ we define $\mathrm{Tw}_{\chi_\phi^i}$ to be the composition of Tw_{χ_ϕ} with itself i times and $\mathrm{Tw}_{\chi_\phi^{-i}} = \mathrm{Tw}^{-1}_{\chi_\phi^i}$.

Remark. The fact that $\Omega^{-1}\partial_\phi$ defines an automorphism of $\left(\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}\right)^{\psi_q=0}$ follows from proposition 2.12 of Fourquaux–Xie's article [22].

8.5.5 The operators Θ_b and Ξ_b

By lemma 97, we have an operator N_∇ on $\mathbf{B}^+_{\mathrm{rig},\mathbb{C}_p}$ given by $N_\nabla = \log_\phi(Z)\partial_\phi$ (cf. the calculations in the proof of lemma 2.1.4 of Kisin–Ren's article [28]). It turns out that this operator corresponds to the action of a distribution in $D(\Gamma_L, \mathbb{C}_p)$ that we call again N_∇. Indeed, we can compute

$$N_\nabla(\eta(1,Z)) = \log_\phi(Z) \cdot \frac{d}{d\log_\phi(Z)}\left[\exp\left(\Omega \log_\phi(Z)\right)\right] = \Omega \log_\phi(Z)\eta(1,Z),$$

which corresponds to $N_\nabla \cdot \delta_1$ via the isomorphism $\mathbf{B}^+_{\mathrm{rig},\mathbb{C}_p} \cong D(\mathscr{O}_K, \mathbb{C}_p)$. Writing

$$\Omega \log_\phi(Z)\eta(1,Z) = \varphi_q\left(\varphi_q^{-1}\left(\frac{\Omega}{\pi_L}\right)\log_\phi(Z)\right)\eta(1,Z),$$

we see that this power series belongs to $\left(\mathbf{B}^+_{\mathrm{rig},\mathbb{C}_p}\right)^{\psi_q=0}$ by corollary 156. Therefore, $N_\nabla = \mathfrak{M}^{-1}\left(\Omega \log_\phi(Z)\eta(1,Z)\right) \in \mathbf{D}(\Gamma_L, \mathbb{C}_p) \subset \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L)$.

Remark. The operator N_∇ is often called ∇ in the literature. Moreover, in section 2.1.2 of their preprint [35], Schneider and Venjakob define ∇ directly as a distribution given by the element of $\mathrm{Lie}(\Gamma_L)$ corresponding to 1 via the isomorphism $\mathrm{Lie}(\Gamma_L) \cong \mathrm{Lie}(\mathscr{O}_K^\times) = K$ induced by χ_ϕ and then one can check that it acts as our N_∇ (see lemma 2.14 and corollary 2.15 of Berger–Schneider–Xie's article [8]). In fact, since $\mathrm{Lie}(\Gamma_L) = \mathrm{Lie}(\Gamma_{L_n})$ for any $n \in \mathbb{Z}_{\geq 1}$, one can view $\nabla \in D(\Gamma_{L_n}, \mathbb{C}_p)$ too.

Proposition 162. *Consider $n \in \mathbb{Z}_{\geq 1}$ and let $b = (b_1, \ldots, b_m)$ be a $\mathbb{Z}_p$–basis of Γ_{L_n}. If n is large enough,*

$$\Theta_b = q^{-n}\prod_{j=1}^{m}\left(\log(\chi_\phi(b_j))\frac{N_\nabla}{\delta_{b_j} - 1}\right)$$

(where δ_{b_j} denotes the Dirac distribution supported on b_j) is a well-defined element of $D(\Gamma_{L_n}, \mathbb{C}_p) \subset \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_{L_n})$ and we can write

$$\mathfrak{M}(\Theta_b) = \varphi_q^n(\xi_b)\eta(1,Z)$$

with

$$\xi_b \equiv q^{-n}\frac{\log_\phi(Z)}{Z} = q^{-n}\lambda(Z) \mod \log_\phi(Z)\mathbf{B}^+_{\mathrm{rig},\mathbb{C}_p}.$$

In particular,

$$\Xi_b = \frac{\Theta_b}{N_\nabla}$$

is a well-defined element of $\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}(\Gamma_{L_n})$*.*

Proof. This is analogous to remark 2.34 of Schneider–Venjakob's preprint [35] (but observe that the definition of Θ_b in loc. cit. does not include the normalizing factors that we used). We adapt it here to the *relative* Lubin–Tate case for the convenience of the reader.

Consider the power series

$$F(Z) = \frac{Z}{\exp(Z) - 1} = 1 + \cdots \in \mathbb{Q}_p[\![Z]\!],$$

which has a positive radius of convergence. Thus, we choose $r \in p^{\mathbb{Q}}$ small enough and work in $\mathbf{B}^{\dagger,[0,r]}_{\mathrm{rig},\mathbb{C}_p}$. As operators, using lemma 97 for $n \gg 0$ we can write

$$\begin{aligned} \log(\chi_\phi(b_j))\frac{N_\nabla}{\delta_{b_j} - 1} = \frac{\log(\chi_\phi(b_j))N_\nabla}{\exp(\log(\chi_\phi(b_j))N_\nabla) - 1} &= F(\log(\chi_\phi(b_j))N_\nabla) \\ &= 1 + \log(\chi_\phi(b_j))N_\nabla \cdot g_j(\log(\chi_\phi(b_j))N_\nabla) \end{aligned}$$

for some $g_j \in \mathbf{B}^{\dagger,[0,r]}_{\mathrm{rig},\mathbb{C}_p}$. Multiplying these equations for $1 \leq j \leq m$, we can express

$$\Theta_b = q^{-n} + N_\nabla \cdot g\big(\log(\chi_\phi(b_1))N_\nabla, \ldots, \log(\chi_\phi(b_m))N_\nabla\big)$$

for some power series g. Since

$$N_\nabla(\eta(1, Z)) = \Omega \log_\phi(Z)\eta(1, Z) \quad \text{and} \quad N_\nabla(\Omega \log_\phi(Z)) = \Omega \log_\phi(Z),$$

iterated uses of Leibniz rule show that

$$N^n_\nabla(\eta(1, Z)) = \left[\prod_{i=0}^{n-1}(\Omega \log_\phi(Z) + i)\right] \cdot \eta(1, Z).$$

All in all, we can express

$$\mathfrak{M}(\Theta_b) = \big[q^{-n} + \Omega \log_\phi(Z)f(Z)\big] \cdot \eta(1, Z) \quad \text{with } f(Z) \in \mathbf{B}^{\dagger,[0,r]}_{\mathrm{rig},\mathbb{C}_p}.$$

On the other hand, the calculations of lemma 2.4.1 and the first part of lemma

2.4.2 of Berger–Fourquaux's article [7] show that we can express

$$\Theta_b(\eta(1,Z)) = \frac{\log_\phi(Z)}{\varphi_q^n(Z)} h(Z) = \varphi_q^n\left(\frac{\log_\phi(Z)}{\varphi_q^{-n}(\pi_L)\cdots\varphi_q^{-1}(\pi_L)Z}\right) h(Z)$$

for some $h(Z) \in \mathbf{B}^+_{\mathrm{rig},\mathbb{C}_p}$. In fact, since $\psi_q\big(\Theta_b(\eta(1,Z))\big) = 0$, we deduce from the projection formula that $h(Z) \in \big(\mathbf{B}^+_{\mathrm{rig},\mathbb{C}_p}\big)^{\psi_q=0}$ and it must be of the form

$$h(Z) = \sum_{a\in(\mathscr{O}_K/\pi_K\mathscr{O}_K)^\times} \varphi_q(h_a(Z))\eta(a,Z)$$

by corollary 156. But we may view $\Theta_b \in D(\Gamma_{L_n},\mathbb{C}_p) \subset D(\Gamma_L,\mathbb{C}_p)$, whence

$$\mathfrak{M}(\Theta_b) \in \varphi_q^n\big(\mathbf{B}^+_{\mathrm{rig},\mathbb{C}_p}\big)\eta(1,Z).$$

This is possible only if $h(Z) = \varphi_q(h_1(Z))\eta(1,Z)$ and

$$\frac{\log_\phi(Z)}{\varphi_q^n(Z)}\varphi_q(h_1(Z)) = \varphi_q^n(\widetilde{h}(Z)) \quad \text{for some } \widetilde{h}(Z) \in \mathbf{B}^+_{\mathrm{rig},\mathbb{C}_p}.$$

In particular,

$$\varphi_q(h_1(Z)) = \varphi_q^n\left(\varphi_q^{-n}(\pi_L)\cdots\varphi_q^{-1}(\pi_L)\frac{Z\widetilde{h}(Z)}{\log_\phi(Z)}\right)$$

and we can define $c(Z) = Z\widetilde{h}(Z)/\log_\phi(Z) \in \mathbf{B}^+_{\mathrm{rig},\mathbb{C}_p}$. Then

$$\mathfrak{M}(\Theta_b) = \varphi_q^n\left(\frac{\log_\phi(Z)}{Z}c(Z)\right)\eta(1,Z) = \varphi_q^n\big(\lambda(Z)c(Z)\big)\eta(1,Z).$$

Comparing the two expressions of $\mathfrak{M}(\Theta_b)$, we see that

$$\varphi_q^n\big(\lambda(Z)c(Z)\big) = q^{-n} + \Omega\log_\phi(Z)f(Z)$$

and, evaluating both sides at $Z=0$, we see that $c(Z) = q^{-n} + \cdots$, which concludes the proof. □

Proposition 163. *Keeping the notation and assumptions of proposition 162, the image*

$\widetilde{\Xi}_b$ of Ξ_b under $\ell_{n,*}\colon \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}(\Gamma_{L_n}) \to \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}$ is

$$\widetilde{\Xi}_b = q^{-n}\Big(\frac{\Omega}{\pi_K^n}\log_\phi(Z)\Big)^{m-1}\prod_{j=1}^{m}\frac{\log(\chi_\phi(b_j))}{\eta(\ell_n(b_j),Z)-1}.$$

In particular,

$$\widetilde{\Xi}_b \equiv \frac{\pi_K^n}{q^n\Omega Z} \mod \mathbf{B}^{+}_{\mathrm{rig},\mathbb{C}_p}.$$

Proof. This is analogous to remark 2.13 in Schneider–Venjakob's preprint [35] (but note that the definition of Ξ_b in loc. cit. does not include the normalizing factors that we used). We reproduce the proof here for the convenience of the reader.

By corollary 147, the isomorphism

$$D(\Gamma_{L_n},\mathbb{C}_p) \xrightarrow{\ell_{n,*}} D(\mathscr{O}_K,\mathbb{C}_p) \cong \mathbf{B}^{+}_{\mathrm{rig},\mathbb{C}_p}$$

sends $\mu \in D(\Gamma_{L_n},\mathbb{C}_p)$ to

$$\widetilde{A}_\mu(Z) = \int_{\Gamma_L}\eta(\ell_n(\gamma),Z)\,\mu(\gamma)$$

Applying this to δ_{b_j} for $1 \le j \le m$, we obtain the denominators in the formula for $\widetilde{\Xi}_b$. It remains to prove that $\widetilde{A}_{N_\nabla}(Z) = \Omega\log_\phi(Z)/\pi_K^n$. But indeed, using χ_ϕ to identify Γ_L and $\mathscr{O}_K^\times$, we can regard the distribution N_∇ as the element 1 of $\mathrm{Lie}(\Gamma_L) \cong L$ and then

$$\begin{aligned} N_\nabla\big(\eta(\pi_K^{-n}\log(\,\cdot\,),Z)\big) &= \frac{d}{dt}\Big[\eta\big(\pi_K^{-n}\log(\exp(1\cdot t)),Z\big)\Big]\Big|_{t=0} \\ &= \frac{d}{dt}\Big[\exp\Big(\frac{\Omega}{\pi_K^n}t\log_\phi(Z)\Big)\Big]\Big|_{t=0} = \frac{\Omega}{\pi_K^n}\log_\phi(Z). \end{aligned}$$

The last formula follows easily by plugging the expansions

$$\eta(\ell_n(b_j),Z)-1 = \frac{\Omega}{\pi_K^n}\log(\chi_\phi(b_j))Z+\cdots \quad \text{and} \quad \log_\phi(Z) = Z+\cdots$$

in the formula for $\widetilde{\Xi}_b$. □

9 Iwasawa cohomology and duality

In this section we adapt the results of Schneider–Venjakob's article [34] to express Iwasawa cohomology with respect to the *relative* Lubin–Tate tower L_∞/L in terms of (φ_q, Γ_L)–modules. Conversely, we can construct cohomology classes from (φ_q, Γ_L)–modules using results of Berger–Fourquaux's article [7] and Schneider–Venjakob's preprint [35].

Apart from that, we adapt other kinds of dualities defined in Schneider–Venjakob's preprint [35] in terms of the Robba rings introduced in the previous sections.

9.1 Cohomology of representations

9.1.1 Iwasawa cohomology

Definition 164.

(1) Let $T \in \operatorname{Ob}\big(\operatorname{Rep}_{\mathscr{O}_K}(G_L)\big)$. We define the *Iwasawa cohomology groups*

$$\mathrm{H}^i_{\mathrm{Iw}}(L_\infty/L, T) = \varprojlim_{n\geq 1} \mathrm{H}^i(L_n, T),$$

where the projective limit is taken with respect to the corestriction maps.

(2) Let $V \in \operatorname{Ob}\big(\operatorname{Rep}_K(G_L)\big)$ and let T be a G_L–stable $\mathscr{O}_K$–lattice of V. We define the *Iwasawa cohomology groups*

$$\mathrm{H}^i_{\mathrm{Iw}}(L_\infty/L, V) = \mathrm{H}^i_{\mathrm{Iw}}(L_\infty/L, T) \otimes_{\mathscr{O}_K} K.$$

(This is independent of the choice of lattice T.)

Lemma 165 (Shapiro). *Let $T \in \operatorname{Ob}\big(\operatorname{Rep}_{\mathscr{O}_K}(G_L)\big)$. There are canonical isomorphisms*

$$\mathrm{H}^i_{\mathrm{Iw}}(L_\infty/L, T) \cong \mathrm{H}^i(L, \Lambda_{\mathscr{O}_K}(\Gamma_L) \otimes_{\mathscr{O}_K} T)$$

for all $i \in \mathbb{Z}$, where $\Lambda_{\mathscr{O}_K}(\Gamma_L) = \mathscr{O}_K[\![\Gamma_L]\!]$.

Proof. See lemma 5.8 of Schneider–Venjakob's article [34]. □

Proposition 166. *The Iwasawa cohomology $\mathrm{H}^*_{\mathrm{Iw}}(L_\infty/L, \cdot\,)$ is a cohomological δ–functor on $\operatorname{Rep}_{\mathscr{O}_K}(G_L)$.*

Proof. See lemma 5.9 of Schneider–Venjakob's article [34]. □

9.1.2 Local Tate duality

In this subsection we recall the duality between Galois cohomology groups of a representation and of some kind of dual representation. The notion of *dual representation* depends on the kind of representation that we consider; namely, we have to distinguish between *free* and *torsion* (finite) $\mathscr{O}_K$–modules.

Definition 167. The *Pontryagin dual* of $T \in \mathrm{Ob}\big(\mathrm{Rep}_{\mathscr{O}_K,\mathrm{tor}}(G_L)\big)$ is the representation

$$T^\vee = \mathrm{Hom}_{\mathscr{O}_K}(T, K/\mathscr{O}_K).$$

Remark. Lemma 5.3 (and the paragraph preceding it) of Schneider–Venjakob's article [34] shows that one can define Pontryagin duals using either $\mathbb{Q}_p/\mathbb{Z}_p$ or $K/\mathscr{O}_K$. Note that, since T is of finite length, it is endowed with the discrete topology and so our definition really coincides with that of Schneider and Venjakob.

Theorem 168 (local Tate duality for torsion representations). *Let L' be a finite extension of L and let $T \in \mathrm{Ob}\big(\mathrm{Rep}_{\mathscr{O}_K,\mathrm{tor}}(G_L)\big)$. The cup product and the local invariant map induce perfect pairings of $\mathscr{O}_K$–modules*

$$\mathrm{H}^i(L', T) \times \mathrm{H}^{2-i}\big(L', \mathrm{Hom}_{\mathbb{Z}_p}\big(T, (\mathbb{Q}_p/\mathbb{Z}_p)(1)\big)\big) \to \mathrm{H}^2\big(L', (\mathbb{Q}_p/\mathbb{Z}_p)(1)\big) \cong \mathbb{Q}_p/\mathbb{Z}_p$$

and

$$\mathrm{H}^i(L', T) \times \mathrm{H}^{2-i}\big(L', \mathrm{Hom}_{\mathscr{O}_K}\big(T, (K/\mathscr{O}_K)(1)\big)\big) \to \mathrm{H}^2\big(L', (K/\mathscr{O}_K)(1)\big) \cong K/\mathscr{O}_K$$

for all $i \in \mathbb{Z}$. Therefore, there are canonical isomorphisms

$$\mathrm{H}^i(L', T) \cong \mathrm{H}^{2-i}(L', T^\vee(1))^\vee.$$

Proof. See proposition 5.7 of Schneider–Venjakob's article [34]. □

Corollary 169. *Let $T \in \mathrm{Ob}\big(\mathrm{Rep}_{\mathscr{O}_K,\mathrm{tor}}(G_L)\big)$. Local Tate duality induces isomorphisms*

$$\mathrm{H}^i_{\mathrm{Iw}}(L_\infty/L, T) \cong \mathrm{H}^{2-i}(L_\infty, T^\vee(1))^\vee$$

for all $i \in \mathbb{Z}$.

Proof. The corollary follows from theorem 168 by taking limits over the L_n for $n \geq 1$. □

Definition 170. The *dual* of $T \in \mathrm{Ob}\big(\mathrm{Rep}_{\mathscr{O}_K,\mathrm{fr}}(G_L)\big)$ is the representation

$$T^* = \mathrm{Hom}_{\mathscr{O}_K}(T, \mathscr{O}_K).$$

Remark. As in the torsion case, the paragraph before lemma 5.3 of Schneider–Venjakob's article [34] shows that one can define duals using $\mathbb{Z}_p$ instead of $\mathscr{O}_K$.

Theorem 171 (local Tate duality for free representations). *Let L' be a finite extension of L and let $T \in \mathrm{Ob}\big(\mathrm{Rep}_{\mathscr{O}_K,\mathrm{fr}}(G_L)\big)$. The cup product and the local invariant map induce perfect pairings of $\mathscr{O}_K$–modules*

$$\mathrm{H}^i(L', T) \times \mathrm{H}^{2-i}\big(L', \mathrm{Hom}_{\mathbb{Z}_p}(T, \mathbb{Z}_p(1))\big) \to \mathrm{H}^2\big(L', \mathbb{Z}_p(1)\big) \cong \mathbb{Z}_p$$

and

$$\mathrm{H}^i(L', T) \times \mathrm{H}^{2-i}\big(L', \mathrm{Hom}_{\mathscr{O}_K}(T, \mathscr{O}_K(1))\big) \to \mathrm{H}^2\big(L', \mathscr{O}_K(1)\big) \cong \mathscr{O}_K$$

for all $i \in \mathbb{Z}$. Therefore, there are canonical isomorphisms

$$\mathrm{H}^i(L', T) \cong \mathrm{H}^{2-i}(L', T^*(1))^*.$$

Proof. See proposition 3.12 of Schneider–Venjakob's preprint [35], where this form of local Tate duality is deduced from theorem 168 by taking projective limits of quotients by π_K^n for $n \geq 1$. □

9.2 The module of differential forms

Definition 172. Let R be any of the rings $\mathbf{A}'_L$, $\mathbf{B}'_L$ or $\mathbf{B}^\dagger_{\mathrm{rig},L}$.

(1) The *module of differential forms over* R is $\Omega^1_R = R\,dZ$; it is endowed with the actions of φ_q and Γ_L given by

$$\varphi_q(f(Z)\,dZ) = f^{\varphi_q}(\phi(Z))\phi'(Z)\pi_L^{-1}\,dZ \quad \text{and}$$
$$\gamma(f(Z)\,dZ) = f\big([\chi_\phi(\gamma)]_\phi(Z)\big)\,[\chi_\phi(\gamma)]'_\phi(Z)\,dZ$$

for all $f(Z)\,dZ \in \Omega^1_R$ and all $\gamma \in \Gamma_L$.

(2) We define (continuous) maps $d\colon R \to \Omega^1_R$ and $\mathrm{Res}\colon \Omega^1_R \to L$ by

$$d(f(Z)) = f'(Z)\,dZ \quad \text{and} \quad \mathrm{Res}\Big(\sum_{k\in\mathbb{Z}} a_k Z^k\,dZ\Big) = a_{-1}.$$

(3) The *module of differential forms over* $\mathbf{A}_L$ is $\Omega^1_{\mathbf{A}_L} = \mathbf{A}_L\,d\omega_\phi$; it is endowed with the actions of φ_q and Γ_L induced from those on $\Omega^1_{\mathbf{A}'_L}$ via the isomorphism

$\Omega^1_{\mathbf{A}'_L} \cong \Omega^1_{\mathbf{A}_L}$ defined by $Z \mapsto \omega_\phi$ and $dZ \mapsto d\omega_\phi$.

Remark. One can check from the definition that $\Omega^1_{\mathbf{A}_L} \in \mathrm{Ob}\big((\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}_{\mathbf{A}_L,\mathrm{fr}}\big)$. Below we show that it is étale by comparing it with another (φ_q, Γ_L)–module.

Recall that the Tate module $\mathrm{T}_\phi \mathfrak{F}_\phi$ is isomorphic to $\mathscr{O}_K(\chi_\phi)$ as an $\mathscr{O}_K$–module with an action of G_L and that we fixed a generator t_0 of $\mathrm{T}_\phi \mathfrak{F}_\phi$. Next we define a (φ_q, Γ_L)–module $\mathbf{A}'_L(\chi_\phi)$ as follows: as an $\mathbf{A}'_L$–module, $\mathbf{A}'_L(\chi_\phi) = \mathbf{A}'_L \otimes_{\mathscr{O}_K} \mathrm{T}_\phi \mathfrak{F}_\phi$ and we always express its elements as $f(Z) \otimes t_0$ with $f(Z) \in \mathbf{A}'_L$. The action of φ_q on $\mathbf{A}'_L(\chi_\phi)$ is given by

$$\varphi_q\big(f(Z) \otimes t_0\big) = \varphi_q\big(f(Z)\big) \otimes t_0 \quad \text{for all } f(Z) \in \mathbf{A}'_L$$

and the action of Γ_L on $\mathbf{A}'_L(\chi_\phi)$ is given by

$$\gamma\big(f(Z) \otimes t_0\big) = \chi_\phi(\gamma)\gamma\big(f(Z)\big) \otimes t_0 \quad \text{for all } f(Z) \in \mathbf{A}'_L \text{ and all } \gamma \in \Gamma_L.$$

That is, $\mathbf{A}'_L(\chi_\phi)$ is simply the module $\mathbf{A}'_L$ with the Γ_L–action twisted by χ_ϕ and, in particular, it is an étale (φ_q, Γ_L)–module over $\mathbf{A}'_L$. By base change, we also obtain $\mathbf{B}^\dagger_{\mathrm{rig},L}(\chi_\phi) \in \mathrm{Ob}\big((\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}_{\mathbf{B}^\dagger_{\mathrm{rig},L},\mathrm{fr}}\big)$.

Lemma 173. *The map*

$$\begin{aligned} \mathbf{A}'_L(\chi_\phi) &\longrightarrow \Omega^1_{\mathbf{A}'_L} \\ f(Z) \otimes t_0 &\longmapsto f(Z)\, \mathrm{dlog}_\phi(Z) \end{aligned}$$

is an isomorphism of (φ_q, Γ_L)–modules over $\mathbf{A}'_L$. (Thus, $\Omega^1_{\mathbf{A}'_L}$ is an étale (φ_q, Γ_L)–module over $\mathbf{A}'_L$.)

Proof. Write $\mathrm{dlog}_\phi(Z) = g_\phi(Z)\, dZ$. Since $g_\phi(Z) = 1 + \cdots$ is invertible in $\mathbf{A}'_L$, the map in the statement of the lemma is well-defined and bijective. The fact that it is compatible with φ_q and Γ_L follows from the relations $\varphi_q(g_\phi(Z))\phi'(Z) = \pi_L g_\phi(Z)$ (see the proof of lemma 65) and $g_\phi([a]_\phi(Z))[a]'_\phi(Z) = a g_\phi(Z)$ for $a \in \mathscr{O}_K$. □

Lemma 174. *The map $d\colon \mathbf{B}^\dagger_{\mathrm{rig},L} \to \Omega^1_{\mathbf{B}^\dagger_{\mathrm{rig},L}}$ satisfies the following properties:*

(1) $\pi_L \cdot \varphi_q \circ d = d \circ \varphi_q$;
(2) $\gamma \circ d = d \circ \gamma$ *for all* $\gamma \in \Gamma_L$;
(3) $d(\,\cdot\,) \circ [a]_\phi = d(\,\cdot\, \circ [a]_\phi)$ *for all* $a \in \mathscr{O}_K$, *and*
(4) $\psi_q \circ d = \varphi_q^{-1}(\pi_L) \cdot d \circ \psi_q$.

Proof. This is analogous to lemma 3.16 of Schneider–Venjakob's article [34] (the only difference is the factor $\varphi_q^{-1}(\pi_L)$ in identity (4)).

The first three assertions are straight-forward from the definitions. For the last one, we observe that φ_q (on $\Omega^1_{\mathbf{B}^\dagger_{\mathrm{rig},L}}$) is injective and so it suffices to prove the relation after composing with φ_q. Thus, using identity (1), we have to prove that

$$\varphi_q \circ \psi_q \circ d = d \circ \varphi_q \circ \psi_q.$$

But we can translate the computation to $\mathbf{B}^\dagger_{\mathrm{rig},L}(\chi_\phi)$ by means of lemma 173. Then, for $f(Z) \in \mathbf{B}^\dagger_{\mathrm{rig},L}$,

$$\begin{aligned}(\varphi_q \circ \psi_q \circ d)\big(f(Z)\big) &= (\varphi_q \circ \psi_q)\big(f'(Z)\,dZ\big) = (\varphi_q \circ \psi_q)\left(\frac{f'(Z)}{g_\phi(Z)} \otimes t_0\right)\\ &= (\varphi_q \circ \psi_q)\big(\partial_\phi(f)(Z) \otimes t_0\big) = (\varphi_q \circ \psi_q)\big(\partial_\phi(f)(Z)\big) \otimes t_0\end{aligned}$$

and now we can use lemma 65 and move back to $\Omega^1_{\mathbf{B}^\dagger_{\mathrm{rig},L}}$:

$$\begin{aligned}(\varphi_q \circ \psi_q)\big(\partial_\phi(f)(Z)\big) \otimes t_0 &= \partial_\phi\big((\varphi_q \circ \psi_q)(f(Z))\big) \otimes t_0\\ &= \partial_\phi\big((\varphi_q \circ \psi_q)(f(Z))\big)\,\mathrm{dlog}_\phi(Z) = d\big((\varphi_q \circ \psi_q)(f(Z))\big).\end{aligned}$$

Putting everything together, we obtain identity (4). □

Proposition 175. *The residue map* $\mathrm{Res}\colon \Omega^1_{\mathbf{B}^\dagger_{\mathrm{rig},L}} \to L$ *satisfies the following properties:*

(1) $\mathrm{Res} \circ \varphi_q = \frac{q}{\pi_L} \cdot \varphi_q \circ \mathrm{Res}$;
(2) $\mathrm{Res} \circ \gamma = \mathrm{Res}$ *for all* $\gamma \in \Gamma_L$;
(3) $\mathrm{Res}(\,\cdot\, \circ [\pi_K]_\phi) = q \cdot \mathrm{Res}$, *and*
(4) $\mathrm{Res} \circ \psi_q = \varphi_q^{-1} \circ \mathrm{Res}$.

Proof. The proof of proposition 3.17 of Schneider–Venjakob's article [34] works almost verbatim in the *relative* Lubin–Tate situation, now using lemma 174 and taking into account that φ_q does not act trivially on L. □

Corollary 176. *For every* $f(Z) \in \mathbf{B}^\dagger_{\mathrm{rig},L}$,

$$\mathrm{Res}\big(f([\pi_K]_\phi(Z))\,\mathrm{dlog}_\phi(Z)\big) = \frac{q}{\pi_K}\,\mathrm{Res}(f(Z)\,\mathrm{dlog}_\phi(Z)).$$

9.3 Duality of (φ_q, Γ_L)–modules

9.3.1 The internal Hom

In the category $(\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{\text{ét}}_{\mathbf{A}'_L}$ (and so in $(\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{\text{ét}}_{\mathbf{A}_L}$ too) there is an internal Hom functor: given $M, N \in \mathrm{Ob}\big((\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{\text{ét}}_{\mathbf{A}'_L}\big)$, we define actions of φ_q and of Γ_L on $\mathrm{Hom}_{\mathbf{A}'_L}(M, N)$ as follows: for every $\alpha \in \mathrm{Hom}_{\mathbf{A}'_L}(M, N)$ and $\gamma \in \Gamma_L$, the elements $\varphi_q(\alpha), \gamma(\alpha) \in \mathrm{Hom}_{\mathbf{A}'_L}(M, N)$ are the unique maps making the diagrams

$$\begin{array}{ccc} M & \overset{\varphi_q(\alpha)}{\dashrightarrow} & N \\ {\scriptstyle 1\otimes\varphi_q}\uparrow\wr & & \wr\uparrow{\scriptstyle 1\otimes\varphi_q} \\ \mathbf{A}'_L \otimes_{\varphi_q, \mathbf{A}'_L} M & \xrightarrow[1\otimes\alpha]{} & \mathbf{A}'_L \otimes_{\varphi_q, \mathbf{A}'_L} N \end{array} \qquad \text{and} \qquad \begin{array}{ccc} M & \overset{\gamma(\alpha)}{\dashrightarrow} & N \\ {\scriptstyle \gamma}\uparrow\wr & & \wr\uparrow{\scriptstyle \gamma} \\ M & \xrightarrow[\alpha]{} & N \end{array}$$

commutative. The paragraphs after lemma 3.12 and until formula (17) of Schneider and Venjakob's article [34] justify that $\mathrm{Hom}_{\mathbf{A}'_L}(M, N) \in \mathrm{Ob}\big((\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{\text{ét}}_{\mathbf{A}'_L}\big)$.

Analogously, given $\mathscr{M}, \mathscr{N} \in \mathrm{Ob}\big((\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}_{\mathbf{B}^\dagger_{\mathrm{rig},L},\mathrm{fr}}\big)$, the same constructions make $\mathrm{Hom}_{\mathbf{B}^\dagger_{\mathrm{rig},L}}(\mathscr{M}, \mathscr{N}) \in \mathrm{Ob}\big((\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}_{\mathbf{B}^\dagger_{\mathrm{rig},L},\mathrm{fr}}\big)$.

9.3.2 The residue pairing

Schneider and Venjakob introduced several pairings in their article [34] and their preprint [35]. All their pairings are constructed from the following starting point:

Definition 177. Let R be any of the rings $\mathbf{A}'_L$ or $\mathbf{B}^\dagger_{\mathrm{rig},L}$. The *residue pairing for R* is

$$\begin{aligned} \{\,\cdot\,,\,\cdot\,\} = \{\,\cdot\,,\,\cdot\,\}_R \colon R \times \Omega^1_R &\longrightarrow L \\ (f, \omega) &\longmapsto \mathrm{Res}(f\omega) \end{aligned}$$

(where Res is the residue map from definition 172).

Corollary 178. *Let R be any of the rings $\mathbf{A}'_L$ or $\mathbf{B}^\dagger_{\mathrm{rig},L}$. The residue pairing $\{\,\cdot\,,\,\cdot\,\}_R$ satisfies the following properties: for every $f \in R$ and every $\omega \in \Omega^1_R$,*

(1) $\{\varphi_q(f), \varphi_q(\omega)\} = \frac{q}{\pi_L}\varphi_q\big(\{f, \omega\}\big)$,
(2) $\{\gamma(f), \gamma(\omega)\} = \{f, \omega\}$ *for all* $\gamma \in \Gamma_L$,
(3) $\{\psi_q(f), \omega\} = \varphi_q^{-1}\big(\{f, \varphi_q(\omega)\}\big)$ *and*
(4) $\{f, \psi_q(\omega)\} = \varphi_q^{-1}\big(\{\varphi_q(f), \omega\}\big)$.

Proof. These identities follow from proposition 175, the last two in combination with the projection formulae $\psi_q(f\varphi_q(\omega)) = \psi_q(f)\omega$ and $\psi_q(\varphi_q(f)\omega) = f\psi_q(\omega)$. □

9.3.3 Duality for torsion modules over $\mathbf{A}'_L$

Our goal is to study the Iwasawa cohomology of representations in $\mathrm{Rep}_{\mathscr{O}_K}(G_L)$ using their associated (étale) (φ_q, Γ_L)–modules over $\mathbf{A}_L$ (or equivalently over $\mathbf{A}'_L$). Since corollary 169 provides a description of Iwasawa cohomology for torsion representations in terms of Pontryagin duality, we focus on the full subcategory $(\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{\text{ét}}_{\mathbf{A}'_L,\text{tor}}$ of torsion modules in $(\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{\text{ét}}_{\mathbf{A}'_L}$ and mimic Pontryagin duality by means of the residue pairing.

Let $n \in \mathbb{Z}_{\geq 1}$. Observe that the residue pairing induces a pairing

$$\begin{aligned} \mathbf{A}'_L/\pi_L^n\mathbf{A}'_L \times \Omega^1_{\mathbf{A}'_L}/\pi_L^n\Omega^1_{\mathbf{A}'_L} &\longrightarrow L/\mathscr{O}_L \\ (f,\omega) &\longmapsto \pi_K^{-n}\operatorname{Res}(f\omega) \mod \mathscr{O}_L \end{aligned}$$

and so we obtain a continuous $\mathscr{O}_L$–linear map

$$\Omega^1_{\mathbf{A}'_L}/\pi_L^n\Omega^1_{\mathbf{A}'_L} \to \mathrm{Hom}^{\text{cont}}_{\mathscr{O}_L}(\mathbf{A}'_L/\pi_L^n\mathbf{A}'_L, L/\mathscr{O}_L),$$

where $\mathrm{Hom}^{\text{cont}}_{\mathscr{O}_L}$ means the module of continuous $\mathscr{O}_L$–linear maps with the compact-open topology. This last map is an isomorphism (see lemma 3.5 of Schneider–Venjakob's article [34]), which means that we should view $\Omega^1_{\mathbf{A}'_L}/\pi_L^n\Omega^1_{\mathbf{A}'_L}$ as "the Pontryagin dual" of $\mathbf{A}'_L/\pi_L^n\mathbf{A}'_L$. More generally:

Lemma 179. *Let $M \in \mathrm{Ob}\big((\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{\text{ét}}_{\mathbf{A}'_L,\text{tor}}\big)$, so that M is killed by π_L^n for some $n \geq 1$. The map*

$$\begin{aligned} \mathrm{Hom}_{\mathbf{A}'_L}(M, \Omega^1_{\mathbf{A}'_L}/\pi_L^n\Omega^1_{\mathbf{A}'_L}) &\longrightarrow \mathrm{Hom}^{\text{cont}}_{\mathscr{O}_L}(M, L/\mathscr{O}_L) \\ g &\longmapsto \pi_K^{-n}\operatorname{Res}\big(g(\,\cdot\,)\big) \mod \mathscr{O}_L \end{aligned}$$

is an isomorphism of topological $\mathscr{O}_L$–modules.

Proof. See lemma 3.6 of Schneider–Venjakob's article [34]. □

Definition 180. For $M \in \mathrm{Ob}\big((\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{\text{ét}}_{\mathbf{A}'_L,\text{tor}}\big)$ and $n \in \mathbb{Z}_{\geq 1}$ such that M is killed by π_L^n, we set

$$M^{\vee_n} = \mathrm{Hom}_{\mathbf{A}'_L}(M, \Omega^1_{\mathbf{A}'_L}/\pi_L^n\Omega^1_{\mathbf{A}'_L}) \in \mathrm{Ob}\big((\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{\text{ét}}_{\mathbf{A}'_L,\text{tor}}\big)$$

and define the pairing

$$\{\,\cdot\,,\,\cdot\,\} = \{\,\cdot\,,\,\cdot\,\}_{M,n} \colon M \times M^{\vee_n} \longrightarrow L/\mathscr{O}_L$$

by

$$\{m, g\} = \pi_K^{-n} \operatorname{Res}(g(m)) \mod \mathscr{O}_L \quad \text{for all } m \in M \text{ and } g \in M^{\vee_n}.$$

Remark. There is a slight abuse of notation here: the actual Pontryagin dual of M is $\operatorname{Hom}_{\mathscr{O}_L}^{\text{cont}}(M, L/\mathscr{O}_L)$, but we use lemma 179 to obtain "a Pontryagin dual" (depending on n) that is again a (φ_q, Γ_L)–module.

Proposition 181. *Let $M \in \operatorname{Ob}\big((\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{\text{ét}}_{\mathbf{A}'_L,\text{tor}}\big)$ and let $n \in \mathbb{Z}_{\geq 1}$ such that M is killed by π_L^n. The residue pairing $\{\,\cdot\,,\,\cdot\,\}_{M,n}$ satisfies the following properties: for every $m \in M$ and every $g \in M^{\vee_n}$,*

(1) $\{\varphi_q(m), \varphi_q(g)\} = \frac{q}{\pi_L}\varphi_q(\{m, g\})$;
(2) $\{\gamma(m), \gamma(g)\} = \{m, g\}$ *for all* $\gamma \in \Gamma_L$;
(3) $\{\psi_q(m), g\} = \varphi_q^{-1}(\{m, \varphi_q(g)\})$, *and*
(4) $\{m, \psi_q(g)\} = \varphi_q^{-1}(\{\varphi_q(m), g\})$.

Proof. These identities follow easily from proposition 175 and the definition of $M^{\vee_n}$ as an étale (φ_q, Γ_L)–module (see section 9.3.1). See proposition 3.19 of Schneider–Venjakob's article [34] for the full details of the proofs of identities (3) and (4). □

9.3.4 Duality for torsion modules over $\mathbf{A}_L$

Next we translate the results of section 9.3.3 to the category $(\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{\text{ét}}_{\mathbf{A}_L,\text{tor}}$ of étale torsion (φ_q, Γ_L)–modules over $\mathbf{A}_L$ using the equivalence provided by proposition 68.

Let $M \in \operatorname{Ob}\big((\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{\text{ét}}_{\mathbf{A}_L,\text{tor}}\big)$ and assume that M is killed by π_L^n with $n \geq 1$. Combining the isomorphism

$$\begin{aligned} \mathbf{A}_L(\chi_\phi) &\longrightarrow \Omega^1_{\mathbf{A}_L} \\ f(\omega_\phi) \otimes t_0 &\longmapsto f(\omega_\phi)\, \operatorname{dlog}_\phi(\omega_\phi) \end{aligned}$$

from lemma 173 with (the analogue of) the isomorphism in lemma 179, we obtain an isomorphism of topological $\mathscr{O}_L$–modules

$$\operatorname{Hom}_{\mathbf{A}_L}(M, \mathbf{A}_L/\pi_L^n \mathbf{A}_L)(\chi_\phi) \xrightarrow{\cong} \operatorname{Hom}_{\mathscr{O}_L}^{\text{cont}}(M, L/\mathscr{O}_L)$$

defined by

$$g \otimes t_0 \mapsto \pi_K^{-n} \operatorname{Res}\big(g(\,\cdot\,)\, \operatorname{dlog}_\phi(\omega_\phi)\big) \mod \mathscr{O}_L.$$

Thus, we define the Pontryagin dual

$$M^\vee = \operatorname{Hom}_{\mathscr{O}_L}^{\text{cont}}(M, L/\mathscr{O}_L)$$

and identify it with

$$M^{\vee_n} = \mathrm{Hom}_{\mathbf{A}_L}(M, \mathbf{A}_L/\pi_L^n\mathbf{A}_L)(\chi_\phi)$$

by means of the pairing

$$\begin{aligned}\{\,\cdot\,,\,\cdot\,\} = \{\,\cdot\,,\,\cdot\,\}_{M,n} \colon M \times M^{\vee_n} &\longrightarrow L/\mathscr{O}_L \\ (m, g\otimes t_0) &\longmapsto \pi_K^{-n}\,\mathrm{Res}\big(g(m)\,\mathrm{dlog}_\phi(\omega_\phi)\big) \mod \mathscr{O}_L\end{aligned}$$

with the properties described in proposition 181. In particular, the operators φ_q and ψ_q are *essentially* adjoint with respect to $\{\,\cdot\,,\,\cdot\,\}$.

Proposition 182. *Let $T \in \mathrm{Ob}\big(\mathrm{Rep}_{\mathscr{O}_K,\mathrm{tor}}(G_L)\big)$ and let $n \in \mathbb{Z}_{\geq 1}$ such that $\pi_K^n T = 0$. There is a natural functorial isomorphism of topological $\mathscr{O}_L$–modules*

$$\mathbf{D}(T)^\vee \cong \mathbf{D}\big(T^\vee(\chi_\phi)\big)$$

which is independent of n and through which the operator ψ_q on $\mathbf{D}\big(T^\vee(\chi_\phi)\big)$ corresponds to $\varphi_q^{-1}\circ\varphi_q^\vee(\,\cdot\,)$, where $\varphi_q^\vee$ denotes the dual of φ_q on $\mathbf{D}(T)$) and $\varphi_q^{-1}\colon L/\mathscr{O}_L \to L/\mathscr{O}_L$ is induced by the inverse of the Frobenius in $\mathrm{Gal}(L/K)$.

Proof. This is analogous to remark 5.6 of Schneider–Venjakob's article [34]. We adapt it here to the *relative* Lubin–Tate situation for the convenience of the reader.

First, observe that π_L and π_K differ (multiplicatively) by an element of $\mathscr{O}_L^\times$. Therefore, an $\mathscr{O}_L$–module is killed by π_L^n if and only if it is killed by π_K^n. In particular, one checks easily that $\mathbf{D}(\mathscr{O}_K/\pi_K^n\mathscr{O}_K) = \mathbf{A}_L/\pi_L^n\mathbf{A}_L$.

Now, by the identifications described before the proposition and by the compatibility of the functor $\mathbf{D}$ with duals and tensor products, we can write

$$\begin{aligned}\mathbf{D}(T)^\vee \cong \mathbf{D}(T)^{\vee_n} &= \mathrm{Hom}_{\mathbf{A}_L}\big(\mathbf{D}(T), \mathbf{A}_L/\pi_L^n\mathbf{A}_L\big)(\chi_\phi) \\ &\cong \mathrm{Hom}_{\mathbf{A}_L}\big(\mathbf{D}(T), \mathbf{D}(\mathscr{O}_K/\pi_K^n\mathscr{O}_K)\big)(\chi_\phi) \\ &\cong \mathbf{D}\big(\mathrm{Hom}_{\mathscr{O}_L}(T, \mathscr{O}_K/\pi_K^n\mathscr{O}_K)\big)(\chi_\phi) \cong \mathbf{D}(T^\vee)(\chi_\phi) \cong \mathbf{D}\big(T^\vee(\chi_\phi)\big).\end{aligned}$$

Here, the first and the second-to-last isomorphisms depend on n, but these two dependences "compensate each other". The correspondence between $\varphi_q^{-1}\circ\varphi_q^\vee(\,\cdot\,)$ and ψ_q is a consequence of the relation $\varphi_q^{-1}\circ\{\varphi_q(\,\cdot\,),\,\cdot\,\}_{\mathbf{D}(T),n} = \{\,\cdot\,,\psi_q(\,\cdot\,)\}_{\mathbf{D}(T),n}$ (cf. part (4) of proposition 181) used in the first isomorphism. □

9.3.5 Duality for modules over $\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}$

The definitions and results given in this section over $\mathbf{B}^{\dagger}_{\mathrm{rig},L}$ have obvious analogues over $\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}$. As already seen in section 8.5, working over $\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}$ is more convenient because it allows us to study distributions.

Lemma 183. *The residue pairing $\{\cdot,\cdot\}_{\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}}$ from definition 177 identifies $\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}$ with the topological dual of $\Omega^1_{\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}}$. That is to say, the residue pairing induces isomorphisms*

$$\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p} \cong \mathrm{Hom}^{\mathrm{cont}}_{\mathbb{C}_p}\big(\Omega^1_{\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}}, \mathbb{C}_p\big) \quad \text{and} \quad \Omega^1_{\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}} \cong \mathrm{Hom}^{\mathrm{cont}}_{\mathbb{C}_p}(\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}, \mathbb{C}_p).$$

Proof. This is lemma 2.35 of Schneider–Venjakob's preprint [35]. □

Definition 184. For $\mathscr{M} \in \mathrm{Ob}\big((\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}_{\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p},\mathrm{fr}}\big)$, we set

$$\mathscr{M}^{\vee} = \mathrm{Hom}_{\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}}\big(\mathscr{M}, \Omega^1_{\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}}\big)$$

and define the pairing

$$\{\cdot,\cdot\} = \{\cdot,\cdot\}_{\mathscr{M}} \colon \mathscr{M} \times \mathscr{M}^{\vee} \longrightarrow \mathbb{C}_p$$

by

$$\{m, g\} = \mathrm{Res}(g(m)) \quad \text{for all } m \in \mathscr{M} \text{ and } g \in \mathscr{M}^{\vee}.$$

Remark. By lemma 173 (or, rather, its base change from $\mathbf{A}'_L$ to $\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}$), we have a canonical isomorphism $\mathscr{M}^{\vee} \cong \mathrm{Hom}_{\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}}(\mathscr{M}, \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p})(\chi_\phi) = \mathscr{M}^*(\chi_\phi)$.

Proposition 185. *Let $\mathscr{M} \in \mathrm{Ob}\big((\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}_{\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p},\mathrm{fr}}\big)$. The pairing*

$$\{\cdot,\cdot\} \colon \mathscr{M} \times \mathscr{M}^{\vee} \to \mathbb{C}_p$$

induces an isomorphism

$$\mathscr{M}^{\vee} \cong \mathrm{Hom}^{\mathrm{cont}}_{\mathbb{C}_p}(\mathscr{M}, \mathbb{C}_p).$$

Proof. This is a consequence of lemma 183, as $\mathscr{M}$ is free over $\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}$. □

Proposition 186. *Let $\mathscr{M} \in \mathrm{Ob}\big((\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}_{\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p},\mathrm{fr}}\big)$. The residue pairing $\{\cdot,\cdot\}_{\mathscr{M}}$ satisfies the following properties: for every $m \in \mathscr{M}$ and every $g \in \mathscr{M}^{\vee}$,*

(1) $\{\varphi_q(m), \varphi_q(g)\} = \frac{q}{\pi_L}\varphi_q\big(\{m, g\}\big)$;

(2) $\{\gamma(m), \gamma(g)\} = \{m, g\}$ *for all* $\gamma \in \Gamma_L$;

(3) $\{\psi_q(m), g\} = \varphi_q^{-1}(\{m, \varphi_q(g)\})$, *and*
(4) $\{m, \psi_q(g)\} = \varphi_q^{-1}(\{\varphi_q(m), g\})$.

Proof. The proof is identical to that of proposition 181. □

9.3.6 The pairing for $\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L)$

Using the isomorphism $\Omega^1_{\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}} \cong \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\chi_\phi)$ from lemma 173, the residue pairing $\{\,\cdot\,,\,\cdot\,\}\colon \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p} \times \Omega^1_{\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}} \longrightarrow \mathbb{C}_p$ induces a pairing

$$\begin{aligned} \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p} \times \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p} &\longrightarrow \mathbb{C}_p \\ (f, g) &\longmapsto \mathrm{Res}(fg\,\mathrm{dlog}_\phi) \end{aligned}$$

(of $\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}$–modules, ignoring the actions of φ_q and Γ_L) that we can try to translate to $\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L)$ by means of the Mellin transform. We follow subsections 2.3.2 and 2.3.3 of Schneider–Venjakob's preprint [35].

Definition 187.

(1) We define $\varrho\colon \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L) \to \mathbb{C}_p$ to be the composition of the maps

$$\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_{L_{n_0}}) \rtimes_{\Gamma_{L_{n_0}}} \Gamma_L \twoheadrightarrow \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_{L_{n_0}}) \overset{\ell_{n_0,*}}{\underset{\cong}{\rightarrow}} \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\mathscr{O}_K) \cong \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p} \longrightarrow \mathbb{C}_p,$$

where the last arrow is given by

$$g \mapsto \left(\frac{q}{\pi_K}\right)^{n_0} \mathrm{Res}(g\,\mathrm{dlog}_\phi).$$

(2) We define the pairing

$$\langle\,\cdot\,,\,\cdot\,\rangle\colon \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L) \times \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L) \longrightarrow \mathbb{C}_p$$

by

$$\langle \lambda, \mu \rangle = \varrho(\lambda\mu) \quad \text{for all } \lambda, \mu \in \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L).$$

Remark. Alternatively, we could define ϱ as the composition

$$\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_{L_{n_0}}) \rtimes_{\Gamma_{L_{n_0}}} \Gamma_L \twoheadrightarrow \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_{L_{n_0}}) \xrightarrow{(\log\circ\chi_\phi)_*} \varphi_q^{n_0}(\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}) \xrightarrow{\mathrm{Res}(\,\cdot\,\mathrm{dlog}_\phi)} \mathbb{C}_p,$$

which justifies the appearance of the factor $(q/\pi_K)^{n_0}$ by corollary 176.

Lemma 188. *The definition of ϱ is independent of the choice of (large enough) n_0.*

Proof. We have to check that, for $m \geq n \geq n_0$, the diagram

$$\begin{array}{ccccc} \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_{L_m}) \rtimes_{\Gamma_{L_m}} \Gamma_{L_n} & \twoheadrightarrow & \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_{L_m}) & \xrightarrow[\cong]{\ell_{m,*}} & \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p} \\ \downarrow \wr\Vert & & & & \downarrow \frac{q^m}{\pi_K^m}\mathrm{Res}(\,\cdot\,\mathrm{dlog}_\phi) \\ \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_{L_n}) & \xrightarrow[\cong]{\ell_{n,*}} & \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p} & \xrightarrow{\frac{q^n}{\pi_K^n}\mathrm{Res}(\,\cdot\,\mathrm{dlog}_\phi)} & \mathbb{C}_p \end{array}$$

is commutative.

First consider $f(\mu_{Z,m}) \otimes [\gamma] \in \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_{L_m}) \rtimes_{\Gamma_{L_m}} \Gamma_{L_n}$ with $\gamma \notin \Gamma_{L_m}$, which is sent to 0 by the first horizontal arrow. Using the lower row, this element maps to

$$f([\pi_K^{m-n}]_\phi(\mu_{Z,n}))\delta_\gamma \mapsto f([\pi_K^{m-n}]_\phi(Z))\eta(\ell_n(\gamma), Z) \in \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}.$$

By lemma 149, we can write $f([\pi_K^{m-n}]_\phi(Z)) = \varphi_q^{m-n}(\widetilde{f}(Z))$ with $\widetilde{f}(Z) \in \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}$. Now the projection formula and identity (5) of lemma 153 imply that

$$\psi_q^{m-n}\big(\varphi_q^{m-n}(\widetilde{f}(Z))\eta(\ell_n(\gamma), Z)\big) = 0$$

because $\ell_n(\gamma) \notin \pi_K^{m-n}\mathscr{O}_K$. We conclude that

$$\mathrm{Res}\big(f([\pi_K^{m-n}]_\phi(Z))\eta(\ell_n(\gamma), Z)\,\mathrm{dlog}_\phi(Z)\big) = 0$$

by the last identity of proposition 175.

Next consider $f(\mu_{Z,m}) \otimes [\gamma_1] \in \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_{L_m}) \rtimes_{\Gamma_{L_m}} \Gamma_{L_n}$, where $\gamma_1 \in \Gamma_{L_m}$ is characterized by $\chi_\phi(\gamma_1) = 1$. This element is mapped to

$$f(\mu_{Z,m})\delta_{\gamma_1} \mapsto f(Z)\eta(0, Z) = f(Z) \mapsto \Big(\frac{q}{\pi_K}\Big)^m \mathrm{Res}\big(f(Z)\,\mathrm{dlog}_\phi(Z)\big)$$

via the upper row of the initial diagram and to

$$f([\pi_K^{m-n}]_\phi(\mu_{Z,n}))\delta_{\gamma_1} \mapsto f([\pi_K^{m-n}]_\phi(Z)) \mapsto \Big(\frac{q}{\pi_K}\Big)^n \mathrm{Res}\big(f([\pi_K^{m-n}]_\phi(Z))\,\mathrm{dlog}_\phi(Z)\big)$$

via the lower row. These two expressions coincide by corollary 176. □

Proposition 189. *The pairing $\langle\,\cdot\,,\,\cdot\,\rangle\colon \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L) \times \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L) \to \mathbb{C}_p$ induces isomorphisms*

$$\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L) \cong \mathrm{Hom}^{\mathrm{cont}}_{\mathbb{C}_p}\big(\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L), \mathbb{C}_p\big)$$

and

$$D(\Gamma_L, \mathbb{C}_p) \cong \mathrm{Hom}^{\mathrm{cont}}_{\mathbb{C}_p}\big(\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L)/D(\Gamma_L, \mathbb{C}_p), \mathbb{C}_p\big).$$

Proof. This is proposition 2.44 of Schneider–Venjakob's preprint [35]. □

Proposition 190. *For every* $\lambda, \mu \in \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L)$*,*

$$\langle \mathrm{Tw}_{\chi_\phi}(\lambda), \mathrm{Tw}_{\chi_\phi}(\mu)\rangle = \langle \lambda, \mu\rangle.$$

Proof. See lemmata 2.47, 2.48 and 2.49 and corollary 2.50 of Schneider–Venjakob's preprint [35]. Note that, in the *relative* Lubin–Tate situation, one must use π_K in place of the π_L in loc. cit. □

There is an alternative definition of the pairing $\langle\,\cdot\,,\,\cdot\,\rangle$ from which the relation with the residue pairing $\{\,\cdot\,,\,\cdot\,\}$ becomes clearer.

Definition 191.

(1) The *twisted Mellin transform* $\mathfrak{M}_{\chi_\phi}\colon \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L) \to \big(\Omega^1_{\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}}\big)^{\psi_q=0}$ is given by

$$\mathfrak{M}_{\chi_\phi}(\lambda) = \lambda\big(\eta(1,Z)\,\mathrm{dlog}_\phi(Z)\big) = \mathrm{Tw}_{\chi_\phi}(\lambda)(\eta(1,Z))\,\mathrm{dlog}_\phi(Z)$$

for all $\lambda \in \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L)$.

(2) We define $\varsigma\colon \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L) \to \mathbb{C}_p$ by

$$\varsigma(\lambda) = \mathrm{Res}\big(\mathfrak{M}(\gamma_{-1})\mathfrak{M}_{\chi_\phi}(\lambda)\big) \quad \text{for all } \lambda \in \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L),$$

where γ_{-1} denotes the element of Γ_L mapping to -1 under χ_ϕ (that we identify with the corresponding Dirac distribution).

Theorem 192 (Schneider–Venjakob). *The maps* ϱ *from definition 187 and* ς *from definition 191 are equal.*

Proof. This is theorem 2.51 of Schneider–Venjakob's preprint [35]. We sketch the proof here (in the *relative* Lubin–Tate situation) for the convenience of the reader.

Since $\mathrm{Res}\big(\Omega^1_{\mathbf{B}^{+}_{\mathrm{rig},\mathbb{C}_p}}\big) = 0$, we see that ς factors through $\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L)/D(\Gamma_L, \mathbb{C}_p)$. By proposition 189, there exists $\mu \in D(\Gamma_L, \mathbb{C}_p)$ such that $\varsigma(\lambda) = \langle\lambda, \mu\rangle$ for all $\lambda \in \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L)$. Using lemmata 2.53 and 2.54 of Schneider–Venjakob's preprint (whose proofs work verbatim in the *relative* Lubin–Tate situation), one checks that μ must be constant. Therefore, $\varsigma(\lambda) = \langle\lambda, \mu\rangle = \varrho(\mu\lambda) = \mu\varrho(\lambda)$ for all $\lambda \in \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L)$. It suffices to find one $\lambda \in \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L)$ such that $\varsigma(\lambda) \neq 0 \neq \varrho(\lambda)$ to compute μ.

Next we choose a large $n_0 \in \mathbb{Z}_{\geq 1}$ and a $\mathbb{Z}_p$–basis $b = (b_1, \dots, b_m)$ of $\Gamma_{L_{n_0}}$ and evaluate ϱ and ς at the element Ξ_b introduced in proposition 162, following lemma 2.55 of Schneider–Venjakob's preprint [35]. (Note that Ξ_b is called $\widehat{\Xi}_b$ in loc. cit.)

On the one hand, by proposition 163

$$\ell_{n_0,*}(\Xi_b) \equiv \Big(\frac{\pi_K}{q}\Big)^{n_0} \frac{1}{\Omega Z} \mod \mathbf{B}^{+}_{\mathrm{rig},\mathbb{C}_p}$$

and so

$$\varrho(\Xi_b) = \Big(\frac{q}{\pi_K}\Big)^{n_0} \mathrm{Res}\big(\ell_{n_0,*}(\Xi_b)\,\mathrm{dlog}_\phi(Z)\big) = \mathrm{Res}\Big(\frac{\mathrm{dlog}_\phi(Z)}{\Omega Z}\Big) = \frac{1}{\Omega}.$$

On the other hand, by definition 161 and the fact that N_∇ acts on $\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}$ as $\log_\phi(Z)\partial_\phi$, we can express

$$\begin{aligned}
\varsigma(\Xi_b) &= \mathrm{Res}\big(\mathfrak{M}(\gamma_{-1})\mathfrak{M}(\mathrm{Tw}_{\chi_\phi}(\Xi_b))\,\mathrm{dlog}_\phi(Z)\big) \\
&= \mathrm{Res}\Big(\eta(-1,Z)\frac{\partial_\phi}{\Omega}\mathfrak{M}\Big(\frac{\Theta_b}{N_\nabla}\Big)\,\mathrm{dlog}_\phi(Z)\Big) \\
&= \frac{1}{\Omega}\mathrm{Res}\Big(\eta(-1,Z)\frac{\mathfrak{M}(\Theta_b)}{\log_\phi(Z)}\,\mathrm{dlog}_\phi(Z)\Big).
\end{aligned}$$

Now by proposition 162, we conclude that

$$\begin{aligned}
\varsigma(\Xi_b) &= \frac{1}{\Omega}\mathrm{Res}\Big(\frac{\varphi_q^{n_0}(\xi_b)}{\log_\phi(Z)}\,\mathrm{dlog}_\phi(Z)\Big) \\
&= \frac{\varphi_q^{n_0-1}(\pi_L)\cdots\varphi_q(\pi_L)\pi_L}{\Omega}\mathrm{Res}\Big(\varphi_q^{n_0}\Big(\frac{\xi_b}{\log_\phi(Z)}\Big)\,\mathrm{dlog}_\phi(Z)\Big) \\
&= \frac{q^{n_0}}{\Omega}\varphi_q^{n_0}\Big(\mathrm{Res}\Big(q^{-n_0}\frac{1}{Z}\,\mathrm{dlog}_\phi(Z)\Big)\Big) = \frac{1}{\Omega}.
\end{aligned}$$

In conclusion, $\varrho(\Xi_b) = \varsigma(\Xi_b)$ as claimed. □

Corollary 193. *The diagram*

$$\begin{array}{ccc}
\langle\,\cdot\,,\,\cdot\,\rangle\colon \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L) \times \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L) & \longrightarrow & \mathbb{C}_p \\
\gamma_{-1}\circ\mathfrak{M}\circ\iota \downarrow \qquad\qquad \downarrow \mathfrak{M}_{\chi_\phi} & & \| \\
\{\,\cdot\,,\,\cdot\,\}\colon \big(\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}\big)^{\psi_q=0} \times \big(\Omega^1_{\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}}\big)^{\psi_q=0} & \longrightarrow & \mathbb{C}_p
\end{array}$$

is commutative.

Proof. Let $\lambda, \mu \in \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L)$. We can express

$$\langle \lambda, \mu \rangle = \varsigma(\lambda\mu) = \mathrm{Res}\big(\mathfrak{M}(\gamma_{-1})\mathfrak{M}_{\chi_\phi}(\lambda\mu)\big) = \{\mathfrak{M}(\gamma_{-1}), \mathfrak{M}_{\chi_\phi}(\lambda\mu)\}.$$

But by the definition of $\mathfrak{M}_{\chi_\phi}$ in terms of an action of $\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L)$, we see that $\mathfrak{M}_{\chi_\phi}(\lambda\mu) = \lambda\big(\mathfrak{M}_{\chi_\phi}(\mu)\big)$.

Now identity (2) of corollary 178 implies that

$$\{\,\cdot\,, \gamma(\,\cdot\,)\} = \{\gamma^{-1}(\,\cdot\,), \,\cdot\,\} = \{\iota(\gamma)(\,\cdot\,), \,\cdot\,\} \quad \text{for all } \gamma \in \Gamma_L$$

and this adjointness with respect to ι can be extended to the action of $\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L)$ by a continuity argument (see lemma 2.39 of Schneider–Venjakob's preprint [35]).

Putting everything together, we conclude that

$$\begin{aligned} \langle \lambda, \mu \rangle &= \{\iota(\lambda)(\mathfrak{M}(\gamma_{-1})), \mathfrak{M}_{\chi_\phi}(\mu)\} = \{\mathfrak{M}(\iota(\lambda)\gamma_{-1}), \mathfrak{M}_{\chi_\phi}(\mu)\} \\ &= \{\mathfrak{M}(\gamma_{-1}\iota(\lambda)), \mathfrak{M}_{\chi_\phi}(\mu)\} = \{\gamma_{-1}(\mathfrak{M}(\iota(\lambda))), \mathfrak{M}_{\chi_\phi}(\mu)\}, \end{aligned}$$

where in the second and the last equalities we used again that $\mathfrak{M}$ is defined in terms of an action of $\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L)$. □

9.3.7 The Iwasawa pairing

Definition 194. Let $\mathscr{M} \in \mathrm{Ob}\big((\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{\mathrm{an}}_{\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p},\mathrm{fr}}\big)$. We define the pairing

$$\{\,\cdot\,,\,\cdot\,\}'_{\mathrm{Iw}} = \{\,\cdot\,,\,\cdot\,\}'_{\mathscr{M},\mathrm{Iw}} \colon \mathscr{M}^{\psi_q=0} \times (\mathscr{M}^\vee)^{\psi_q=0} \to \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L)$$

by requiring that the diagram

$$\begin{array}{ccc} \mathscr{M}^{\psi_q=0} \times (\mathscr{M}^\vee)^{\psi_q=0} \times \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L) & \xrightarrow{(m,g,\lambda)\mapsto(m,\lambda(g))} & \mathscr{M}^{\psi_q=0} \times (\mathscr{M}^\vee)^{\psi_q=0} \\ {\scriptstyle \{\cdot,\cdot\}'_{\mathscr{M},\mathrm{Iw}}} \downarrow \quad \Vert & & \downarrow {\scriptstyle \{\cdot,\cdot\}_{\mathscr{M}}} \\ \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L) \times \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L) & \xrightarrow{\langle\,\cdot\,,\,\cdot\,\rangle} & \mathbb{C}_p \end{array}$$

be commutative.

Remark. Given $m \in \mathscr{M}^{\psi_q=0}$ and $g \in (\mathscr{M}^\vee)^{\psi_q=0}$, the condition

$$\big\langle \{m, g\}'_{\mathrm{Iw}}, \lambda \big\rangle = \{m, \lambda(g)\} \quad \text{for all } \lambda \in \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L)$$

does uniquely determine $\{m, g\}'_{\mathrm{Iw}}$ by proposition 189.

Also, as we saw in the proof of corollary 193, $\{m, \lambda(g)\} = \{\iota(\lambda)(m), g\}$ and so we could define $\{m, g\}'_{\mathrm{Iw}}$ equivalently by the condition

$$\big\langle \{m, g\}'_{\mathrm{Iw}}, \lambda \big\rangle = \{\iota(\lambda)(m), g\} \quad \text{for all } \lambda \in \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L).$$

Proposition 195. *Let* $\mathscr{M} \in \mathrm{Ob}\big((\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{\mathrm{an}}_{\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p},\mathrm{fr}}\big)$*. The pairing*

$$\{\,\cdot\,,\,\cdot\,\}'_{\mathrm{Iw}} \colon \mathscr{M}^{\psi_q=0} \times (\mathscr{M}^{\vee})^{\psi_q=0} \to \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L)$$

induces an isomorphism

$$(\mathscr{M}^{\vee})^{\psi_q=0} \cong \mathrm{Hom}_{\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L)}(\mathscr{M}^{\psi_q=0}, \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L))^{\iota},$$

where the superscript ι *means that the action of* $\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L)$ *is twisted by* ι*.*

Proof. The isomorphism of proposition 185 restricts to an isomorphism

$$(\mathscr{M}^{\vee})^{\psi_q=0} \cong \mathrm{Hom}^{\mathrm{cont}}_{\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}}(\mathscr{M}^{\psi_q=0}, \mathbb{C}_p)^{\iota}$$

by the "adjointness" between φ_q and ψ_q with respect to $\{\,\cdot\,,\,\cdot\,\}$ and the decompositions $\mathscr{M} = \varphi_q(\mathscr{M}) \oplus \mathscr{M}^{\psi_q=0}$ and $\mathscr{M}^{\vee} = \varphi_q(\mathscr{M}^{\vee}) \oplus (\mathscr{M}^{\vee})^{\psi_q=0}$. Then the pairing $\{\,\cdot\,,\,\cdot\,\}'_{\mathrm{Iw}}$ induces

$$\begin{array}{ccc} \mathscr{M}^{\psi_q=0} \cong \mathrm{Hom}^{\mathrm{cont}}_{\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}}(\mathscr{M}, \mathbb{C}_p)^{\iota} & \longleftarrow & \mathrm{Hom}_{\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L)}(\mathscr{M}, \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L))^{\iota} \\ \varrho \circ g & \longleftarrow\!\shortmid & g \end{array}$$

by definition. This morphism is in fact an isomorphism by proposition 2.45 of Schneider–Venjakob's preprint [35]. □

In our study of Iwasawa cohomology of a representation, we do not need to consider the whole $(\psi_q = 0)$–part of (φ_q, Γ_L)–modules, but only the image of the $(\psi_q = 1)$–part under $1 - \frac{\pi_L}{q}\varphi_q$. That is why we introduce the following pairing.

Definition 196. Let $\mathscr{M} \in \mathrm{Ob}\big((\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{\mathrm{an}}_{\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p},\mathrm{fr}}\big)$. The *Iwasawa pairing for* $\mathscr{M}$ is the pairing

$$\{\,\cdot\,,\,\cdot\,\}_{\mathrm{Iw}} = \{\,\cdot\,,\,\cdot\,\}_{\mathscr{M},\mathrm{Iw}} \colon \mathscr{M}^{\psi_q=1} \times (\mathscr{M}^{\vee})^{\psi_q=\frac{q}{\varphi_q^{-1}(\pi_L)}} \to \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L)$$

that makes the diagram

$$\begin{array}{ccc} \mathscr{M}^{\psi_q=1} \times \left(\mathscr{M}^\vee\right)^{\psi_q=\frac{q}{\varphi_q^{-1}(\pi_L)}} & \overset{\{\,\cdot\,,\,\cdot\,\}_{\mathrm{Iw}}}{\dashrightarrow} & \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L) \\ {\scriptstyle 1-\frac{\pi_L}{q}\varphi_q}\big\downarrow \qquad \big\downarrow{\scriptstyle 1-\varphi_q} & & \big\| \\ \mathscr{M}^{\psi_q=0} \times (\mathscr{M}^\vee)^{\psi_q=0} & \xrightarrow{\{\,\cdot\,,\,\cdot\,\}'_{\mathrm{Iw}}} & \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L) \end{array}$$

commutative.

Remark. Given $m \in \mathscr{M}^{\psi_q=1}$ and $g \in (\mathscr{M}^\vee)^{\psi_q=q/\varphi_q^{-1}(\pi_L)}$, $\{m,g\}_{\mathrm{Iw}} \in \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L)$ is characterized by

$$\big\langle \{m,g\}_{\mathrm{Iw}}, \lambda \big\rangle = \left\{ \Big(1 - \frac{\pi_L}{q}\varphi_q\Big)(m), \lambda\big((1-\varphi_q)(g)\big) \right\} \quad \text{for all } \lambda \in \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L).$$

We claim that

$$\big\langle \{m,g\}_{\mathrm{Iw}}, \mu \big\rangle = (1-\varphi_q)\big(\{m,\mu(g)\}\big) \quad \text{for all } \mu \in D(\Gamma_L, \mathbb{C}_p).$$

By a density argument, it suffices to prove it for Dirac distributions $\delta_\gamma \in D(\Gamma_L, \mathbb{C}_p)$ (that act as $\gamma \in \Gamma_L$ and so commute with φ_q and ψ_q). Then, using proposition 186,

$$\begin{aligned} \big\langle \{m,g\}_{\mathrm{Iw}}, \delta_\gamma \big\rangle &= \big\{m, \gamma\big((1-\varphi_q)(g)\big)\big\} - \frac{\pi_L}{q}\big\{\varphi_q(m), \gamma\big((1-\varphi_q)(g)\big)\big\} \\ &= \big\{m, \gamma\big((1-\varphi_q)(g)\big)\big\} - \frac{\pi_L}{q}\varphi_q\big(\big\{m, \gamma\big(\psi_q \circ (1-\varphi_q)(g)\big)\big\}\big) \\ &= \{m,\gamma(g)\} - \{m, \varphi_q(\gamma(g))\} = \{m,\gamma(g)\} - \varphi_q\big(\{\psi_q(m), \gamma(g)\}\big) \\ &= \{m,\gamma(g)\} - \varphi_q\big(\{m,\gamma(g)\}\big). \end{aligned}$$

In the *non-relative* Lubin–Tate situation, where φ_q is the identity on $\mathbb{C}_p$, we see that $\big\langle \{m,g\}_{\mathrm{Iw}}, \cdot \big\rangle$ factors through $\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L)/D(\Gamma_L,\mathbb{C}_p)$ and, by proposition 189, we deduce that $\{m,g\}_{\mathrm{Iw}} \in D(\Gamma_L,\mathbb{C}_p)$. I do not know how to obtain an Iwasawa pairing with values in $D(\Gamma_L,\mathbb{C}_p)$ in general…

Proposition 197. *Let* $\mathscr{M} \in \mathrm{Ob}\big((\varphi_q,\Gamma_L)\text{–}\mathrm{Mod}^{\mathrm{an}}_{\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p},\mathrm{fr}}\big)$ *and consider the* $\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}$*–linear map*

$$\begin{aligned} \mathrm{Tw}_{\chi_\phi} \colon \mathscr{M} &\longrightarrow \mathscr{M}(\chi_\phi) \\ m &\longmapsto m \otimes t_0 \end{aligned}$$

(which is not $\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L)$–linear). More generally, for $i \in \mathbb{Z}_{\geq 0}$, define $\mathrm{Tw}_{\chi_\phi^i} = \left(\mathrm{Tw}_{\chi_\phi}\right)^i$ and $\mathrm{Tw}_{\chi_\phi^{-i}} = \left(\mathrm{Tw}_{\chi_\phi^i}\right)^{-1}$. For every $i \in \mathbb{Z}$, the diagram

$$\begin{array}{ccc}
\mathscr{M}^{\psi_q=1} \times \left(\mathscr{M}^\vee\right)^{\psi_q=\frac{q}{\varphi_q^{-1}(\pi_L)}} & \xrightarrow{\{\,\cdot\,,\,\cdot\,\}_{\mathrm{Iw}}} & \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L) \\
\mathrm{Tw}_{\chi_\phi^i}\Big\downarrow \qquad\qquad \Big\downarrow\mathrm{Tw}_{\chi_\phi^{-i}} & & \Big\downarrow\mathrm{Tw}_{\chi_\phi^i} \\
\mathscr{M}(\chi_\phi^i)^{\psi_q=1} \times \left(\mathscr{M}^\vee(\chi_\phi^{-i})\right)^{\psi_q=\frac{q}{\varphi_q^{-1}(\pi_L)}} & \xrightarrow{\{\,\cdot\,,\,\cdot\,\}_{\mathrm{Iw}}} & \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L)
\end{array}$$

is commutative.

Proof. See lemma 2.62 of Schneider–Venjakob's preprint [35], whose proof works verbatim in the *relative* Lubin–Tate situation. □

Proposition 198. *Let D be a φ_q–module over L (e.g., $D \in \mathrm{Ob}\big((\mathrm{Fil}, \varphi_q)\text{–}\mathrm{Mod}_L\big)$). Consider $\mathscr{M} = \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p} \otimes_L D \in \mathrm{Ob}\big((\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{\mathrm{an}}_{\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p},\mathrm{fr}}\big)$, so that*

$$\mathscr{M}^\vee = \mathrm{Hom}_{\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}}\left(\mathscr{M}, \Omega^1_{\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}}\right) \cong \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p} \otimes_L D^*, \quad \textit{where } D^* = \mathrm{Hom}_L(D, L).$$

The natural (evaluation) pairing between D and D^* makes the diagram

$$\begin{array}{ccc}
\left(\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L) \otimes_L D\right) \times \left(\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L) \otimes_L D^*\right) & \dashrightarrow & \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L) \\
\gamma_{-1}\circ\mathfrak{M}\circ\iota\otimes\mathrm{id}_D\Big\downarrow \qquad\qquad \Big\downarrow\mathfrak{M}_{\chi_\phi}\otimes\mathrm{id}_{D^*} & & \Big\| \\
\left(\left(\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}\right)^{\psi_q=0} \otimes_L D\right) \times \left(\left(\Omega^1_{\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}}\right)^{\psi_q=0} \otimes_L D^*\right) & & \Big\| \\
\cap\!\!\mid \qquad\qquad\qquad \cap\!\!\mid & & \Big\| \\
\left(\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p} \otimes_L D\right)^{\psi_q=0} \times \left(\Omega^1_{\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}} \otimes_L D^*\right)^{\psi_q=0} & \xrightarrow{\{\,\cdot\,,\,\cdot\,\}'_{\mathrm{Iw}}} & \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L)
\end{array}$$

commutative.

Proof. A direct calculation using corollary 193 yields this result; see lemma 2.66 of Schneider–Venjakob's preprint [35] for the details. □

9.4 Cohomology from (φ_q, Γ_L)–modules

Finally, we can put together the results of the previous subsections to compute different kinds of cohomology groups of a representation in terms of its associated (φ_q, Γ_L)–modules.

9.4.1 Iwasawa cohomology groups as duals

In this subsection we explain the main technical result of Schneider–Venjakob's article [34] adapted to the *relative* Lubin–Tate situation.

Lemma 199. *Let* $T \in \mathrm{Ob}\big(\mathrm{Rep}_{\mathscr{O}_K,\mathrm{tor}}(G_L)\big)$*. The long exact sequence of cohomology of* H_L *associated with the short exact sequence*

$$0 \longrightarrow T \longrightarrow \mathbf{A} \otimes_{\mathscr{O}_K} T \xrightarrow{\varphi_q - 1} \mathbf{A} \otimes_{\mathscr{O}_K} T \longrightarrow 0$$

is

$$0 \longrightarrow \mathrm{H}^0(L_\infty, T) \longrightarrow \mathbf{D}(T) \xrightarrow{\varphi_q - 1} \mathbf{D}(T) \longrightarrow \mathrm{H}^1(L_\infty, T) \longrightarrow 0.$$

Proof. The short exact sequence in the statement arises from lemma 69 after applying $\cdot \otimes_{\mathscr{O}_K} T$ (and is again exact because $\mathbf{A}$ is flat over $\mathscr{O}_K$). Then, as in lemma 5.2 of Schneider–Venjakob's article [34], by taking generators of T it suffices to prove that $\mathrm{H}^i(L_\infty, \mathbf{A}/\pi_L^n \mathbf{A}) = 0$ for all $i, n \geq 1$. An induction argument on n reduces the claim to the case $n = 1$; that is, to $\mathrm{H}^i(H_L, \mathbf{E}) = 0$ for all $i \geq 1$, which is clear thanks to the isomorphism $H_L \cong \mathrm{Gal}(\mathbf{E}/\mathbf{E}_L)$. ☐

Theorem 200 (Schneider–Venjakob). *Let* $T \in \mathrm{Ob}\big(\mathrm{Rep}_{\mathscr{O}_K}(G_L)\big)$ *and consider the character* $\tau = \chi_{\mathrm{cyc}} \cdot \chi_\phi^{-1}$*. There is an exact sequence*

$$0 \longrightarrow \mathrm{H}^1_{\mathrm{Iw}}(L_\infty/L, T) \longrightarrow \mathbf{D}(T(\tau^{-1})) \xrightarrow{\psi_q - 1} \mathbf{D}(T(\tau^{-1})) \longrightarrow \mathrm{H}^2_{\mathrm{Iw}}(L_\infty/L, T) \longrightarrow 0$$

that is functorial in T*. Furthermore,* $\mathrm{H}^i_{\mathrm{Iw}}(L_\infty/L, T) = 0$ *for all* $i \in \mathbb{Z} \setminus \{1, 2\}$*.*

Proof. The proof for the *relative* Lubin–Tate situation is very similar to the proofs of lemma 5.12 and theorem 5.13 of Schneider and Venjakob's article [34]. Here we just summarize the general strategy.

For every $n \in \mathbb{Z}_{\geq 1}$, set $T_n = T/\pi_K^n T$. Applying lemma 199 to $T_n^\vee(1)$ and taking Pontryagin duals (see the next paragraphs for some subtleties) yields an exact sequence

$$0 \to \mathrm{H}^1\big(L_\infty, T_n^\vee(1)\big)^\vee \to \mathbf{D}\big(T_n^\vee(1)\big)^\vee \xrightarrow{\varphi_q^{-1} \circ \varphi_q^\vee(\cdot) - 1} \mathbf{D}\big(T_n^\vee(1)\big)^\vee \to \mathrm{H}^0\big(L_\infty, T_n^\vee(1)\big)^\vee \to 0.$$

Technically, we should take Pontryagin duals over $\mathscr{O}_K$ everywhere and get $\varphi_q^\vee - 1$ in the middle arrow. However, lemma 5.3 of Schneider–Venjakob's article

[34] gives an isomorphism $\operatorname{Hom}^{\mathrm{cont}}_{\mathscr{O}_K}(\,\cdot\,, K/\mathscr{O}_K) \cong \operatorname{Hom}^{\mathrm{cont}}_{\mathscr{O}_L}(\,\cdot\,, L/\mathscr{O}_L)$ and then we have to add the composition with φ_q^{-1} to preserve $\mathscr{O}_L$–linearity. Tracing through the isomorphisms in the paragraph immediately preceding loc. cit., we check next that $\varphi_q^{-1} \circ \varphi_q^{\vee}(\,\cdot\,) - 1$ is the correct dual of $\varphi_q - 1$ over $\mathscr{O}_L$:

- Let D be an $\mathscr{O}_L$–module whose two Pontryagin duals we want to compare.
- Since L/K is separable (characteristic 0),

$$\begin{aligned} \operatorname{tr}_{L/K} \colon L \times L &\longrightarrow K \\ (x, y) &\longmapsto \operatorname{tr}_{L/K}(xy) \end{aligned}$$

  is a perfect pairing. If

$$\mathfrak{d}_{L/K}^{-1} = \{\, x \in L : \operatorname{tr}_{L/K}(x\mathscr{O}_L) \subseteq \mathscr{O}_K \,\} = \pi_K^{-s}\mathscr{O}_L,$$

  then we get a perfect pairing

$$\begin{aligned} \operatorname{tr}_{L/K}(\pi_K^{-s}\,\cdot) \colon \mathscr{O}_L \times \mathscr{O}_L &\longrightarrow \mathscr{O}_K \\ (x, y) &\longmapsto \operatorname{tr}_{L/K}(\pi_K^{-s}xy) \end{aligned}$$

  which induces an isomorphism of $\mathscr{O}_L$–modules $\mathscr{O}_L \cong \operatorname{Hom}_{\mathscr{O}_K}(\mathscr{O}_L, \mathscr{O}_K)$.
- Taking $\otimes_{\mathscr{O}_K} (K/\mathscr{O}_K)$ we obtain

$$\begin{array}{ccc} L/\mathscr{O}_L \cong \operatorname{Hom}_{\mathscr{O}_K}(\mathscr{O}_L, \mathscr{O}_K) \otimes_{\mathscr{O}_K} (K/\mathscr{O}_K) & \cong & \operatorname{Hom}_{\mathscr{O}_K}(\mathscr{O}_L, K/\mathscr{O}_K) \\ x \longmapsto & & \operatorname{tr}_{L/K}(\pi_K^{-s}x\,\cdot) \end{array}$$

  (all isomorphims of $\mathscr{O}_L$–modules).
- But $\operatorname{Hom}_{\mathscr{O}_K}(\mathscr{O}_L, \,\cdot\,)$ is right adjoint to restriction of scalars from $\mathscr{O}_L$ to $\mathscr{O}_K$ via the following isomorphisms:

$$\begin{aligned} \operatorname{Hom}_{\mathscr{O}_K}(D, K/\mathscr{O}_K) &\cong \operatorname{Hom}_{\mathscr{O}_L}\big(D, \operatorname{Hom}_{\mathscr{O}_K}(\mathscr{O}_L, K/\mathscr{O}_K)\big) \\ f &\longmapsto \big[d \mapsto f(d\,\cdot)\big] \\ g(\,\cdot\,)(1) &\longleftarrow g \end{aligned}$$

- Combining everything, we obtain the isomorphism

$$\begin{array}{ccc} \operatorname{Hom}_{\mathscr{O}_L}(D, L/\mathscr{O}_L) \cong \operatorname{Hom}_{\mathscr{O}_L}\big(D, \operatorname{Hom}_{\mathscr{O}_K}(\mathscr{O}_L, K/\mathscr{O}_K)\big) & \cong & \operatorname{Hom}_{\mathscr{O}_K}(D, K/\mathscr{O}_K) \\ f \longmapsto & & \operatorname{tr}_{L/K}(\pi_K^{-s} f(\,\cdot\,)) \end{array}$$

of $\mathscr{O}_K$–modules. If, moreover, D is a (φ_q, Γ_L)–module over $\mathbb{A}_L$, then $\varphi_q^\vee$ makes sense on $\mathrm{Hom}_{\mathscr{O}_K}(D, K/\mathscr{O}_K)$ but not on $\mathrm{Hom}_{\mathscr{O}_L}(D, L/\mathscr{O}_L)$, where we have to use $\left(\varphi_q^\vee\right)^{\varphi_q^{-1}}$ instead.

Via the identifications provided by corollary 169 and by proposition 182 applied to $T_n^\vee(1)$, the previous exact sequence can be rewritten as

$$0 \longrightarrow \mathrm{H}^1_{\mathrm{Iw}}(L_\infty/L, T_n) \longrightarrow \mathbf{D}(T_n(\tau^{-1})) \xrightarrow{\psi_q - 1} \mathbf{D}(T_n(\tau^{-1})) \longrightarrow \mathrm{H}^2_{\mathrm{Iw}}(L_\infty/L, T_n) \longrightarrow 0$$

and so one only needs to prove using general results that the projective limit of these sequences for $n \in \mathbb{Z}_{\geq 1}$ is exact. □

9.4.2 Analytic cohomology

Consider a finite extension L' of L contained in L_∞ (e.g., $L' = L$ or $L' = L_n$ for some $n \geq 1$) and set $\Gamma_{L'} = \mathrm{Gal}(L_\infty/L')$. Let $V \in \mathrm{Rep}_K^{\mathrm{an}}(G_L)$. We can identify $\mathrm{H}^1(L', V)$ with $\mathrm{Ext}^1_{\mathrm{Rep}_K(G_{L'})}(K, V)$. Then we define the *analytic* and the *overconvergent* (first) cohomology subgroups to be the subgroups of the usual (continuous) cohomology that classify analytic and overconvergent extensions, respectively. That is to say, we define $\mathrm{H}^1_{\mathrm{an}}(L', V)$ and $\mathrm{H}^1_\dagger(L', V)$ to make the diagram

$$\begin{array}{ccc} \mathrm{H}^1(L', V) & \cong & \mathrm{Ext}^1_{\mathrm{Rep}_K(G_{L'})}(K, V) \\ \cup\! \mid & & \cup\! \mid \\ \mathrm{H}^1_\dagger(L', V) & \cong & \mathrm{Ext}^1_{\mathrm{Rep}^\dagger_K(G_{L'})}(K, V) \\ \cup\! \mid & & \cup\! \mid \\ \mathrm{H}^1_{\mathrm{an}}(L', V) & \cong & \mathrm{Ext}^1_{\mathrm{Rep}^{\mathrm{an}}_K(G_{L'})}(K, V) \end{array}$$

commutative.

By theorem 91, one should be able to compute $\mathrm{H}^1_{\mathrm{an}}(L', V)$ by means of the $(\varphi_q, \Gamma_{L'})$–module $\mathbf{D}^\dagger_{\mathrm{rig}}(V)$. To do that, we use the theory of analytic cohomology as in sections 2.1 and 2.2 of Berger–Fourquaux's article [7] (cf. section 5 of Colmez's article [17] and section 4 of Fourquaux–Xie's article [22]).

Definition 201. Let G be a locally K–analytic semigroup and let

$$M = \varinjlim_{r \in R} M_r = \varinjlim_{r \in R} \varprojlim_{s \in S_r} M_{r,s}$$

be an LF space with a K–proanalytic action of G. We write $C^\bullet(G, M)$ for the inhomogeneous continuous cochain complex of G with coefficients in M and $C^\bullet_{\mathrm{an}}(G, M)$ for the subcomplex of locally K–analytic cochains. More precisely,

$C^i_{\mathrm{an}}(G,M)$ is the subspace of $C^i(G,M)$ of locally analytic functions in the sense that they locally have values in some M_r and then the compositions with the projection to $M_{r,s}$ are analytic for all $s \in S_r$. We define

$$\mathrm{H}^*(G,M) = \mathrm{H}^*\big(C^\bullet(G,M)\big) \quad \text{and} \quad \mathrm{H}^*_{\mathrm{an}}(G,M) = \mathrm{H}^*\big(C^\bullet_{\mathrm{an}}(G,M)\big).$$

Remark. We want to work with $M = \mathscr{M} \in \mathrm{Ob}\big((\varphi_q,\Gamma_L)\text{–}\mathrm{Mod}^{\mathrm{an}}_{\mathbf{B}^\dagger_{\mathrm{rig},L'},\mathrm{fr}}\big)$ and $G = \Gamma_{L'}$, $\Phi \times \Gamma_{L'}$ or $\Psi \times \Gamma_{L'}$, where

$$\Phi = \langle \varphi_q \rangle = \{\, \varphi_q^n : n \in \mathbb{Z}_{\geq 0} \,\}$$

and

$$\Psi = \Big\langle \frac{\varphi_q^{-1}(\pi_L)}{q}\psi_q \Big\rangle = \Big\{ \Big(\frac{\varphi_q^{-1}(\pi_L)}{q}\psi_q\Big)^n : n \in \mathbb{Z}_{\geq 0} \Big\}$$

are discrete semigroups and the K–analytic structure comes from Γ_L. In particular, we want to study $\mathrm{H}^1_{\mathrm{an}}(G,M) = Z^1_{\mathrm{an}}(G,M)/B^1_{\mathrm{an}}(G,M)$, where

- $Z^1_{\mathrm{an}}(G,M)$ is the subset of $f \in C^1_{\mathrm{an}}(G,M)$ such that $f(gh) = f(g) + g(f(h))$ for all $g,h \in G$ and
- $B^1_{\mathrm{an}}(G,M)$ is the subset of $f \in C^1_{\mathrm{an}}(G,M)$ of the form $g \mapsto (g-1)(m)$ for some $m \in M$.

Theorem 202 (Berger–Fourquaux). *Let* $\mathscr{M} \in \mathrm{Ob}\big((\varphi_q,\Gamma_L)\text{–}\mathrm{Mod}^{\mathrm{an}}_{\mathbf{B}^\dagger_{\mathrm{rig},L'},\mathrm{fr}}\big)$. *For* $i = 0$ *or* 1,

$$\mathrm{H}^i_{\mathrm{an}}(\Phi \times \Gamma_{L'}, \mathscr{M}) \cong \mathrm{H}^i_{\mathrm{an}}(\Psi \times \Gamma_{L'}, \mathscr{M}).$$

Proof. See theorem 2.2.2 and corollary 2.2.3 of Berger–Fourquaux's article [7], whose proofs work in the same way in the *relative* Lubin–Tate situation (taking into account that ψ_q is normalized in a different way and so the $(\psi_q = 1)$–parts of modules in loc. cit. correspond to $\big(\psi_q = q/\varphi_q^{-1}(\pi_L)\big)$–parts in our notation). □

Proposition 203. *Let* $V \in \mathrm{Ob}\big(\mathrm{Rep}^{\mathrm{an}}_K(G_L)\big)$. *There are natural isomorphisms*

$$\mathrm{H}^1_{\mathrm{an}}(L',V) \cong \mathrm{H}^1_{\mathrm{an}}(\Phi \times \Gamma_{L'}, \mathbf{D}^\dagger_{\mathrm{rig}}(V)) \cong \mathrm{H}^1_{\mathrm{an}}(\Psi \times \Gamma_{L'}, \mathbf{D}^\dagger_{\mathrm{rig}}(V)).$$

Proof. See proposition 2.2.1 of Berger–Fourquaux's article [7]. □

9.4.3 The operator Θ_b and construction of cocycles

Choose $n \in \mathbb{Z}_{\geq 1}$ large enough so that $\ell_n = \pi_K^{-n} \log \circ \chi_\phi$ defines an isomorphism $\Gamma_{L_n} \cong \mathscr{O}_K$ (cf. section 8.5.2). Let $b = (b_1, \dots, b_r)$ be a $\mathbb{Z}_p$–basis of Γ_{L_n}. In section 8.5.5 we defined an operator

$$\Theta_b = q^{-n} \prod_{j=1}^{r} \Big(\log(\chi_\phi(b_j)) \frac{N_\nabla}{b_j - 1} \Big)$$

on $\mathscr{O}_K$–analytic (φ_q, Γ_L)–modules using the operator N_∇ from lemma 90. Our goal in this subsection is to use the operator Θ_b to construct cocycles in $Z^1_{\mathrm{an}}(\Gamma_{L_n}, M)$ for certain spaces M.

For each $i \in \mathbb{Z}$, write $N_{\nabla,i} = N_\nabla - i$.

Lemma 204. *Let $f(Z) \in \big(\mathbf{B}^+_{\mathrm{rig},L}\big)^{\psi_q = 0}$. For every $h \in \mathbb{Z}_{\geq 1}$,*

$$(N_{\nabla,h-1} \circ \cdots N_{\nabla,1} \circ \Theta_b)(f)(Z) \in \left(\frac{\log_\phi(Z)}{\varphi_q^n(Z)} \right)^h \mathbf{B}^+_{\mathrm{rig},L}.$$

Proof. This result is analogous to lemma 2.4.2 of Berger–Fourquaux's article [7]. We explain its proof here for the *relative* Lubin–Tate situation for the convenience of the reader.

We can prove the lemma by induction on h. For the base case $h = 1$, pick a sequence of $v_m \in \mathfrak{F}_{\phi,m} \setminus \mathfrak{F}_{\phi,m-1}$ for $m \in \mathbb{Z}_{\geq 1}$ with the compatibility condition $v_m = \varphi^{-1}(\phi(v_{m+1}))$ (e.g., by picking a generator $(z_m)_{m \geq 1} \in \mathrm{T}_\phi \mathfrak{F}_\phi$ and setting $v_m = \varphi_q^m(z_m)$).

We claim that $\Theta_b(f)(v_m) = 0$ for all $m > n$. Indeed, lemma 2.4.1 of Berger–Fourquaux's article [7] implies that

$$\Theta_b(f)(v_m) = \frac{1}{q^m} \mathrm{Tr}_{L_m/L_n}(f(v_m))$$

and it suffices to prove that the last trace is 0. But the $\mathrm{Gal}(L_m/L_{m-1})$–conjugates of v_m are the elements of $\mathfrak{F}_{\phi,m} \setminus \mathfrak{F}_{\phi,m-1}$, which can be expressed as $\mathfrak{F}_\phi(v_m, w_1)$ with $w_1 \in \mathfrak{F}_{\phi,1}$. Thus,

$$\mathrm{Tr}_{L_m/L_{m-1}}(f(v_m)) = \sum_{w_1 \in \mathfrak{F}_{\phi,1}} f(\mathfrak{F}_\phi(v_m, w_1)) = \pi_L(\varphi_q \circ \psi_q)(f)(v_m) = 0$$

because $\psi_q(f) = 0$ by hypothesis.

The previous claim implies that $\Theta_b(f)(Z)$ vanishes at all the torsion points in $\mathfrak{F}_{\phi,m}$ for $m > n$ and so it must be divisible by $\log_\phi(Z)/(\varphi_q^n(Z))$ in $\mathbf{B}^+_{\mathrm{rig},L}$.

Now suppose that the result has been proved for h and write

$$(N_{\nabla,h-1} \circ \cdots \circ N_{\nabla,1} \circ \Theta_b)(f) = \log_\phi(Z)^h \frac{f_h(Z)}{(\varphi_q^n(Z))^h} \quad \text{for some } f_h \in \mathbf{B}^+_{\mathrm{rig},L}.$$

We can compute

$$\begin{aligned}(N_{\nabla,h} \circ \cdots \circ N_{\nabla,1} \circ \Theta_b)(f) &= (\log_\phi(Z)\partial_\phi - h)\left(\log_\phi(Z)^h \frac{f_h(Z)}{(\varphi_q^n(Z))^h}\right) \\ &= \log_\phi(Z)^{h+1} \frac{\partial_\phi(f_h(Z))(\varphi_q^n(Z))^h - f_h(Z)\partial_\phi((\varphi_q^n(Z))^h)}{(\varphi_q^n(Z))^{2h}} \\ &= \left(\frac{\log_\phi(Z)}{\varphi_q^n(Z)}\right)^{h+1} f_{h+1},\end{aligned}$$

where

$$f_{h+1} = \partial_\phi(f_h(Z))\varphi_q^n(Z) - hf_h(Z)\partial_\phi(\varphi_q^n(Z)) \in \mathbf{B}^+_{\mathrm{rig},L}. \qquad \square$$

Proposition 205. *Let $\mathscr{M} \in \mathrm{Ob}\big((\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{\mathrm{an}}_{\mathbf{B}^\dagger_{\mathrm{rig},L},\mathrm{fr}}\big)$ and let $b = (b_1, \dots, b_r)$ be a $\mathbb{Z}_p$–basis of Γ_{L_n} as above. For every $y \in \mathscr{M}^{\psi_q = q/\varphi_q^{-1}(\pi_L)}$, there exists a unique cocycle $c_b(y) \in Z^1_{\mathrm{an}}\big(\Gamma_{L_n}, \mathscr{M}^{\psi_q = q/\varphi_q^{-1}(\pi_L)}\big)$ such that*

$$c_b(y)(b_j^k) = q^{-n}\left(\frac{\log(\chi_\phi(b_j))(b_j^k - 1)}{b_j - 1} \prod_{\substack{i=1 \\ i \neq j}}^{r} \frac{\log(\chi_\phi(b_i))N_\nabla}{b_i - 1}\right)(y)$$

for all $j \in \{1, \dots, r\}$ and all $k \in \mathbb{Z}_{\geq 0}$. Moreover,

$$c_b(y)'(1) = \lim_{k\to\infty} \frac{c_b(y)(b_j^k) - c_b(y)(1)}{\log(\chi_\phi(b_j^k))} = \Theta_b(y).$$

Proof. See proposition 2.5.1 of Berger–Fourquaux's article [7], whose proof works verbatim in the *relative* Lubin–Tate situation. □

Lemma 206. *In the situation of proposition 205, if $b' = (b'_1, \dots, b'_r)$ is another $\mathbb{Z}_p$–basis of Γ_{L_n}, then the cocycles $c_b(y)$ and $c_{b'}(y)$ are cohomologous.*

Proof. See lemma 2.5.3 of Berger–Fourquaux's article [7], whose proof works verbatim in the *relative* Lubin–Tate situation. □

Lemma 207. *In the situation of proposition 205, take $m \geq n$ and let $a = (a_1, \dots, a_r)$ be a $\mathbb{Z}_p$–basis of Γ_{L_m}. Then*

$$\mathrm{cor}([c_a(y)]) = [c_b(y)] \quad \textit{in } \mathrm{H}^1_{\mathrm{an}}\big(\Gamma_{L_n}, \mathscr{M}^{\psi_q = q/\varphi_q^{-1}(\pi_L)}\big),$$

where cor *denotes the corestriction from Γ_{L_m} to Γ_{L_n}.*

Proof. See lemma 2.5.4 of Berger–Fourquaux's article [7], whose proof works verbatim in the *relative* Lubin–Tate situation. □

9.4.4 Construction of analytic cohomology classes

Definition 208. Let $V \in \mathrm{Ob}\big(\mathrm{Rep}_K^{\mathrm{an}}(G_L)\big)$ and consider an integer $n \gg 0$ and a $\mathbb{Z}_p$–basis $b = (b_1, \dots, b_r)$ of Γ_{L_n} as in section 9.4.3. We define the map

$$h^1_{L_n,V} \colon \mathbf{D}^\dagger_{\mathrm{rig}}(V)^{\psi_q = \frac{q}{\varphi_q^{-1}(\pi_L)}} \to \mathrm{H}^1_{\mathrm{an}}(L_n, V)$$

to be the composition of the map

$$\begin{aligned} \mathbf{D}^\dagger_{\mathrm{rig}}(V)^{\psi_q = q/\varphi_q^{-1}(\pi_L)} &\longrightarrow \mathrm{H}^1_{\mathrm{an}}\big(\Gamma_{L_n}, \mathbf{D}^\dagger_{\mathrm{rig}}(V)^{\psi_q = q/\varphi_q^{-1}(\pi_L)}\big) \\ y &\longmapsto [c_b(y)] \end{aligned}$$

from proposition 205 with the maps

$$\mathrm{H}^1_{\mathrm{an}}\big(\Gamma_{L_n}, \mathbf{D}^\dagger_{\mathrm{rig}}(V)^{\psi_q = q/\varphi_q^{-1}(\pi_L)}\big) \hookrightarrow \mathrm{H}^1_{\mathrm{an}}\big(\Psi \times \Gamma_{L_n}, \mathbf{D}^\dagger_{\mathrm{rig}}(V)\big) \cong \mathrm{H}^1_{\mathrm{an}}(L_n, V)$$

(see proposition 203 for the last isomorphism).

Remark. Given $m \geq n$, lemma 207 implies that $\mathrm{cor} \circ h^1_{L_m,V} = h^1_{L_n,V}$. Therefore, we may extend the definition of $h^1_{L_n,V}$ to all $n \in \mathbb{Z}_{\geq 1}$ and even to $h^1_{L,V}$ by requiring that these maps be compatible with corestriction.

9.5 (Generalized) Herr complexes

Following section 3.2 of Schneider–Venjakob's preprint [35], we can reinterpret the constructions of the previous subsection by means of (a generalized version of) Herr complexes.

9.5.1 Some constructions in homological algebra

We use the notation introduced in section 9.4.2. In the following, we use the symbol ? for either an (analytic) or nothing. Let G be either Φ or Ψ and let g be the generator of the semigroup G. Consider a finite extension L' of L contained in L_∞ and set $\Gamma_{L'} = \mathrm{Gal}(L_\infty/L')$. Let M be an LF space with a K–proanalytic action of $\Gamma_{L'}$ that commutes with g (e.g., if $M = \mathscr{M}$ is an $\mathscr{O}_K$–analytic (φ_q, Γ_L)–module over $\mathbf{B}^\dagger_{\mathrm{rig},L}$). We define $\mathcal{T}^\bullet_?(G \times \Gamma_{L'}, M)$ to be the total (cohomological) complex of the double complex

$$
\begin{array}{ccc}
\vdots & & \vdots \\
\uparrow & & \uparrow \\
C^2_?(\Gamma_{L'}, M) & \xrightarrow{g-1} & C^2_?(\Gamma_{L'}, M) \\
\uparrow & & \uparrow \\
C^1_?(\Gamma_{L'}, M) & \xrightarrow{g-1} & C^1_?(\Gamma_{L'}, M) \\
\uparrow & & \uparrow \\
C^0_?(\Gamma_{L'}, M) & \xrightarrow{g-1} & C^0_?(\Gamma_{L'}, M) \\
\uparrow & & \uparrow \\
0 & & 0
\end{array}
$$

concentrated horizontally in degrees 0 and 1. That is, if we write $(C^\bullet, d^\bullet)$ for the complex $C^\bullet_?(\Gamma_{L'}, M)$, the complex $\mathcal{T}^\bullet_?(G \times \Gamma_{L'}, M)$ is given by

$$\mathcal{T}^i_?(G \times \Gamma_{L'}, M) = C^i \oplus C^{i-1}$$

with differentials

$$\begin{pmatrix} d^i & 0 \\ g-1 & -d^{i-1} \end{pmatrix} : C^i \oplus C^{i-1} \longrightarrow C^{i+1} \oplus C^i$$

for all $i \in \mathbb{Z}$.

The first filtration (i.e., the one obtained by looking at pieces with horizontal degree $\leq m$ for varying $m \in \mathbb{Z}$) gives rise to a spectral sequence

$${}_{\mathrm{I}}E_2^{i,j} = \mathrm{H}^i\big(G, \mathrm{H}^j_?(\Gamma_{L'}, M)\big) \implies \mathrm{H}^{i+j}\big(\mathcal{T}^\bullet_?(G \times \Gamma_{L'}, M)\big)$$

that degenerates to the short exact sequences

$$0 \longrightarrow \frac{\mathrm{H}^{i-1}_?(\Gamma_{L'}, M)}{(g-1)} \longrightarrow \mathrm{H}^i\big(\mathcal{T}^\bullet_?(G \times \Gamma_{L'}, M)\big) \longrightarrow \mathrm{H}^i_?(\Gamma_{L'}, M)^{g=1} \longrightarrow 0.$$

Remark. Like Schneider and Venjakob, we follow the sign conventions from Nekovář's book [29]. Namely, given two complexes $(X^\bullet, d^\bullet_X)$ and $(Y^\bullet, d^\bullet_Y)$ of R–modules, we define

- the shifted complex $X[n]^\bullet$ given by $X[n]^i = X^{i+n}$ and $d^i_{X[n]} = (-1)^n d_{X^{i+n}}$ (for $n \in \mathbb{Z}$),
- the complex $\mathrm{Hom}^\bullet_R(X, Y)$ given by

  $$\mathrm{Hom}^i_R(X, Y) = \prod_{n \in \mathbb{Z}} \mathrm{Hom}_R(X^n, Y^{n+i})$$

  and

  $$d^i_{\mathrm{Hom}}\big(f\colon X^n \to Y^{n+i}\big) = \big((-1)^{i-1} f \circ d^{n-1}_X, d^{n+i}_Y \circ f\big)$$

  (if Y is concentrated in degree 0, then $\mathrm{Hom}^i_R(X^\bullet, Y) = \mathrm{Hom}_R(X^{-i}, Y)$ and $d^i_{\mathrm{Hom}}(f) = (-1)^{i-1} f \circ d^{-i-1}_X$),
- the complex $(X \otimes_R Y)^\bullet$ given by

  $$(X \otimes_R Y)^i = \bigoplus_{n \in \mathbb{Z}} (X^n \otimes_R Y^{i-n})$$

  and

  $$d^i_{X \otimes Y}(x \otimes y) = d^n_X(x) \otimes y + (-1)^n x \otimes d^{i-n}_Y(y) \quad \text{if } x \in X^n \text{ and } y \in Y^{i-n},$$

  and
- for a morphism of complexes $f^\bullet\colon X^\bullet \to Y^\bullet$, the mapping cone complex $\mathrm{Cone}(f)^\bullet$ given by $\mathrm{Cone}(f)^i = X^{i+1} \oplus Y^i$ and

  $$d^i_{\mathrm{Cone}(f)} = \begin{pmatrix} -d^{i+1}_X & 0 \\ -f^{i+1} & d^i_Y \end{pmatrix}.$$

In particular, we can express

$$\mathcal{T}^\bullet_?(G \times \Gamma_{L'}, M) = \mathrm{Cone}\Big(C^\bullet_?(\Gamma_{L'}, M) \xrightarrow{g-1} C^\bullet_?(\Gamma_{L'}, M)\Big)[-1].$$

9.5.2 Cohomology of Herr complexes

Keep the notation of section 9.5.1 (here and in the following subsections). Next we give some examples of cohomology groups that appeared earlier and which can be computed using complexes of the form $\mathcal{T}_?^\bullet(G \times \Gamma_{L'}, M)$.

Theorem 209 (Thomas). *Let $\mathscr{M} \in \mathrm{Ob}\big((\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{\mathrm{an}}_{\mathbf{B}^\dagger_{\mathrm{rig},L'},\mathrm{fr}}\big)$. For every $i \in \mathbb{Z}$, there are canonical isomorphisms*

$$\mathrm{H}^i_{\mathrm{an}}(\Phi \times \Gamma_{L'}, \mathscr{M}) \cong \mathrm{H}^i\big(\mathcal{T}^\bullet_{\mathrm{an}}(\Phi \times \Gamma_{L'}, \mathscr{M})\big)$$

and

$$\mathrm{H}^i_{\mathrm{an}}(\Psi \times \Gamma_{L'}, \mathscr{M}) \cong \mathrm{H}^i\big(\mathcal{T}^\bullet_{\mathrm{an}}(\Psi \times \Gamma_{L'}, \mathscr{M})\big).$$

Proof. See theorem 11.6 of Thomas's article [40]. □

Corollary 210. *Let $V \in \mathrm{Ob}\big(\mathrm{Rep}^{\mathrm{an}}_K(G_L)\big)$. There is a canonical isomorphism*

$$\mathrm{H}^1_{\mathrm{an}}(L', V) \cong \mathrm{H}^1\big(\mathcal{T}^\bullet_{\mathrm{an}}(\Phi \times \Gamma_{L'}, \mathbf{D}^\dagger_{\mathrm{rig}}(V))\big).$$

Proof. This is a combination of proposition 203 and theorem 209. □

Remark. Using these results, we can reinterpret definition 208 as follows. Let $V \in \mathrm{Ob}\big(\mathrm{Rep}^{\mathrm{an}}_K(G_L)\big)$ and take an integer $n \gg 0$ and a $\mathbb{Z}_p$–basis $b = (b_1, \dots, b_r)$ of Γ_{L_n}. Write $\mathscr{M} = \mathbf{D}^\dagger_{\mathrm{rig}}(V)$. Then $h^1_{L_n,V}$ is the composition of

$$\begin{array}{ccccc} \mathscr{M}^{\psi_q = \frac{q}{\varphi_q^{-1}(\pi_L)}} & \longrightarrow & \mathrm{H}^1\big(\mathcal{T}^\bullet_{\mathrm{an}}(\Psi \times \Gamma_{L_n}, \mathscr{M})\big) & \cong & \mathrm{H}^1\big(\mathcal{T}^\bullet_{\mathrm{an}}(\Phi \times \Gamma_{L_n}, \mathscr{M})\big) \\ y & \longmapsto & [(c_b(y), 0)] & \longmapsto & [(c_b(y), m_{b,y})] \end{array}$$

with the isomorphism in corollary 210, where

$$m_{b,y} = \Xi_b(\varphi_q - 1)(y)$$

is the unique element in $\mathscr{M}^{\psi_q = 0}$ such that

$$(\varphi_q - 1)c_b(y)(\gamma) = (\gamma - 1)m_{b,y} \quad \text{for all } \gamma \in \Gamma_{L_n}$$

(cf. proposition 205).

On the other hand, given $T \in \mathrm{Ob}\big(\mathrm{Rep}_{\mathscr{O}_K}(G_L)\big)$, $V = K \otimes_{\mathscr{O}_K} T$ and an integer $n \geq 0$, theorem 200 induces a "projection" morphism

$$\mathrm{pr}_{L_n,V} \colon \mathbf{D}(V(\tau^{-1}))^{\psi_q=1} \cong \mathrm{H}^1_{\mathrm{Iw}}(L_\infty/L, V) \longrightarrow \mathrm{H}^1(L_n, V),$$

where the last arrow is induced by the natural projection

$$\mathrm{H}^1_{\mathrm{Iw}}(L_\infty/L, T) = \varprojlim_{m\geq 1} \mathrm{H}^1(L_m, T) \twoheadrightarrow \mathrm{H}^1(L_n, V).$$

One might hope to express $\mathrm{pr}_{L_n,V}$ similarly in terms of, say, the cohomology of the complex $\mathcal{T}^\bullet(\Psi \times \Gamma_{L_n}, \mathbf{D}^\dagger_{\mathrm{rig}}(V(\tau^{-1})))$ if $V(\tau^{-1})$ is K–analytic. But this is not so easy because the isomorphism deduced from theorem 200 was defined by local Tate duality. At least we have the following result:

Proposition 211. *Let $V \in \mathrm{Ob}\big(\mathrm{Rep}^\dagger_K(G_L)\big)$. There is a natural isomorphism*

$$\mathrm{H}^1\big(\mathcal{T}^\bullet(\Phi \times \Gamma_{L'}, \mathbf{D}^\dagger_{\mathrm{rig}}(V))\big) \cong \mathrm{H}^1_\dagger(L', V).$$

Proof. See theorem 3.6 and lemma 3.7 of Schneider–Venjakob's preprint [35]. □

9.5.3 Duality in terms of Herr complexes

Let $W \in \mathrm{Ob}\big(\mathrm{Rep}^{\mathrm{an}}_K(G_L)\big)$ and set $\mathscr{M} = \mathbf{D}^\dagger_{\mathrm{rig}}(W)$. Let $\mathcal{T} = \mathcal{T}^\bullet = \mathcal{T}^\bullet(\Phi \times \Gamma_{L'}, \mathscr{M})$ and $\mathcal{T}^* = \mathrm{Hom}^{\mathrm{cont}}_K(\mathcal{T}, K)$ (this is the dual complex, as introduced in the remark of section 9.5.1). More generally, we use the notation $\cdot^*$ for the topological dual of any topological K–vector space. For every $i \in \mathbb{Z}$, there is a canonical morphism

$$\mathrm{H}^{-i}(\mathcal{T}^*) = \frac{\mathrm{Ker}(d^{-i}_{\mathcal{T}^*})}{\mathrm{Im}(d^{-i-1}_{\mathcal{T}^*})} = \frac{\mathrm{Ker}\big((d^{i-1}_{\mathcal{T}})^*\big)}{\mathrm{Im}\big((d^{i}_{\mathcal{T}})^*\big)} \longrightarrow \left(\frac{\mathrm{Ker}(d^{i}_{\mathcal{T}})}{\mathrm{Im}(d^{i-1}_{\mathcal{T}})}\right)^* = \mathrm{H}^i(\mathcal{T})^*$$

given by

$$\big(f \colon \mathcal{T}^i \to K\big) \longmapsto \left(\frac{\mathrm{Ker}(d^{i}_{\mathcal{T}})}{\mathrm{Im}(d^{i-1}_{\mathcal{T}})} \xrightarrow{f} K\right).$$

By lemma 3.10 and remark 3.11 of Schneider–Venjakob's preprint [35], this map is surjective in general and even an isomorphism if $\mathrm{H}^{i+1}(\mathcal{T})$ is finite-dimensional over K. In particular, we obtain a morphism

$$\mathrm{H}^1(\mathcal{T}^*[-2]) = \mathrm{H}^{-1}(\mathcal{T}^*) \twoheadrightarrow \mathrm{H}^1(\mathcal{T})^*$$

that is an isomorphism whenever $\mathrm{H}^2(\mathcal{T})$ has finite dimension over K.

By proposition 211, we know that $\mathrm{H}^1(\mathcal{T}) \cong \mathrm{H}^1_\dagger(L', W) \subset \mathrm{H}^1(L', W)$. In addition, theorem 171 gives a perfect pairing

$$\langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{Tate}}\colon\ \mathrm{H}^1(L', W) \times \mathrm{H}^1(L', W^*(1)) \to K$$

that we can use to identify the dual $\mathrm{H}^1_\dagger(L', W)^*$ with a quotient $\mathrm{H}^1_{/\dagger}(L', W^*(1))$ of $\mathrm{H}^1_\dagger(L', W^*(1))$. Therefore, we obtain a canonical morphism

$$\mathrm{H}^1(\mathcal{T}^*[-2]) \twoheadrightarrow \mathrm{H}^1(\mathcal{T})^* \cong \mathrm{H}^1_{/\dagger}(L', W^*(1))$$

making the diagram

$$\begin{array}{ccc}
\mathrm{H}^1(\mathcal{T}) \times \mathrm{H}^1(\mathcal{T}^*[-2]) & \longrightarrow & K \\
\wr\!\!\downarrow \qquad\qquad \downarrow\!\!\!\downarrow & & \| \\
\mathrm{H}^1_\dagger(L', W) \times \mathrm{H}^1_{/\dagger}(L', W^*(1)) & \xrightarrow{\langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{Tate}}} & K
\end{array}$$

commutative, where the upper pairing is induced by evaluation at the level of cochains.

To understand the morphism $\mathrm{H}^1(\mathcal{T}^*[-2]) \to \mathrm{H}^1_{/\dagger}(L', W^*(1))$, we will need to study the complex $\mathcal{T}^*[-2]$. But there is a duality of complexes that is very explicit: the "self-duality" of Koszul complexes.

9.5.4 Koszul complexes

Now assume that $L' = L_n$ for $n \gg 0$ and pick a $\mathbb{Z}_p$–basis $b = (b_1, \ldots, b_r)$ of Γ_{L_n}. Keep the rest of the notation as in section 9.5.3. Let $R = \mathbb{Z}_p[\![\Gamma_{L_n}]\!]$ and consider the free R–module $N = R^r$ with standard basis $e_1, \ldots, e_r$. The elements $[b_1] - 1, \ldots, [b_r] - 1 \in R$ give rise to a (homological) Koszul complex

$$K_\bullet(b)\colon\ \cdots \longrightarrow \bigwedge^{m+1} N \xrightarrow{d_{m+1}} \bigwedge^{m} N \xrightarrow{d_m} \bigwedge^{m-1} N \longrightarrow \cdots$$

with differentials given by

$$d_m(e_{i_1} \wedge \cdots \wedge e_{i_m}) = \sum_{k=1}^{m} (-1)^{k-1}([b_{i_k}] - 1) e_{i_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_m}$$

(where the hat over e_{i_k} means that this element is omitted). Write $K^\bullet(b)$ for the (cohomological) Koszul complex defined by $K^m(b) = K_{-m}(b)$ and $d^m = d_{-m}$ and consider the dual complex $K^{*,\bullet}(b) = \mathrm{Hom}_R^\bullet(K_\bullet(b), R)$ (i.e., the complex given by $K^{*,m} = \mathrm{Hom}_R(K_m(b), R)$ and $d^{*,m} = (-1)^{m+1} d^*_{m+1}$). There is a "self-duality" isomorphism of complexes

$$K^\bullet(b) \cong K^{*,\bullet}(b)[r]$$

that can be described explicitly as follows: the basis $e_1, \ldots, e_r$ of N induces an identification

$$\begin{aligned} R &\cong \bigwedge^r N \\ 1 &\mapsto e_1 \wedge e_2 \wedge \cdots \wedge e_r \end{aligned}$$

that we can use to define isomorphisms

$$\begin{array}{ccc} \bigwedge^m N & \longrightarrow & \mathrm{Hom}_R\Big(\bigwedge^{r-m} N, \bigwedge^r N\Big) \cong \mathrm{Hom}_R\Big(\bigwedge^{r-m} N, R\Big) \\ e_{i_1} \wedge \cdots \wedge e_{i_m} & \longmapsto & \Big(e_{j_1} \wedge \cdots \wedge e_{j_{d-m}} \mapsto e_{i_1} \wedge \cdots \wedge e_{i_m} \wedge e_{j_1} \wedge \cdots \wedge e_{j_{d-m}}\Big) \end{array}$$

for all $m \in \mathbb{Z}$.

Given an $\mathscr{O}_K$–analytic (φ_q, Γ_L)–module $\mathscr{M}$ over $\mathbf{B}^\dagger_{\mathrm{rig},L}$), we define

$$\begin{aligned} K_\bullet(b, \mathscr{M}) &= K_\bullet(b) \otimes_R \mathscr{M}, \\ K^\bullet(b, \mathscr{M}) &= K^\bullet(b) \otimes_R \mathscr{M}, \\ K^{*,\bullet}(b, \mathscr{M}) &= K^{*,\bullet}(b) \otimes_R \mathscr{M}. \end{aligned}$$

Section 4.2 of Colmez and Nizioł's article [18] explains that $K_\bullet(b)$ is a projective resolution of $\mathbb{Z}_p$ in the category of topological R–modules, like the completed standard complex $X_\bullet$ given by $X_m = R^{\otimes(m+1)}$ that can be used to compute continuous group cohomology via homogeneous cochains, and so there is a unique (up to homotopy) quasi-isomorphism $X_\bullet \to K_\bullet(b)$. In this way, we obtain a quasi-isomorphism

$$K^{*,\bullet}(b, \mathscr{M}) \cong \mathrm{Hom}_R(K_\bullet(b), \mathscr{M}) \to \mathrm{Hom}_R^{\mathrm{cont}}(X_\bullet, \mathscr{M}) \to C^\bullet(\Gamma_{L_n}, \mathscr{M}).$$

Therefore, we can replace the Herr complex

$$\mathcal{T}^\bullet(\Phi \times \Gamma_{L_n}, \mathscr{M}) = \mathrm{Cone}\Big(C^\bullet(\Gamma_{L_n}, \mathscr{M}) \xrightarrow{\varphi_q - 1} C^\bullet(\Gamma_{L_n}, \mathscr{M})\Big)[-1]$$

with the complex

$$\mathcal{K}^{*,\bullet}(\Phi \times \Gamma_{L_n}, \mathscr{M}) = \mathrm{Cone}\Big(K^{*,\bullet}(b, \mathscr{M}) \xrightarrow{\varphi_q - 1} K^{*,\bullet}(b, \mathscr{M})\Big)[-1]$$

in our study of the cohomology of (K–analytic) representations because we have an induced quasi-isomorphism

$$\mathcal{K}^{*,\bullet}(\Phi \times \Gamma_{L_n}, \mathscr{M}) \to \mathcal{T}^{\bullet}(\Phi \times \Gamma_{L_n}, \mathscr{M}).$$

Since we wanted to study the dual of the latter complex, we will need the following result:

Proposition 212. *In the situation above, the "self-duality" of Koszul complexes induces a natural isomorphism*

$$\big(K^{*,\bullet}(b, \mathscr{M})\big)^* \cong K^{*,\bullet}(b, \mathscr{M}^\vee)[r].$$

Proof. As mentioned in the proof of theorem 200, the paragraph before lemma 5.3 of Schneider–Venjakob's article [34] shows that we can use the trace pairing to identify $L \cong \mathrm{Hom}_K(L, K)$ and then we can identify L–duals and K–duals of L–vector spaces. Therefore,

$$\begin{aligned}
\big(K^{*,\bullet}(b, \mathscr{M})\big)^* &\cong \mathrm{Hom}_L^{\mathrm{cont}}\big(K^{*,\bullet}(b) \otimes_R \mathscr{M}, L\big) \\
&\cong \mathrm{Hom}_R\big(\mathrm{Hom}_R(K_\bullet(b), R), \mathrm{Hom}_L^{\mathrm{cont}}(\mathscr{M}, L)\big) \\
&\cong \mathrm{Hom}_R^\bullet\big(\mathrm{Hom}_R(K_\bullet(b), R), R\big) \otimes_R \mathrm{Hom}_L^{\mathrm{cont}}(\mathscr{M}, L).
\end{aligned}$$

Now, by paragraph 1.2.8 of Nekovář's book [29], there is a biduality isomorphism

$$K^\bullet(b) \cong \mathrm{Hom}_R^\bullet\big(\mathrm{Hom}_R(K_\bullet(b), R), R\big)$$

given by $x \mapsto (-1)^{\deg(x)} x^{**}$. In addition, by proposition 185, the residue pairing induces an isomorphism

$$\mathrm{Hom}_L^{\mathrm{cont}}(\mathscr{M}, L) \cong \mathrm{Hom}_{\mathbf{B}^\dagger_{\mathrm{rig},L}}\big(\mathscr{M}, \Omega^1_{\mathbf{B}^\dagger_{\mathrm{rig},L}}\big) = \mathscr{M}^\vee.$$

Combining these two isomorphisms with the previous ones, we deduce that

$$\big(K^{*,\bullet}(b, \mathscr{M})\big)^* \cong K^\bullet(b) \cong \mathscr{M}^\vee.$$

Finally, we can use the "self-duality" $K^\bullet(b) \cong K^{*,\bullet}(b)[r]$ to express

$$\big(K^{*,\bullet}(b,\mathscr{M})\big)^* \cong K^{*,\bullet}(b)[r] \otimes_R \mathscr{M}^\vee \cong K^{*,\bullet}(b,\mathscr{M}^\vee)[r]. \qquad \square$$

9.5.5 Duality in terms of Herr complexes, revisited

In this subsection, take $V \in \operatorname{Ob}\big(\operatorname{Rep}_K(G_L)\big)$ such that $W = V(\tau^{-1})$ is K–analytic. Continue with the notation from section 9.5.3. As mentioned above, we may replace the Herr complex $\mathcal{T}$ with

$$\mathcal{K}^* = \mathcal{K}^{*,\bullet} = \mathcal{K}^{*,\bullet}(\Phi \times \Gamma_{L_n}, \mathscr{M}),$$

which computes the same cohomology, and our goal is to describe the map

$$\mathrm{H}^1\big((\mathcal{K}^{*,\bullet})^*[-2]\big) \twoheadrightarrow \mathrm{H}^1_{/\dagger}(L_n, W^*(1))$$

induced by local Tate duality. But, by proposition 212, we have an isomorphism

$$\begin{aligned}(\mathcal{K}^{*,\bullet})^*[-2] &\cong \operatorname{Cone}\Big(\big(K^{*,\bullet}(b,\mathscr{M})\big)^* \xrightarrow{\varphi_q^{-1}\circ\varphi_q^*(\cdot)-1} \big(K^{*,\bullet}(b,\mathscr{M})\big)^*\Big)[-1] \\ &\cong \operatorname{Cone}\Big(K^{*,\bullet}(b,\mathscr{M}^\vee)[r] \xrightarrow{\psi_q-1} K^{*,\bullet}(b,\mathscr{M}^\vee)[r]\Big)[-1].\end{aligned}$$

TODO: Apparently, Schneider and Venjakob can prove that the complex appearing in the last line (or something very similar) computes exactly $\mathrm{H}^1_{/\dagger}(L_n, W^*(1))$ and they use Tate duality again to obtain a factorization of $\mathrm{pr}_{L_n,V}\colon \mathscr{M}^{\psi_q=1} \to \mathrm{H}^1(L_n, V)$ through a very simple map to the cohomology of a complex of this kind. This part is not written at all in the version that I have of their preprint [35], but they told me that the final version will have it.

Proposition 213. *Let $V \in \operatorname{Ob}\big(\operatorname{Rep}_K(G_L)\big)$ such that $V(\tau^{-1})$ is K–analytic. For every $i \in \mathbb{Z}$, the diagram*

$$\begin{array}{ccc} \mathbf{D}(V(\tau^{-1}))^{\psi_q=1} \times \mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1))^{\psi_q=\frac{q}{\varphi_q^{-1}(\pi_L)}} & \xrightarrow{\{\cdot,\cdot\}_{\mathrm{Iw}}} & \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L) \\ \Big\downarrow \mathrm{pr}_{L,V(\chi_\phi^{-i})}\circ\mathrm{Tw}_{\chi_\phi^{-i}} \quad\quad \Big\downarrow h^1_{L,V^*(1)(\chi_\phi^i)}\circ\mathrm{Tw}_{\chi_\phi^i} & & \Big\downarrow \tfrac{-1}{\Omega}\mathrm{ev}_{\chi_\phi^{-i}} \\ \mathrm{H}^1(L, V(\chi_\phi^{-i})) \times \mathrm{H}^1(L, V^*(1)(\chi_\phi^i)) & \xrightarrow{\langle\cdot,\cdot\rangle_{\mathrm{Tate}}} & L \subseteq \mathbb{C}_p \end{array}$$

is commutative, where $\mathrm{ev}_{\chi_\phi^{-i}}$ is the map given by evaluation of distributions at χ_ϕ^{-i}. (Here, the maps $\mathrm{Tw}_{\chi_\phi^i}$ and $\mathrm{Tw}_{\chi_\phi^{-i}}$ are defined as in proposition 197.)

Proof. TODO: the proof depends on the part that is not written in the preprint [35]. The final version shouldn't change much in any case. □

10 The regulator

This section contains the most important results of this part. The main result of Schneider and Venjakob's preprint [35] is the construction of a regulator map with an interpolation formula analogous to those of Lei–Loeffler–Zerbes for cyclotomic extensions. The regulator map will be essentially "dual" to the big logarithm map introduced in Berger–Fourquaux's article [7], and the interpolation formula for the regulator is a consequence of the interpolation formula for the big logarithm and that "duality". All the constructions introduced so far are the necessary pieces to prove such results, that we generalize here to the *relative* Lubin–Tate setting.

10.1 The definition of the regulator map

Recall that we defined $\tau = \chi_{\text{cyc}}\chi_\phi^{-1}$. The Lubin–Tate character χ_ϕ is $\mathscr{O}_K$–analytic with (non-trivial) Hodge–Tate weight 1. In contrast, the base change $\mathscr{O}_K(1)$ of the cyclotomic character χ_{cyc} has all its Hodge–Tate weights equal to 1, so it cannot be $\mathscr{O}_K$–analytic unless $K = \mathbb{Q}_p$.

Definition 214. Let $T \in \mathrm{Ob}\big(\mathrm{Rep}^{\text{crys}}_{\mathscr{O}_K,\text{fr}}(G_L)\big)$ and consider $V = K \otimes_{\mathscr{O}_K} T$. Suppose that $T(\tau^{-1}) \in \mathrm{Ob}\big(\mathrm{Rep}^{\text{crys,an},\geq 0}_{\mathscr{O}_K,\text{fr}}(G_L)\big)$ and that $\mathbf{D}_{\text{crys},K}(V(\tau^{-1}))^{\varphi_q=1} = 0$. We define the *p–adic regulator*

$$\mathbf{L}_T \colon \mathrm{H}^1_{\mathrm{Iw}}(L_\infty/L, T) \longrightarrow D(\Gamma_L, \mathbb{C}_p) \otimes_L \mathbf{D}_{\text{crys},K}(V(\tau^{-1}))$$

to be the composition of the following maps:

(1) the isomorphism $\mathrm{H}^1_{\mathrm{Iw}}(L_\infty/L, T) \cong \mathbf{D}(T(\tau^{-1}))^{\psi_q=1}$ from theorem 200;

(2) the equality $\mathbf{D}(T(\tau^{-1}))^{\psi_q=1} = \mathbf{N}(T(\tau^{-1}))^{\psi_q=1}$ from proposition 132;

(3) the map

$$1 - \frac{\pi_L}{q}\varphi_q \colon \mathbf{N}(T(\tau^{-1}))^{\psi_q=1} \longrightarrow \mathbf{N}^{(\varphi_q)}(V(\tau^{-1}))^{\psi_q=0}$$

(well-defined thanks to the relation

$$\psi_q \circ \varphi_q = \frac{q}{\varphi_q^{-1}(\pi_L)}$$

and because $\mathbf{N}(T(\tau^{-1})) \subseteq \mathbf{N}^{(\varphi_q)}(V(\tau^{-1}))$ by lemma 123);

(4) the inclusion $\mathbf{N}^{(\varphi_q)}(V(\tau^{-1}))^{\psi_q=0} \hookrightarrow (\mathbf{B}^+_{\text{rig},L})^{\psi_q=0} \otimes_L \mathbf{D}_{\text{crys},K}(V(\tau^{-1}))$ given by the last part of proposition 127;

(5) the inclusion

$$(\mathbf{B}^+_{\mathrm{rig},L})^{\psi_q=0} \otimes_L \mathbf{D}_{\mathrm{crys},K}(V(\tau^{-1})) \subseteq (\mathbf{B}^+_{\mathrm{rig},\mathbb{C}_p})^{\psi_q=0} \otimes_L \mathbf{D}_{\mathrm{crys},K}(V(\tau^{-1}))$$

induced by $\mathbf{B}^+_{\mathrm{rig},L} \subseteq \mathbf{B}^+_{\mathrm{rig},\mathbb{C}_p}$, and

(6) the isomorphism

$$(\mathbf{B}^+_{\mathrm{rig},\mathbb{C}_p})^{\psi_q=0} \otimes_L \mathbf{D}_{\mathrm{crys},K}(V(\tau^{-1})) \cong D(\Gamma_L, \mathbb{C}_p) \otimes_L \mathbf{D}_{\mathrm{crys},K}(V(\tau^{-1}))$$

induced by the Mellin transform $\mathfrak{M}\colon D(\Gamma_L, \mathbb{C}_p) \cong (\mathbf{B}^+_{\mathrm{rig},\mathbb{C}_p})^{\psi_q=0}$ (see definition 151).

10.2 The big exponential map

Lemma 215. *Let $V \in \mathrm{Ob}\big(\mathrm{Rep}^{\mathrm{crys,an}}_K(G_L)\big)$ and take $h \in \mathbb{Z}_{\geq 1}$ such that the Hodge–Tate weights of V are $\leq h$ or, equivalently, $\mathrm{Fil}^{-h}\,\mathbf{D}_{\mathrm{crys},K}(V) = \mathbf{D}_{\mathrm{crys},K}(V)$. There is an exact sequence*

$$0 \longrightarrow \bigoplus_{k=0}^{h} t^k_\phi \mathbf{D}_{\mathrm{crys},K}(V)^{\varphi_q=\pi_L^{-k}} \longrightarrow \big(\mathbf{B}^+_{\mathrm{rig},L} \otimes_L \mathbf{D}_{\mathrm{crys},K}(V)\big)^{\psi_q=\frac{q}{\varphi_q^{-1}(\pi_L)}} \xrightarrow{1-\varphi_q}$$

$$\to (\mathbf{B}^+_{\mathrm{rig},L})^{\psi_q=0} \otimes_L \mathbf{D}_{\mathrm{crys},K}(V) \xrightarrow{\Delta} \bigoplus_{k=0}^{h} \frac{\mathbf{D}_{\mathrm{crys},K}(V)}{1-\pi_L^k\varphi_q} \longrightarrow 0,$$

where the morphism Δ is given by

$$f(Z) \otimes \delta \longmapsto \Big(\partial^k_\phi(f)(0) \cdot \delta \mod (1-\pi_L^k\varphi_q)(\mathbf{D}_{\mathrm{crys},K}(V))\Big)_{0 \leq k \leq h}.$$

Proof. See lemma 3.5.1 of Berger–Fourquaux's article [7], whose proof works verbatim in the *relative* Lubin–Tate situation. □

Remark. For every $f \in \big((\mathbf{B}^+_{\mathrm{rig},L})^{\psi_q=0} \otimes_L \mathbf{D}_{\mathrm{crys},K}(V)\big)^{\Delta=0}$, lemma 215 shows that there is $y \in \big(\mathbf{B}^+_{\mathrm{rig},L} \otimes_L \mathbf{D}_{\mathrm{crys},K}(V)\big)^{\psi_q=q/\varphi_q^{-1}(\pi_L)}$ such that $f = (1-\varphi_q)(y)$. Observe that $(N_{\nabla,h-1} \circ \cdots \circ N_{\nabla,0})(y)$ is independent of the choice of such a preimage y if $\mathbf{D}_{\mathrm{crys},K}(V)^{\varphi_q=\pi_L^{-h}} = 0$ because

$$N_{\nabla,h-1} \circ \cdots \circ N_{\nabla,0} \text{ annihilates } \bigoplus_{k=0}^{h-1} t^k_\phi \mathbf{D}_{\mathrm{crys},K}(V)^{\varphi_q=\pi_L^{-k}}.$$

Moreover, $(N_{\nabla,h-1} \circ \cdots \circ N_{\nabla,0})(y) \in \log_\phi^h \mathbf{B}^+_{\mathrm{rig},L} \otimes_L \mathbf{D}_{\mathrm{crys},K}(V)$ (cf. the proof of lemma 204).

Definition 216. Let $V \in \mathrm{Ob}\big(\mathrm{Rep}_K^{\mathrm{crys,an}}(G_L)\big)$ and take $h \in \mathbb{Z}_{\geq 1}$ such that the Hodge–Tate weights of V are $\leq h$ and $\mathbf{D}_{\mathrm{crys},K}(V)^{\varphi_q = \pi_L^{-h}} = 0$. By the diagram at the end of section 7.5.3, we can identify $\mathbf{D}^\dagger_{\mathrm{rig}}(V)$ with $\mathbf{B}^\dagger_{\mathrm{rig},L} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathscr{M}(\mathbf{D}_{\mathrm{crys},K}(V))$ and then view $\log_\phi^h \mathbf{B}^\dagger_{\mathrm{rig},L} \otimes_L \mathbf{D}_{\mathrm{crys},K}(V) \subseteq \mathbf{D}^\dagger_{\mathrm{rig}}(V)$ (cf. the proof of lemma 125). The *big exponential map*

$$\mathbf{\Omega}_{V,h} \colon \big((\mathbf{B}^+_{\mathrm{rig},L})^{\psi_q=0} \otimes_L \mathbf{D}_{\mathrm{crys},K}(V)\big)^{\Delta=0} \to \mathbf{D}^\dagger_{\mathrm{rig}}(V)^{\psi_q = \frac{q}{\varphi_q^{-1}(\pi_L)}}$$

is defined by

$$\mathbf{\Omega}_{V,h}(f) = (N_{\nabla,h-1} \circ \cdots \circ N_{\nabla,1} \circ N_{\nabla,0})(y),$$

where y is an element of $\big(\mathbf{B}^+_{\mathrm{rig},L} \otimes_L \mathbf{D}_{\mathrm{crys},K}(V)\big)^{\psi_q = q/\varphi_q^{-1}(\pi_L)}$ with the property that $f = (1-\varphi_q)(y)$ (see lemma 215 and the remark after it).

10.3 Interpolation of exponentials and duals

Definition 217. Let D be a φ_q–module over L (e.g., an object in $(\mathrm{Fil}, \varphi_q)$–$\mathrm{Mod}_L$).

(1) For each $n \in \mathbb{Z}_{\geq 1}$, we define the morphism

$$\begin{aligned} \varphi_q^{-n} \colon \mathbf{B}^+_{\mathrm{rig},L}[\log_\phi^{-1}] \otimes_L D &\longrightarrow L_n((t_\phi)) \otimes_L D \\ \frac{f(Z)}{\log_\phi(Z)^h} \otimes \delta &\longmapsto \frac{1}{t_{\phi,n}^h} f^{\varphi_q^{-n}}\Big(\mathfrak{F}_\phi^{\varphi_q^{-n}}\Big(z_n, \exp_\phi^{\varphi_q^{-n}}(t_{\phi,n})\Big)\Big) \otimes \varphi_q^{-n}(\delta) \end{aligned}$$

where

$$t_{\phi,n} = \frac{t_\phi}{\varphi_q^{-n}(\pi_L) \cdots \varphi_q^{-1}(\pi_L)}.$$

(2) We define $\partial_D \colon L_n((t_\phi)) \otimes_L D \to L_n \otimes_L D$ to be the map that takes the constant coefficient of a formal series in t_ϕ.

Lemma 218. *Let D be a φ_q–module over L and let $m, n \in \mathbb{Z}_{\geq 0}$ with $m \geq n$. For every $y \in \big(\mathbf{B}^+_{\mathrm{rig},L}[\log_\phi^{-1}] \otimes_L D\big)^{\psi_q = q/\varphi_q^{-1}(\pi_L)}$,*

$$q^{-m}\,\mathrm{Tr}_{L_m/L_n}\big(\partial_D(\varphi_q^{-m}(y))\big) = \begin{cases} q^{-n}\partial_D(\varphi_q^{-n}(y)) & \text{if } n \geq 1, \\ (1 - q^{-1}\varphi_q^{-1})\partial_D(y) & \text{if } n = 0. \end{cases}$$

Proof. This result is analogous to lemma 2.4.3 of Berger–Fourquaux's article [7]. We adapt the proof to the *relative* Lubin–Tate situation for the convenience of the reader.

Write
$$y = \frac{1}{\log_\phi(Z)^h} \sum_{k=0}^{\infty} Z^k \otimes a_k \quad \text{with } a_k \in D \text{ for all } k \geq 0,$$
so that
$$\varphi_q^{-m}(y) = t_{\phi,m}^{-h} \sum_{k=0}^{\infty} \Big(\mathfrak{F}_\phi^{\varphi_q^{-m}} \big(z_m, \exp_\phi^{\varphi_q^{-m}}(t_{\phi,m})\big)\Big)^k \otimes \varphi_q^{-m}(a_k).$$
Write $v_m = \varphi_q^m(z_m)$. Observe that

$$\mathfrak{F}_\phi^{\varphi_q^{-m}} \big(z_m, \exp_\phi^{\varphi_q^{-m}}(t_{\phi,m})\big) = \varphi_q^{-m}\big(\mathfrak{F}_\phi(v_m, \exp_\phi(t_\phi))\big) \quad \text{in } L_m((t_\phi)).$$

If $m \geq 2$, the $\mathrm{Gal}(L_m/L_{m-1})$–conjugates of v_m are the $\mathfrak{F}_\phi(v_m, w_1)$ for $w_1 \in \mathfrak{F}_{\phi,1}$. Therefore,

$$\begin{aligned}
&\mathrm{Tr}_{L_m/L_{m-1}} \big(\partial_D(\varphi_q^{-m}(y))\big) \\
&\quad = \partial_D \bigg(\sum_{w_1 \in \mathfrak{F}_{\phi,1}} t_{\phi,m}^{-h} \sum_{k=0}^{\infty} \varphi_q^{-m} \Big(\mathfrak{F}_\phi\big(\mathfrak{F}_\phi(v_m, w_1), \exp_\phi(t_\phi)\big)\Big)^k \otimes \varphi_q^{-m}(a_k) \bigg) \\
&\quad = \partial_D \Big(\varphi_q^{-m}\big(\pi_L(\varphi_q \circ \psi_q)(y)\big)\Big) = \partial_D \bigg(\varphi_q^{-m} \Big(\pi_L \varphi_q \Big(\frac{q}{\varphi_q^{-1}(\pi_L)} y \Big)\Big)\bigg) \\
&\quad = \partial_D(q\varphi_q^{-m+1}(y)).
\end{aligned}$$

Similarly, if $m = 1$, the $\mathrm{Gal}(L_1/L)$–conjugates of v_1 are the $\mathfrak{F}_\phi(v_1, w_1)$ for $w_1 \in \mathfrak{F}_{\phi,1} \setminus \{0\}$. An analogous calculation summing over all $w_1 \in \mathfrak{F}_{\phi,1}$ and subtracting the summand for $w_1 = 0$ shows that

$$\mathrm{Tr}_{L_1/L}\big(\partial_D(\varphi_q^{-1}(y))\big) = \partial_D(qy - \varphi_q^{-1}(y)). \qquad \square$$

Lemma 219. *Let $V \in \mathrm{Ob}\big(\mathrm{Rep}_K^{\mathrm{crys,an}}(G_L)\big)$. By the diagram at the end of section 7.5.3, we can identify $\mathbf{D}_{\mathrm{rig}}^\dagger(V)$ with $\mathbf{B}_{\mathrm{rig},L}^\dagger \otimes_{\mathbf{B}_{\mathrm{rig},L}^+} \mathscr{M}(\mathbf{D}_{\mathrm{crys},K}(V))$ and we obtain an inclusion*

$$\mathbf{D}_{\mathrm{rig}}^\dagger(V)^{\psi_q = q/\varphi_q^{-1}(\pi_L)} \subseteq \mathbf{B}_{\mathrm{rig},L}^+[\log_\phi^{-1}] \otimes_L \mathbf{D}_{\mathrm{crys},K}(V).$$

Proof. See theorem 3.1.1 of Berger–Fourquaux's article [7], whose proof works almost verbatim in the *relative* Lubin–Tate situation. $\square$

Let $V \in \mathrm{Ob}\big(\mathrm{Rep}_K^{\mathrm{an}}(G_L)\big)$ and let L' be a finite extension of L contained in L_∞.

We write $\exp_{L',V}\colon (\mathbf{B}_{\mathrm{dR}}/\mathbf{B}_{\mathrm{dR}}^{+}\otimes_K V)^{G_{L'}} \to \mathrm{H}^1(L',V)$ for Bloch–Kato's exponential map for V regarded as a representation of $G_{L'}$. By proposition A.1 in the appendix of Schneider–Venjakob's article [34], this map can also be constructed more directly from the fundamental exact sequence

$$0 \longrightarrow K \longrightarrow \mathbf{B}_{\mathrm{crys},K}^{\varphi_q=1} \longrightarrow \mathbf{B}_{\mathrm{dR}}/\mathbf{B}_{\mathrm{dR}}^{+} \longrightarrow 0.$$

Let $\exp^*_{L',V^*(1)}\colon \mathrm{H}^1(L',V) \to (\mathbf{B}_{\mathrm{dR}}\otimes_K V)^{G_{L'}}$ denote Bloch–Kato's dual exponential map for V regarded as a representation of $G_{L'}$ (i.e., the dual of $\exp_{L',V^*(1)}$ using local Tate duality on cohomology and the natural duality on $\mathbf{D}_{\mathrm{dR},K}$).

Theorem 220 (Berger–Fourquaux). *Let $V \in \mathrm{Ob}\big(\mathrm{Rep}_K^{\mathrm{crys,an}}(G_L)\big)$ and let $n \in \mathbb{Z}_{\geq 0}$. Write ∂_V for the map $\partial_{\mathbf{D}_{\mathrm{crys},K}(V)}$ from definition 217. For every $y \in \mathbf{D}_{\mathrm{rig}}^{\dagger}(V)^{\psi_q=q/\varphi_q^{-1}(\pi_L)}$,*

$$\exp^*_{L_n,V^*(1)}\big(h^1_{L_n,V}(y)\big) = \begin{cases} q^{-n}\partial_V(\varphi_q^{-n}(y)) & \text{if } n \geq 1,\\ (1-q^{-1}\varphi_q^{-1})\partial_V(y) & \text{if } n = 0.\end{cases}$$

Proof. See theorem 3.3.1 of Berger–Fourquaux's article [7], whose proof works exactly in the same way for the *relative* Lubin–Tate situation (now using proposition 205 and lemma 218). □

Theorem 221 (Berger–Fourquaux). *Let $V \in \mathrm{Ob}\big(\mathrm{Rep}_K^{\mathrm{crys,an}}(G_L)\big)$ and let $n \in \mathbb{Z}_{\geq 0}$. Take $h \in \mathbb{Z}_{\geq 0}$ such that $\mathrm{Fil}^{-h}\,\mathbf{D}_{\mathrm{crys},K}(V) = \mathbf{D}_{\mathrm{crys},K}(V)$ (i.e., the Hodge–Tate weights of V are $\leq h$) and write ∂_V for the map $\partial_{\mathbf{D}_{\mathrm{crys},K}(V)}$ from definition 217. Identifying $\mathbf{D}_{\mathrm{rig}}^{\dagger}(V)$ with $\mathbf{B}_{\mathrm{rig},L}^{\dagger}\otimes_{\mathbf{B}_{\mathrm{rig},L}^{+}}\mathscr{M}(\mathbf{D}_{\mathrm{crys},K}(V))$ we view $\log_\phi^h \mathbf{B}_{\mathrm{rig},L}^{+}\otimes_L \mathbf{D}_{\mathrm{crys},K}(V) \subset \mathbf{D}_{\mathrm{rig}}^{\dagger}(V)$ (cf. the proof of lemma 125). Then, for every $y \in (\mathbf{B}_{\mathrm{rig},L}^{+}\otimes_L \mathbf{D}_{\mathrm{crys},K}(V))^{\psi_q=q/\varphi_q^{-1}(\pi_L)}$,*

$$y_h = (N_{\nabla,h-1}\circ\cdots\circ N_{\nabla,1}\circ N_{\nabla,0})(y) \in \mathbf{D}_{\mathrm{rig}}^{\dagger}(V)^{\psi_q=q/\varphi_q^{-1}(\pi_L)}$$

and

$$h^1_{L_n,V}(y_h) = \begin{cases} (-1)^{h-1}(h-1)!\exp_{L_n,V}\big(q^{-n}\partial_V(\varphi_q^{-n}(y))\big) & \text{if } n \geq 1,\\ (-1)^{h-1}(h-1)!\exp_{L,V}\big((1-q^{-1}\varphi_q^{-1})\partial_V(y)\big) & \text{if } n = 0.\end{cases}$$

Proof. See theorem 3.3.2 of Berger–Fourquaux's article [7], whose proof works exactly in the same way for the *relative* Lubin–Tate situation (now using proposition 205 and lemma 218). □

10.4 Interpolation formula for the big exponential

Let $V \in \mathrm{Ob}\big(\mathrm{Rep}_K^{\mathrm{crys,an}}(G_L)\big)$ and take $h \in \mathbb{Z}_{\geq 1}$ such that the Hodge–Tate weights of V are $\leq h$. For every $i \in \mathbb{Z}$, the twist $V(\chi_\phi^i)$ has Hodge–Tate weights $\leq h+i$ and we can identify

$$\mathbf{D}_{\mathrm{crys},K}(V(\chi_\phi^i)) \cong \mathbf{D}_{\mathrm{crys},K}(V) \otimes_L \mathbf{D}_{\mathrm{crys},K}(K(\chi_\phi^i)) = \mathbf{D}_{\mathrm{crys},K}(V) \otimes (t_\phi^{-i} \otimes t_0^i)$$

and

$$\mathbf{D}_{\mathrm{rig}}^\dagger(V(\chi_\phi^i)) \cong \mathbf{D}_{\mathrm{rig}}^\dagger(V) \otimes_{\mathbf{B}_{\mathrm{rig},L}^\dagger} \mathbf{D}_{\mathrm{rig}}^\dagger(K(\chi_\phi^i)) = t_\phi^{-i}\mathbf{D}_{\mathrm{rig}}^\dagger(V) \otimes t_0^i,$$

where t_0 is a generator of $K(\chi_\phi)$.

Theorem 222 (Berger–Fourquaux). *With the notations and assumptions introduced above, let $y \in \big(\mathbf{B}_{\mathrm{rig},L}^+ \otimes_L \mathbf{D}_{\mathrm{crys},K}(V)\big)^{\psi_q = q/\varphi_q^{-1}(\pi_L)}$ and $f = (1-\varphi_q)(y)$. Let $n \in \mathbb{Z}_{\geq 0}$.*

(1) *If $h+i \geq 1$, then*

$$h^1_{L_n,V(\chi_\phi^i)}\big(\mathbf{\Omega}_{V,h}(f) \otimes t_0^i\big) = (-1)^{h+i-1}(h+i-1)! \cdot \\ \cdot \begin{cases} \exp_{L_n,V(\chi_\phi^i)}\Big(q^{-n}\partial_{V(\chi_\phi^i)}\big(\varphi_q^{-n}(\partial_\phi^{-i} y \otimes (t_\phi^{-i} \otimes t_0^i))\big)\Big) & \text{if } n \geq 1, \\ \exp_{L,V(\chi_\phi^i)}\Big((1-q^{-1}\varphi_q^{-1})\partial_{V(\chi_\phi^i)}(\partial_\phi^{-i} y \otimes (t_\phi^{-i} \otimes t_0^i))\Big) & \text{if } n = 0. \end{cases}$$

(2) *If $h+i \leq 0$, then*

$$\exp^*_{L_n,V(\chi_\phi^i)^*(1)}\Big(h^1_{L_n,V(\chi_\phi^i)}\big(\mathbf{\Omega}_{V,h}(f) \otimes t_0^i\big)\Big) = \frac{1}{(-h-i)!} \cdot \\ \cdot \begin{cases} q^{-n}\partial_{V(\chi_\phi^i)}\big(\varphi_q^{-n}(\partial_\phi^{-i} y \otimes (t_\phi^{-i} \otimes t_0^i))\big) & \text{if } n \geq 1, \\ (1-q^{-1}\varphi_q^{-1})\partial_{V(\chi_\phi^i)}(\partial_\phi^{-i} y \otimes (t_\phi^{-i} \otimes t_0^i)) & \text{if } n = 0. \end{cases}$$

Proof. This result is an application of theorems 220 and 221. For the details, see theorem 3.5.3 of Berger–Fourquaux's article [7], whose proof works verbatim in the *relative* Lubin–Tate situation. □

10.5 The abstract reciprocity formula

Consider $T \in \mathrm{Ob}\big(\mathrm{Rep}_{\mathscr{O}_K,\mathrm{fr}}^{\mathrm{crys}}(G_L)\big)$ such that $T(\tau^{-1}) \in \mathrm{Ob}\big(\mathrm{Rep}_{\mathscr{O}_K,\mathrm{fr}}^{\mathrm{crys,an}}(G_L)\big)$. Write $V = K \otimes_{\mathscr{O}_K} T$. By the large diagram of functors in section 7.5.3, we obtain

$$\mathscr{M} = \mathbf{D}_{\mathrm{rig},\mathbb{C}_p}^\dagger(V(\tau^{-1})) = \mathbf{B}_{\mathrm{rig},\mathbb{C}_p}^\dagger \otimes_{\mathbf{A}_L^+} \mathbf{N}(T(\tau^{-1})) \in \mathrm{Ob}\big((\varphi_q, \Gamma_L)\text{–}\mathrm{Mod}^{0,\mathrm{an}}_{\mathbf{B}_{\mathrm{rig},\mathbb{C}_p}^\dagger,\mathrm{fr}}\big),$$

which is a base change of $\mathscr{M}\big(\mathbf{D}_{\mathrm{crys},K}(V(\tau^{-1}))\big)$. The goal of this subsection is to interpret the pairings of section 9.3.7 in terms of $\mathbf{D}_{\mathrm{crys},K}$, following section 2.3.5 of Schneider–Venjakob's preprint [35].

Write $D = \mathbf{D}_{\mathrm{crys},K}(V(\tau^{-1}))$ and identify it with $\boldsymbol{D}\big(\mathbf{B}^+_{\mathrm{rig},L} \otimes_{\mathbf{A}^+_L} \mathbf{N}(T(\tau^{-1}))\big)$ as in section 7.5.5. The map $\xi\colon D \to \big(\mathbf{B}^+_{\mathrm{rig},L} \otimes_{\mathbf{A}^+_L} \mathbf{N}(T(\tau^{-1}))\big)[\lambda^{-1}]$ from lemma 101 induces an isomorphism

$$1 \otimes \xi\colon \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}[\log_\phi^{-1}] \otimes_L D \to \mathscr{M}[\log_\phi^{-1}]$$

(observe that, over $\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}$, inverting $\log_\phi$ is the same as inverting λ). Let

$$\mathrm{comp}_{\mathscr{M}}\colon \mathscr{M}[\log_\phi^{-1}] \to \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}[\log_\phi^{-1}] \otimes_L D$$

denote the inverse of $1 \otimes \xi$.

Observe that the actions of Γ_L, φ_q and ψ_q on $\mathscr{M}$ extend to $\mathscr{M}[\log_\phi^{-1}]$: for every $m \in \mathscr{M}$ and $k \in \mathbb{Z}_{\geq 0}$,

$$\begin{aligned}
\gamma\Big(\frac{m}{\log_\phi^k}\Big) &= \frac{\chi_\phi^{-k}(\gamma)\gamma(m)}{\log_\phi^k} \quad \text{for } \gamma \in \Gamma_L,\\
\varphi_q\Big(\frac{m}{\log_\phi^k}\Big) &= \frac{\pi_L^{-k}\varphi_q(m)}{\log_\phi^k}, \text{ and}\\
\psi_q\Big(\frac{m}{\log_\phi^k}\Big) &= \frac{\varphi_q^{-1}(\pi_L^k)\psi_q(m)}{\log_\phi^k}.
\end{aligned}$$

Also, $\big(\mathscr{M}[\log_\phi^{-1}]\big)^{\psi_q=0} = \mathscr{M}^{\psi_q=0}[\log_\phi]$ and the continuous action of $\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L)$ on $\mathscr{M}^{\psi_q=0}$ extends to a continuous action on $\big(\mathscr{M}[\log_\phi^{-1}]\big)^{\psi_q=0}$ (see lemma 2.63 of Schneider–Venjakob's preprint [35]).

On the other hand, as $\Omega^1_{\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}} \cong \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\chi_\phi)$, we can identify

$$\begin{aligned}
\mathscr{M}^\vee = \mathrm{Hom}_{\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}}\big(\mathscr{M}, \Omega^1_{\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}}\big) &\cong \mathbf{D}^\dagger_{\mathrm{rig},\mathbb{C}_p}(V(\tau^{-1}))^*(\chi_\phi)\\
&\cong \mathbf{D}^\dagger_{\mathrm{rig},\mathbb{C}_p}(V^*(\tau))(\chi_\phi) \cong \mathbf{D}^\dagger_{\mathrm{rig},\mathbb{C}_p}(V^*(1)) = \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p} \otimes_{\mathbf{A}^+_L} \mathbf{N}(T^*(1)).
\end{aligned}$$

This suggests that we have to work with

$$\mathbf{D}_{\mathrm{crys},K}(V^*(1)) = \mathbf{D}_{\mathrm{crys},K}(V^*(\chi_\phi\tau)) \cong \mathbf{D}_{\mathrm{crys},K}(K(\chi_\phi)) \otimes_L \mathbf{D}_{\mathrm{crys},K}(V(\tau^{-1})^*)$$

and we set $D^* = \mathbf{D}_{\mathrm{crys},K}(V(\tau^{-1})^*)$ and $D_0 = \mathbf{D}_{\mathrm{crys},K}(K(\chi_\phi))$ for convenience.

Then we can express

$$\mathrm{comp}_{\mathscr{M}^\vee} : \mathscr{M}^\vee[\log_\phi^{-1}] \to \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}[\log_\phi^{-1}] \otimes_L D_0 \otimes_L D^*$$

as the composition

$$\begin{array}{c}
\mathscr{M}^\vee[\log_\phi^{-1}] \cong \mathrm{Hom}_{\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}[\log_\phi^{-1}]}\big(\mathscr{M}[\log_\phi^{-1}], \Omega^1_{\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}}[\log_\phi^{-1}]\big) \\
\Big\downarrow \cdot \circ \mathrm{comp}_{\mathscr{M}}^{-1} \\
\mathrm{Hom}_{\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}[\log_\phi^{-1}]}\big(\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}[\log_\phi^{-1}] \otimes_L D, \Omega^1_{\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}}[\log_\phi^{-1}]\big) \\
\wr\Vert \\
\Omega^1_{\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}}[\log_\phi^{-1}] \otimes_L D^* \\
\Big\downarrow \mathrm{comp}_{\Omega^1_{\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}}} \otimes \mathrm{id}_{D^*} \\
\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}[\log_\phi^{-1}] \otimes_L D_0 \otimes_L D^*,
\end{array}$$

where

$$\begin{aligned}
\mathrm{comp}_{\Omega^1_{\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}}} : \Omega^1_{\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}}[\log_\phi^{-1}] &\longrightarrow \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}[\log_\phi^{-1}] \otimes_L D_0 \\
\frac{\mathrm{dlog}_\phi}{\log_\phi} &\longmapsto 1 \otimes (t_\phi^{-1} \otimes t_0)
\end{aligned}$$

is obtained from the identification $\Omega^1_{\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}} \cong \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\chi_\phi)$ (see lemma 173).

Let $[\,\cdot\,,\,\cdot\,]_{\mathrm{crys}} = [\,\cdot\,,\,\cdot\,]_{D,\mathrm{crys}} : D \times D^* \to \mathbf{D}_{\mathrm{crys},K}(K) = L$ denote the natural (evaluation) pairing or, by abuse of notation, any base change of it.

Lemma 223. *The diagram*

$$\begin{array}{ccc}
\mathscr{M}^{\psi_q=0} \times (\mathscr{M}^\vee)^{\psi_q=0} & \xrightarrow{\{\,\cdot\,,\,\cdot\,\}'_{\mathrm{Iw}}} & \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L) \\
\swarrow \qquad \searrow & & \| \\
\mathscr{M}^{\psi_q=0}[\log_\phi^{-1}] \times (\mathscr{M}^\vee)^{\psi_q=0}[\log_\phi^{-1}] & & \| \\
\mathrm{comp}_{\mathscr{M}} \downarrow \cong \qquad \cong \downarrow \cdot \circ \mathrm{comp}_{\mathscr{M}}^{-1} & & \| \\
\Big((\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p})^{\psi_q=0}[\log_\phi^{-1}] \otimes_L D\Big) \times \Big((\Omega^1_{\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}})^{\psi_q=0}[\log_\phi^{-1}] \otimes_L D^*\Big) & & \| \\
\uparrow \qquad \uparrow & & \| \\
\Big((\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p})^{\psi_q=0} \otimes_L D\Big) \times \Big((\Omega^1_{\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}})^{\psi_q=0} \otimes_L D^*\Big) & \xrightarrow{\{\,\cdot\,,\,\cdot\,\}'_{\mathrm{Iw}}} & \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L) \\
\gamma_{-1}\circ\mathfrak{M}\circ\iota\otimes\mathrm{id}_D \uparrow \cong \qquad \cong \uparrow \mathfrak{M}_{\chi_\phi}\otimes\mathrm{id}_{D^*} & & \| \\
\big(\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L) \otimes_L D\big) \times \big(\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L) \otimes_L D^*\big) & \xrightarrow{[\,\cdot\,,\,\cdot\,]_{\mathrm{crys}}} & \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L)
\end{array}$$

is commutative on the vertical intersections.

Proof. For the upper half of the diagram, see lemma 2.65 of Schneider–Venjakob's preprint [35] (whose proof works verbatim in the *relative* Lubin–Tate situation). The lower half of the diagram is proposition 198. □

Recall that

$$\mathrm{comp}_{\mathscr{M}^\vee} = \big(\mathrm{comp}_{\Omega^1_{\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}}} \otimes \mathrm{id}_{D^*}\big) \circ \big(\,\cdot\, \circ \mathrm{comp}_{\mathscr{M}}^{-1}\big)$$

and lemma 223 uses $\cdot \circ \mathrm{comp}_{\mathscr{M}}^{-1}$, which is why the term D_0 does not appear. But $D_0 = L(t_\phi^{-1} \otimes t_0)$, so we can use the fixed basis element $t_\phi^{-1} \otimes t_0$ to account for D_0 if we use $\mathrm{comp}_{\mathscr{M}^\vee}$.

Lemma 224. *The diagram*

$$\begin{array}{ccc}
\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L) \otimes_L D^* & \xrightarrow[\cong]{\lambda\otimes d^* \mapsto \lambda\otimes(t_\phi^{-1}\otimes t_0)\otimes d^*} & \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L) \otimes_L D_0 \otimes_L D^* \\
\mathfrak{M}_{\chi_\phi}\otimes\mathrm{id}_{D^*} \downarrow & & \downarrow \frac{N_\nabla}{\Omega}\mathfrak{M}\otimes\mathrm{id}_{D_0\otimes D^*} \\
(\Omega^1_{\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}})^{\psi_q=0}[\log_\phi^{-1}] \otimes_L D^* & \xrightarrow[\cong]{\mathrm{comp}_{\Omega^1_{\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}}} \otimes \mathrm{id}_{D^*}} & (\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}))^{\psi_q=0}[\log_\phi^{-1}] \otimes_L D_0 \otimes_L D^*
\end{array}$$

is commutative.

Proof. Knowing that $\mathfrak{M}_{\chi_\phi}(\lambda) = \mathfrak{M}(\mathrm{Tw}_{\chi_\phi}(\lambda))\,\mathrm{dlog}_\phi$ and that the bottom arrow maps dlog_ϕ to $\log_\phi \otimes (t_\phi^{-1} \otimes t_0)$, a straight-forward computation shows that the dashed arrow is defined by

$$\lambda \otimes (t_\phi^{-1} \otimes t_0) \otimes d^* \mapsto \log_\phi(Z)\mathfrak{M}(\mathrm{Tw}_{\chi_\phi}(\lambda)) \otimes (t_\phi^{-1} \otimes t_0) \otimes d^*.$$

But, by definition 161,

$$\mathfrak{M}(\mathrm{Tw}_{\chi_\phi}(\lambda)) = \frac{\partial_\phi}{\Omega}\mathfrak{M}(\lambda).$$

The lemma follows from this and from the fact that N_∇ acts on $\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}$ as $\log_\phi(Z)\partial_\phi$ (cf. lemma 90). $\square$

Theorem 225 (Schneider–Venjakob). *With the notation introduced in section 10.5, the diagram*

$$\begin{array}{ccc}
\mathscr{M}^{\psi_q=0} \times (\mathscr{M}^\vee)^{\psi_q=0} & \xrightarrow{\{\cdot,\cdot\}'_{\mathrm{Iw}}} & \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L) \\
\swarrow \mathrm{comp}_{\mathscr{M}} \qquad \mathrm{comp}_{\mathscr{M}^\vee} \searrow & & \Big\downarrow \wr\, \otimes(t_\phi^{-1}\otimes t_0) \\
\Big((\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p})^{\psi_q=0}[\log_\phi^{-1}] \otimes_L D\Big) \times \Big((\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p})^{\psi_q=0}[\log_\phi^{-1}] \otimes_L D_0 \otimes_L D^*\Big) & & \\
\uparrow \gamma_{-1}\circ\mathfrak{M}\circ\iota\otimes\mathrm{id}_D \qquad \uparrow \frac{N_\nabla}{\Omega}\mathfrak{M}\otimes\mathrm{id}_{D_0\otimes D^*} & & \\
\big(\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L) \otimes_L D\big) \times \big(\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L) \otimes_L D_0 \otimes_L D^*\big) & \xrightarrow{[\cdot,\cdot]_{\mathrm{crys}}} & \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L) \otimes_L D_0
\end{array}$$

is commutative on the vertical intersections.

Proof. This theorem is a combination of lemmata 223 and 224. $\square$

10.6 Relation between the regulator and the big exponential

The regulator map introduced in definition 214 and the big exponential map from definition 216 are essentially adjoint via the abstract reciprocity map. More precisely:

Theorem 226 (Schneider–Venjakob). *Let* $T \in \mathrm{Ob}\big(\mathrm{Rep}^{\mathrm{crys}}_{\mathscr{O}_K,\mathrm{fr}}(G_L)\big)$ *and consider the crystalline representation* $V = K \otimes_{\mathscr{O}_K} T$. *Suppose that*

(i) $V(\tau^{-1})$ *is K–analytic with Hodge–Tate weights* ≥ 0 *and*

(ii) $\mathbf{D}_{\mathrm{crys},K}(V(\tau^{-1}))^{\varphi_q=1} = 0$

and that

(i') $V^*(1)$ *is K–analytic with Hodge–Tate weights* ≤ 1,

(ii') $\mathbf{D}_{\mathrm{crys},K}(V^*(1))^{\varphi_q=\pi_L^{-1}} = 0$ *and*

(iii′) $\mathbf{D}_{\mathrm{crys},K}(V^*(1))^{\varphi_q=0} = 0$.

(Observe that conditions (i) and (ii) are equivalent to conditions (i′) and (ii′), respectively.)

We use the same notation as in section 10.5, namely:

- $\mathscr{M} = \mathbf{D}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}(V(\tau^{-1}))$,
- $\mathscr{M}^{\vee} = \mathbf{D}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}(V^*(1))$,
- $D = \mathbf{D}_{\mathrm{crys},K}(V(\tau^{-1}))$,
- $D^* = \mathbf{D}_{\mathrm{crys},K}(V^*(1))$ *and*
- $D_0 = \mathbf{D}_{\mathrm{crys},K}(K(\chi_\phi))$.

Then, the diagram

$$
\begin{array}{ccccc}
\mathscr{M}^{\psi_q=1} & \times & (\mathscr{M}^{\vee})^{\psi_q=\frac{q}{\varphi_q^{-1}(\pi_L)}} & \xrightarrow{\{\cdot,\cdot\}_{\mathrm{Iw}}} & \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L) \\
\cup & & \| & & \cup \\
\mathbf{D}(T(\tau^{-1}))^{\psi_q=1} & & \mathbf{D}^{\dagger}_{\mathrm{rig},\mathbb{C}_p}(V^*(1))^{\psi_q=\frac{q}{\varphi_q^{-1}(\pi_L)}} & & D(\Gamma_L,\mathbb{C}_p) \\
\wr\| & & & & \\
\mathrm{H}^1_{\mathrm{Iw}}(L_\infty/L,T) & & \uparrow \boldsymbol{\Omega}_{V^*(1),1} & & \wr\| \downarrow \cdot\otimes(t_\phi^{-1}\otimes t_0) \\
\downarrow \Omega\mathbf{L}_T & & & & \\
\big(D(\Gamma_L,\mathbb{C}_p)\otimes_L D\big) & & \big((\mathbf{B}^{+}_{\mathrm{rig},\mathbb{C}_p})^{\psi_q=0}\otimes_L D_0\otimes_L D^*\big) & & \\
\wr\| \downarrow \gamma_{-1}\iota & & \wr\| \uparrow \mathfrak{M}\otimes\mathrm{id}_{D_0\otimes D^*} & & \\
\big(D(\Gamma_L,\mathbb{C}_p)\otimes_L D\big) & \times & \big(D(\Gamma_L,\mathbb{C}_p)\otimes_L D_0\otimes_L D^*\big) & \xrightarrow{[\cdot,\cdot]_{\mathrm{crys}}} & D(\Gamma_L,\mathbb{C}_p)\otimes_L D_0
\end{array}
$$

is commutative.

Proof. This is analogous to corollary 3.3 of Schneider–Venjakob's preprint [35] (but note that the normalizations of the lower pairing and some vertical maps are not the same there!).

On the one hand, by definition 214, the diagram

$$
\begin{array}{ccc}
\mathbf{D}(T(\tau^{-1}))^{\psi_q=1} & \hookrightarrow & \mathscr{M}^{\psi_q=1} \\
\wr\| & & \downarrow 1-\frac{\pi_L}{q}\varphi_q \\
\mathrm{H}^1_{\mathrm{Iw}}(L_\infty/L,T) & & \mathscr{M}^{\psi_q=0} \\
\downarrow \mathbf{L}_T & & \downarrow \mathrm{comp}_{\mathscr{M}} \\
D(\Gamma_L,\mathbb{C}_p)\otimes_L D & & \\
\wr\| & & \\
(\mathbf{B}^{+}_{\mathrm{rig},\mathbb{C}_p})^{\psi_q=0}\otimes_L D & \subset & (\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{C}_p})^{\psi_q=0}[\log_\phi^{-1}]\otimes_L D
\end{array}
$$

is commutative. On the other hand, by definition 216, the diagram

$$\begin{array}{ccc}
(\mathscr{M}^\vee)^{\psi_q=\frac{q}{\varphi_q^{-1}(\pi_L)}} & & \\
\Big\downarrow{\scriptstyle 1-\varphi_q} & \nwarrow^{\mathbf{\Omega}_{V^*(1),1}} & \\
(\mathscr{M}^\vee)^{\psi_q=0} & & \\
\Big\downarrow{\scriptstyle \mathrm{comp}_{\mathscr{M}^\vee}} & & \\
(\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p})^{\psi_q=0}\otimes_L D_0\otimes_L D^* & \xleftarrow[N_\nabla]{} & (\mathbf{B}^+_{\mathrm{rig},\mathbb{C}_p})^{\psi_q=0}\otimes_L D_0\otimes_L D^* \\
 & & \Big\uparrow{\scriptstyle \mathfrak{M}\otimes\mathrm{id}_{D_0\otimes D^*}} \\
 & & D(\Gamma_L,\mathbb{C}_p)\otimes_L D_0\otimes_L D^*
\end{array}$$

is also commutative. But the abstract reciprocity formula (see theorem 225) and the definition of the Iwasawa pairing (see definition 196) yield the commutative diagram

$$\begin{array}{ccc}
\mathscr{M}^{\psi_q=1}\times(\mathscr{M}^\vee)^{\psi_q=\frac{q}{\varphi_q^{-1}(\pi_L)}} & \xrightarrow{\{\cdot,\cdot\}_{\mathrm{Iw}}} & \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L) \\
{\scriptstyle 1-\frac{\pi_L}{q}\varphi_q}\Big\downarrow \qquad \Big\downarrow{\scriptstyle 1-\varphi_q} & & \Big\| \\
\mathscr{M}^{\psi_q=0}\times(\mathscr{M}^\vee)^{\psi_q=0} & \xrightarrow{\{\cdot,\cdot\}'_{\mathrm{Iw}}} & \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L) \\
{\scriptstyle \mathrm{comp}_{\mathscr{M}}}\swarrow \qquad \searrow{\scriptstyle \mathrm{comp}_{\mathscr{M}^\vee}} & & \wr\Big\downarrow{\scriptstyle \otimes(t_\phi^{-1}\otimes t_0)} \\
\Big((\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p})^{\psi_q=0}[\log_\phi^{-1}]\otimes_L D\Big)\times\Big((\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p})^{\psi_q=0}[\log_\phi^{-1}]\otimes_L D_0\otimes_L D^*\Big) & & \\
{\scriptstyle \gamma_{-1}\circ\mathfrak{M}\circ\iota\otimes\mathrm{id}_D}\Big\uparrow \qquad \Big\uparrow{\scriptstyle \frac{N_\nabla}{\Omega}\mathfrak{M}\otimes\mathrm{id}_{D_0\otimes D^*}} & & \\
\big(\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L)\otimes_L D\big)\times\big(\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L)\otimes_L D_0\otimes_L D^*\big) & \xrightarrow{[\cdot,\cdot]_{\mathrm{crys}}} & \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{C}_p}(\Gamma_L)\otimes_L D_0 \\
\cup \qquad\qquad \cup & & \cup \\
\big(D(\Gamma_L,\mathbb{C}_p)\otimes_L D\big)\times\big(D(\Gamma_L,\mathbb{C}_p)\otimes_L D_0\otimes_L D^*\big) & \xrightarrow{[\cdot,\cdot]_{\mathrm{crys}}} & D(\Gamma_L,\mathbb{C}_p)\otimes_L D_0
\end{array}$$

(commutative only on vertical intersections). The theorem follows by comparing these three diagrams. □

10.7 Interpolation formula for the regulator

We are are finally in a position to prove an interpolation formula for the regulator map with the ingredients introduced in the previous sections. In analogy to the regulator maps for cyclotomic extensions, the interpolation formula will be given in terms of Bloch–Kato dual exponentials and logarithms. Since that duality is defined in terms of χ_{cyc} instead of χ_ϕ, we will need to introduce some notation.

Choose a compatible system ε of primitive p^n–th roots of unity for $n \in \mathbb{Z}_{\geq 1}$ and consider the usual period

$$t_\varepsilon = t = \log([\varepsilon]) \in \mathbf{B}_{\mathrm{crys}}.$$

Let t_1 be a basis of $K(1)$ (i.e., $K(1) = K \otimes t_1$ and G_L acts on t_1 by χ_{cyc}). We can identify $\mathbf{D}_{\mathrm{crys},K}(K(1))$ and $\mathbf{D}_{\mathrm{crys},K}(K(\chi_\phi))$ with L using the bases $t_\varepsilon^{-1} \otimes t_1$ and $t_\phi^{-1} \otimes t_0$, respectively. We define $t_{\tau^{-1}} = t_\varepsilon t_\phi^{-1} \otimes t_1^{-1} \otimes t_0 \in \mathbf{D}_{\mathrm{crys},K}(K(\tau^{-1}))$, so that the diagram

$$\begin{array}{ccccc} \mathbf{D}_{\mathrm{crys},K}(K(1)) & = & L(t_\varepsilon^{-1} \otimes t_1) & \cong & L \\ \big\downarrow{\scriptstyle \cdot t_{\tau^{-1}}} & & & & \big\| \\ \mathbf{D}_{\mathrm{crys},K}(K(\chi_\phi)) & = & L(t_\phi^{-1} \otimes t_0) & \cong & L \end{array}$$

is commutative.

Given $T \in \mathrm{Ob}\big(\mathrm{Rep}_{\mathscr{O}_K,\mathrm{fr}}(G_L)\big)$, $x \in \mathrm{H}^1_{\mathrm{Iw}}(L_\infty/L, T)$ and $i \in \mathbb{Z}$, we set as usual $V = K \otimes_{\mathscr{O}_K} T$ and define $x_{\chi_\phi^{-i}} \in \mathrm{H}^1(L, V(\chi_\phi^{-i}))$ to be the image of x under the composition

$$\mathrm{H}^1_{\mathrm{Iw}}(L_\infty/L, T) \xrightarrow{\otimes t_0^i} \mathrm{H}^1_{\mathrm{Iw}}(L_\infty/L, T(\chi_\phi^{-i})) \xrightarrow{\mathrm{cor}} \mathrm{H}^1(L, T(\chi_\phi^{-i})) \longrightarrow \mathrm{H}^1(L, V(\chi_\phi^{-i})).$$

Theorem 227 (Schneider–Venjakob). *Let $T \in \mathrm{Ob}\big(\mathrm{Rep}^{\mathrm{crys}}_{\mathscr{O}_K,\mathrm{fr}}(G_L)\big)$ and consider the crystalline representation $V = K \otimes_{\mathscr{O}_K} T$. Suppose that*

(i) $V(\tau^{-1})$ is K–analytic with Hodge–Tate weights ≥ 0 and

(ii) $\mathbf{D}_{\mathrm{crys},K}(V(\tau^{-1}))^{\varphi_q = 1} = 0$

and that

(i') $V^*(1)$ is K–analytic with Hodge–Tate weights ≤ 1,

(ii') $\mathbf{D}_{\mathrm{crys},K}(V^*(1))^{\varphi_q = \pi_L^{-1}} = 0$ and

(iii') $\mathbf{D}_{\mathrm{crys},K}(V^*(1))^{\varphi_q = 0} = 0$.

(Observe that conditions (i) and (ii) are equivalent to conditions (i') and (ii'), respectively.) Let $x \in \mathbf{D}(T(\tau^{-1}))^{\psi_q = 1}$ and let $i \in \mathbb{Z}$.

(1) *If $i \geq 0$, then*

$$\mathbf{L}_T(x)(\chi_\phi^i) = -i!\,\Omega^{-i} \cdot \\ \cdot \big(1 - \varphi_q^{-1}(\pi_L^{i+1})\varphi_q^{-1}\big)^{-1}\Big(1 - \frac{\pi_L}{q}\varphi_q\Big)\big(t_{\tau^{-1}} \exp^*_{L,V^*(1)(\chi_\phi^i)}(x_{\chi_\phi^{-i}})\big)$$

(where we abuse notation and write $\exp^*_{L,V^*(1)(\chi_\phi^i)}$ for the composition of this dual

exponential with the projection to the identity component).

(2) *If $i \leq -1$, then*

$$\mathbf{L}_T(x)(\chi_\phi^i) = \frac{(-1)^{i+1}\Omega^{-i}}{(-i-1)!} \cdot \\ \cdot \left(1 - \varphi_q^{-1}(\pi_L^{i+1})\varphi_q^{-1}\right)^{-1}\left(1 - \frac{\pi_L}{q}\varphi_q\right)\left(t_{\tau^{-1}} \log_{L,V(\chi_\phi^{-i})}(x_{\chi_\phi^{-i}})\right)$$

*TODO: these formulas are *wrong* for the relative LT case because the period Ω does not commute with operators like $(1-\varphi_q)$. Moreover, I should really check what I wrote in step 5 below.*

Proof. The theorem will be a consequence of theorems 222 and 226. The idea to prove it is to use that the pairing $[\,\cdot\,,\,\cdot\,]_{\mathrm{crys}}$ is non-degenerate and check that both sides of the equalities give the same result when paired with an arbitrary element.

- Step 1. Observe that

$$\left((\gamma_{-1}\iota \circ \Omega\mathbf{L}_T)(x)\right)(\chi_\phi^{-i}) = \Omega\left(\delta_{\gamma_{-1}} \cdot \iota(\mathbf{L}_T(x))\right)(\chi_\phi^{-i}) \\ = \Omega\chi_\phi^{-i}(\gamma_{-1})\mathbf{L}_T(x)(\chi_\phi^i) = (-1)^i\Omega\mathbf{L}_T(x)(\chi_\phi^i),$$

as $\gamma_{-1} \in \Gamma_L$ is defined by $\chi_\phi(\gamma_{-1}) = -1$.

- Step 2. Let $y \in D(\Gamma_L, \mathbb{C}_p) \otimes_L \mathbf{D}_{\mathrm{crys},K}(V^*(1))$. To prove the theorem, it suffices to show that

$$\left[(-1)^i\Omega\mathbf{L}_T(x)(\chi_\phi^i), y(\chi_\phi^{-i})\right]_{\mathrm{crys}} = [TODO, y(\chi_\phi^{-i})]_{\mathrm{crys}}.$$

Indeed, since the distribution $y \in D(\Gamma_L, \mathbb{C}_p) \otimes_L \mathbf{D}_{\mathrm{crys},K}(V^*(1))$ is arbitrary, so is its value $y(\chi_\phi^{-i}) \in \mathbb{C}_p \otimes_L \mathbf{D}_{\mathrm{crys},K}(V^*(1))$; the result will follow by the non-degeneracy of $[\,\cdot\,,\,\cdot\,]_{\mathrm{crys}}$. Now, by step 1 and theorem 226, we can express

$$\left[(-1)^i\Omega\mathbf{L}_T(x)(\chi_\phi^i), y(\chi_\phi^{-i})\right]_{\mathrm{crys}} = \left[(\gamma_{-1}\iota \circ \Omega\mathbf{L}_T)(x), y\right]_{\mathrm{crys}}(\chi_\phi^{-i}) \\ = \left\{x, (\mathbf{\Omega}_{V^*(1),1} \circ (\mathfrak{M} \otimes 1))(y)\right\}_{\mathrm{Iw}}(\chi_\phi^{-i}) \otimes (t_\phi^{-1} \otimes t_0).$$

Then we can use proposition 213 to compute

$$\frac{-1}{\Omega}\{x, (\mathbf{\Omega}_{V^*(1),1} \circ (\mathfrak{M} \otimes 1))(y)\}_{\mathrm{Iw}}(\chi_\phi^{-i}) = \\ = \langle \mathrm{pr}_{L,V(\chi_\phi^{-i})}(x \otimes t_0^{-i}), h^1_{L,V^*(1)(\chi_\phi^i)}(\mathbf{\Omega}_{V^*(1),1}((\mathfrak{M} \otimes 1)(y)) \otimes t_0^i)\rangle_{\mathrm{Tate}}.$$

- Step 3. The interpolation formula for $\left(\mathbf{\Omega}_{V^*(1),1} \circ (\mathfrak{M} \otimes 1)\right)(y) \otimes t_0^i$ from theorem 222 (with $n = 0$) contains the term

$$C(y,i) = \partial_{V^*(1)(\chi_\phi^i)}\Big(\partial_\phi^{-i}\big((1-\varphi_q)^{-1}\circ(\mathfrak{M}\otimes 1)(y)\big)\otimes(t_\phi^{-i}\otimes t_0^i)\Big)$$

that can be computed as follows. By the definition of $\partial_{V^*(1)}$,

$$C(y,i) = \partial_{V^*(1)}\Big(\partial_\phi^{-i}\big((1-\varphi_q)^{-1}\circ(\mathfrak{M}\otimes 1)(y)\big)\Big)$$

and this quantity is the term inside the large parentheses evaluated at $Z = 0$, which under $\mathfrak{M}$ corresponds to the trivial character χ_{triv}. But

$$\partial_\phi^{-i} = \Omega^{-i}\left(\frac{\partial_\phi}{\Omega}\right)^{-i} \quad \text{and} \quad \frac{\partial_\phi}{\Omega}\text{ corresponds to }\mathrm{Tw}_{\chi_\phi}\text{ under }\mathfrak{M},$$

so we will be able to express $C(y,i)$ in terms of $y(\chi_\phi^{-i})$ if we can "swap the positions of ∂_ϕ^{-i} and $(1-\varphi_q)^{-1}$". We know that $\partial_\phi\circ\varphi_q = \pi_L\varphi_q\circ\partial_\phi$ by lemma 65. Thus, working formally,

$$\begin{aligned}\partial_\phi^{-i}\circ(1-\varphi_q)^{-1}\circ(\mathfrak{M}\otimes 1)(y) &= \sum_{m\geq 0}\partial_\phi^{-i}\circ\varphi_q^m\circ(\mathfrak{M}\otimes 1)(y)\\ &= \sum_{m\geq 0}(\pi_L^{-i}\varphi_q)^m\circ\partial_\phi^{-i}\circ(\mathfrak{M}\otimes 1)(y)\\ &= \sum_{m\geq 0}(\pi_L^{-i}\varphi_q)^m\circ\Omega^{-i}(\mathfrak{M}\otimes 1)\big(\mathrm{Tw}_{\chi_\phi^{-i}}y\big)\\ &= (1-\pi_L^{-i}\varphi_q)^{-1}\Big(\Omega^{-i}(\mathfrak{M}\otimes 1)\big(\mathrm{Tw}_{\chi_\phi^{-i}}y\big)\Big).\end{aligned}$$

All in all,

$$C(y,i) = (1-\pi_L^{-i}\varphi_q)^{-1}\big(\Omega^{-i}y(\chi_\phi^{-i})\big).$$

- Step 4. Recall that Bloch–Kato's exponential and dual exponential maps are related by means of local Tate duality and a crystalline pairing that uses the cyclotomic character χ_{cyc} (instead of χ_ϕ as in $[\,\cdot\,,\,\cdot\,]_{\mathrm{crys}}$ above). Observe that the diagram

$$\begin{array}{ccccc}
\mathbf{D}_{\mathrm{crys},K}(V(\chi_\phi^{-i})) & \times & \mathbf{D}_{\mathrm{crys},K}(V^*(1)(\chi_\phi^{i})) & \xrightarrow{[\,\cdot\,,\,\cdot\,]'_{\mathrm{crys}}} & \mathbf{D}_{\mathrm{crys},K}(K(1)) \\
 & & & & \parallel \\
 & & & & L(t_\varepsilon^{-1} \otimes t_1) \\
{\scriptstyle \cdot\, t_\varepsilon t_\phi^{-1} \otimes (t_1^{-1} \otimes t_0)} \Big\downarrow & & \Big\| & & {\scriptstyle \cdot\, (t_\varepsilon^{-1} \otimes t_1)^{-1}(t_\phi^{-1} \otimes t_0)} \Big\downarrow \wr\| \\
 & & & & L(t_\phi^{-1} \otimes t_0) \\
 & & & & \parallel \\
\mathbf{D}_{\mathrm{crys},K}(V(\tau^{-1})(\chi_\phi^{-i})) & \times & \mathbf{D}_{\mathrm{crys},K}(V^*(1)(\chi_\phi^{i})) & \xrightarrow{[\,\cdot\,,\,\cdot\,]_{\mathrm{crys}}} & \mathbf{D}_{\mathrm{crys},K}(K(\chi_\phi))
\end{array}$$

is commutative. Here, we use the prime symbol in the first row just to distinguish it from the map in the last row, as both are the usual duality pairings.

(1) If $i \geq 0$, take $c \in \mathrm{H}^1(L, V(\chi_\phi^{-i}))$ and $\delta \in \mathbf{D}_{\mathrm{crys},K}(V^*(1)(\chi_\phi^{i}))$. We can express

$$\begin{aligned}
\big\langle c, \exp_{L,V^*(1)(\chi_\phi^i)}(\delta) \big\rangle_{\mathrm{Tate}} &= \big[\exp^*_{L,V^*(1)(\chi_\phi^i)}(c), \delta\big]'_{\mathrm{crys}} \cdot (t_\varepsilon^{-1} \otimes t_1)^{-1} \\
&= \big[t_\varepsilon t_\phi^{-1} \exp^*_{V^*(1)(\chi_\phi^i)}(c) \otimes t_1^{-1} \otimes t_0, \delta\big]_{\mathrm{crys}} \cdot (t_\phi^{-1} \otimes t_0)^{-1}.
\end{aligned}$$

(2) If $i \leq -1$, take $c \in \mathrm{H}^1(L, V^*(1)(\chi_\phi^{i}))$ and $\delta \in \mathbf{D}_{\mathrm{crys},K}(V(\chi_\phi^{-i}))$. We can express

$$\begin{aligned}
\big\langle \exp_{L,V(\chi_\phi^{-i})}(\delta), c \big\rangle_{\mathrm{Tate}} &= \big[\delta, \exp^*_{L,V(\chi_\phi^{-i})}(c)\big]'_{\mathrm{crys}} \cdot (t_\varepsilon^{-1} \otimes t_1)^{-1} \\
&= \big[t_\varepsilon t_\phi^{-1} \delta \otimes t_1^{-1} \otimes t_0, \exp^*_{L,V(\chi_\phi^{-i})}(c)\big]_{\mathrm{crys}} \cdot (t_\phi^{-1} \otimes t_0)^{-1}.
\end{aligned}$$

From now on, write $t_{\tau^{-1}} = t_\varepsilon t_\phi^{-1} \otimes t_1^{-1} \otimes t_0$ to simplify the notation.

- Step 5. If we compose the non-degenerate L–bilinear pairing $[\,\cdot\,,\,\cdot\,]_{\mathrm{crys}}$ (resp. $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{Tate}}$) with the map $\mathrm{Tr}_{L/K} \colon L \to K$, we obtain again a non-degenerate (K–bilinear) pairing because L/K is a finite separable extension. In what follows, we work implicitly with the new pairings obtained after composing with $\mathrm{Tr}_{L/K}$. One checks easily that, under $[\,\cdot\,,\,\cdot\,]_{\mathrm{crys}}$,
  - the adjoint of $1 - \pi_L^{-i}\varphi_q$ is $1 - \varphi_q^{-1}(\pi_L^{-i-1})\varphi_q^{-1}$ and
  - the adjoint of $1 - q^{-1}\varphi_q^{-1}$ is $1 - \pi_L q^{-1}\varphi_q$.

TODO: I don't think the adjointness part is so easy once we consider the pairing tensored with $\mathbb{C}_p$. I guess that the periods must become uglier (at least in the relative case, when φ_q has non-trivial action!).

- Step 6.1. Suppose that $i \geq 0$. Combining step 2 with the (first) interpolation formula from theorem 222 and steps 3, 4.1 and 5, we obtain that

$$\begin{aligned}
&\big[(-1)^i \Omega \mathbf{L}_T(x)(\chi_\phi^i), y(\chi_\phi^{-i})\big]_{\mathrm{crys}} = \\
&\quad = -\Omega \langle x_{\chi_\phi^{-i}}, (-1)^i i! \exp_{L,V^*(1)(\chi_\phi^i)}\big((1-q^{-1}\varphi_q^{-1})C(y,i)\big)\rangle_{\mathrm{Tate}} \otimes (t_\phi^{-1} \otimes t_0) \\
&\quad = (-1)^{i+1} i!\, \Omega \big[t_{\tau^{-1}} \exp^*_{L,V^*(1)(\chi_\phi^i)}(x_{\chi_\phi^{-i}}), (1-q^{-1}\varphi_q^{-1})C(y,i)\big]_{\mathrm{crys}} \\
&\quad = (-1)^{i+1} i!\, \Omega \Big[\Big(1-\frac{\pi_L}{q}\varphi_q\Big)\big(t_{\tau^{-1}} \exp^*_{L,V^*(1)(\chi_\phi^i)}(x_{\chi_\phi^{-i}})\big), \\
&\qquad\qquad (1-\pi_L^{-i}\varphi_q)^{-1}\big(\Omega^{-i} y(\chi_\phi^{-i})\big)\Big]_{\mathrm{crys}} \\
&\quad = (-1)^{i+1} i!\, \Omega^{1-i} \cdot \\
&\qquad \cdot \left[\left(1-\frac{1}{\varphi_q^{-1}(\pi_L^{i+1})}\varphi_q^{-1}\right)^{-1}\Big(1-\frac{\pi_L}{q}\varphi_q\Big)\big(t_{\tau^{-1}} \exp^*_{L,V^*(1)(\chi_\phi^i)}(x_{\chi_\phi^{-i}})\big), \right. \\
&\qquad\qquad \left. y(\chi_\phi^{-i}) \right]_{\mathrm{crys}}
\end{aligned}$$

and the first part of the theorem follows by the non-degeneracy of the pairing $[\,\cdot\,,\,\cdot\,]_{\mathrm{crys}}$.

- Step 6.2. Now suppose that $i \leq -1$. Combining step 2 with the (second) interpolation formula from theorem 222 and steps 3, 4.2 and 5, we obtain that

$$\begin{aligned}
&\big[(-1)^i \Omega \mathbf{L}_T(x)(\chi_\phi^i), y(\chi_\phi^{-i})\big]_{\mathrm{crys}} = \\
&\quad = -\Omega \langle \exp_{L,V(\chi_\phi^{-i})}\big(\log_{L,V(\chi_\phi^{-i})}(x_{\chi_\phi^{-i}})\big), \\
&\qquad\qquad h^1_{L,V^*(1)(\chi_\phi^i)}\big(\mathbf{\Omega}_{V^*(1),1}((\mathfrak{M} \otimes 1)(y)) \otimes t_0^i\big)\rangle_{\mathrm{Tate}} \otimes (t_\phi^{-1} \otimes t_0) \\
&\quad = \frac{-\Omega}{(-i-1)!}\big[t_{\tau^{-1}} \log_{L,V(\chi_\phi^{-i})}(x_{\chi_\phi^{-i}}), (1-q^{-1}\varphi_q^{-1})C(y,i)\big]_{\mathrm{crys}} \\
&\quad = \frac{-\Omega}{(-i-1)!}\Big[\Big(1-\frac{\pi_L}{q}\varphi_q\Big)\big(t_{\tau^{-1}} \log_{L,V(\chi_\phi^{-i})}(x_{\chi_\phi^{-i}})\big), \\
&\qquad\qquad (1-\pi_L^{-i}\varphi_q)^{-1}\big(\Omega^{-i} y(\chi_\phi^{-i})\big)\Big]_{\mathrm{crys}} \\
&\quad = \frac{-\Omega^{1-i}}{(-i-1)!} \cdot \\
&\qquad \cdot \Big[\big(1-\varphi_q^{-1}(\pi_L^{-i-1})\varphi_q^{-1}\big)^{-1}\Big(1-\frac{\pi_L}{q}\varphi_q\Big)\big(t_{\tau^{-1}} \log_{L,V(\chi_\phi^{-i})}(x_{\chi_\phi^{-i}})\big), \\
&\qquad\qquad y(\chi_\phi^{-i})\Big]_{\mathrm{crys}}
\end{aligned}$$

and the second part of the theorem follows by the non-degeneracy of the pairing $[\,\cdot\,,\,\cdot\,]_{\mathrm{crys}}$. □

TODO: what happens for characters of finite order?

Part III

Application

TODO!!!

11 The representation associated with a Hida family

11.1 Berthelot's functor

In this subsection we recall Berthelot's functor from formal schemes (satisfying some extra conditions) to rigid analytic spaces in a concrete setting. The construction of this functor is described in full generality in paragraph 0.2.6 of Berthelot's unpublished article [9].

Let R be a noetherian adic $\mathbb{Z}_p$–algebra with an ideal of definition I. Assume that R/I is of finite type over $\mathbb{F}_p$. Fix a set of generators $f_1, \ldots, f_r$ of the ideal I.

For every $n \in \mathbb{Z}_{\geq 1}$, we define

$$R_n = R\Big\langle \frac{f_1^n}{p}, \ldots, \frac{f_r^n}{p} \Big\rangle = R\langle T_{n,1}, \ldots, T_{n,r}\rangle / (f_1^n - pT_{n,1}, \ldots, f_r^n - pT_{n,r}).$$

By the hypothesis on R/I,

$$R_n/pR_n \cong \big(R/(p, f_1^n, \ldots, f_r^n)\big)[T_{n,1}, \ldots, T_{n,r}]$$

is of finite type over $\mathbb{F}_p$ and so R_n is topologically of finite type over $\mathbb{Z}_p$. Thus, $A_n = R_n \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is an affinoid $\mathbb{Q}_p$–algebra. Write $\mathcal{X}_n = \mathrm{Spm}(A_n)$ for the corresponding rigid space.

For $m \geq n \geq 1$, there is a canonical morphism of topological R–algebras $R_m \to R_n$ given by

$$\frac{f_i^m}{p} = T_{m,i} \mapsto f_i^{m-n} T_{n,i} = f_i^{m-n} \frac{f_i^n}{p}.$$

All these morphisms are compatible and induce a projective system of affinoid $\mathbb{Q}_p$–algebras $(A_n)_{n \geq 1}$. We obtain in this way a rigid analytic space

$$\mathcal{X} = \varinjlim_{n \geq 1} \mathcal{X}_n$$

with an admissible covering $(\mathcal{X}_n)_{n \geq 1}$ by affinoids.

Remark. This construction can be globalized to a functor that attaches a rigid analytic space $\mathcal{X}$ to each locally noetherian formal scheme $\mathfrak{S}$ over $\mathrm{Spf}(\mathbb{Z}_p)$ equipped with an ideal sheaf of definition $\mathscr{I}$ with the property that the closed subscheme S_0 defined by $\mathscr{I}$ is locally of finite type over $\mathrm{Spec}(\mathbb{F}_p)$.

Example 228. Let us apply this construction to the Iwasawa algebra $R = \mathbb{Z}_p[\![T]\!]$ with the ideal of definition $I = (p, T)$. For every $n \geq 1$ we get

$$R_n = \mathbb{Z}_p[\![T]\!]\langle T_n \rangle / (T^n - pT_n) \cong \mathbb{Z}_p\Big\langle T, \frac{T^n}{p} \Big\rangle.$$

The affinoid $\mathcal{X}_n$ should be thought of as a p–adic closed disc of radius $p^{-1/n}$ (at least when regarding its $\mathbb{C}_p$–points). Then $\mathcal{X}$ can be interpreted as the open disc of radius 1, as it is the union of the $\mathcal{X}_n$ for $n \geq 1$.

Our main purpose is to use the p–adic Hodge theory on arithmetic families of representations over a rigid analytic space to construct a Perrin-Riou logarithm map as in Castellà's article [12] but without using the results of Ochiai's article [30].

To this aim, we apply Berthelot's construction to the ring $R = \mathbb{I}$ of coefficients of the Hida family with its maximal ideal as ideal of definition. Consider also the representation $\mathbf{T}$ (or a twist $\mathbb{T}$ of it), which is a free $\mathbb{I}$–module of rank 2 with a continuous action of the Galois group G_K. For every $n \in \mathbb{Z}_{\geq 1}$, we define

$$T_n = \mathbb{T} \otimes_R R_n \qquad \text{and} \qquad V_n = T_n \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

The V_n for $n \geq 1$ together form a locally free $\mathscr{O}_{\mathcal{X}}$–module $\mathscr{V} = \mathscr{O}_{\mathcal{X}} \otimes_{R[p^{-1}]} \mathbb{T}[p^{-1}]$ of rank 2. The T_n for $n \geq 1$ provide a formal model $\mathscr{T}$ of $\mathscr{V}$. These sheaves come equipped with a continuous action of G_K and we will want to study $\mathrm{H}^1_{\mathrm{Iw}}(G_K, \mathscr{V})$.

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