Modularity lifting and the Taylor–Wiles method

Course taught by: Patrick Allen

Notes taken by: Francesc Gispert

4th December 2020

Contents

1 Overview

Let $f \in S_k(\Gamma_1(N), \mathbb{C})$ for some weight $k \geq 1$ and some level $N \geq 1$. Suppose that f is an eigenform for all Hecke operators T_ℓ and $\langle \ell \rangle$ for $\ell \nmid N$. Write $\mathrm{T}_\ell f = a_\ell \cdot f$ and $\langle \ell \rangle f = \chi(\ell) \cdot f$, so that $a_\ell \in \overline{\mathbb{Q}}$ and $\chi \colon \left(\mathbb{Z}/N\mathbb{Z}\right)^\times \to \overline{\mathbb{Q}}^\times$.

We implicitly fix embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$.

Theorem 1 (Shimura–Deligne, Deligne–Serre, Ribet). *There exists a continuous irreducible representation*

$$\rho_{f,p} \colon \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$$

satisfying that

- *(i)* *$\rho_{f,p}$ is unramified at every $\ell \nmid Np$ and the characteristic polynomial of $\rho_{f,p}(\mathrm{Frob}_\ell)$ is $X^2 - a_\ell X + \chi(\ell)\ell^{k-1}$.*
- *(ii)* *$\rho_{f,p}$ is potentially semistable at p.*

Remark. The first property determines uniquely $\rho_{f,p}$, by Chebotarev's density theorem.

This course is about the study of a converse to this theorem (and some generalizations).

Conjecture 2 (Fontaine–Mazur). *Let $\rho \colon \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{Q}_p})$ be a continuous irreducible representation that is*

- *(i)* *unramified outside a finite set of primes and*
- *(ii)* *potentially semistable at p.*

If there is no $i \in \mathbb{Z}$ making the twist $\rho \otimes \epsilon_p^i$ (where ϵ_p denotes the p–adic cyclotomic character) an even representation with finite image, then $\rho \cong \rho_{f,p} \otimes \epsilon_p^j$ for some eigenform f and some $j \in \mathbb{Z}$.

Remarks.

1. By Chebotarev's density theorem, $\det(\rho_{f,p}) = \chi\epsilon_p^{1-k}$. In particular, one checks that $\rho_{f,p}$ is odd, which justifies the need of the hypothesis on $\rho \otimes \epsilon_p^i$ in the conjecture. That is, the conjecture states that the only obstructions for a representation to arise from an eigenform are the known ones.
2. Nowadays the conjecture in this form is *almost* completely proved. For example, it is known for regular weights.

Conjecture 3 (Fontaine–Mazur–Langlands). *Let F be a number field. Every continuous irreducible representation $\rho \colon \operatorname{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{Q}_p})$ that is* *(i) unramified outside a finite set of primes and*

(ii) is potentially semistable at p

arises (in some sense) from a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$.

Why do we care?

(1) Philosophically, this theory is a non-abelian form of class field theory (in the sense that it relates some Galois groups to some groups of adèles).

(2) Currently, it is the only way to study analytic properties of certain arithmetic L–functions (e.g., analytic continuation). For example, there are arithmetically defined L–functions (like the L–function of an elliptic curve), which converge on some half-plane, that are only known to have analytic continuation by means of automorphic tools.

Example 4. The conjecture implies the modularity of elliptic curves.

- Over $\mathbb{Q}$ the modularity theorem is already known by the work of Wiles, Taylor–Wiles, . . . , Breuil–Conrad–Diamond–Taylor.
- If F is a totally real number field, it is known that all but finitely many elliptic curves over F are modular, even all in the case that F is a real quadratic field by the work of Freitas–Le Hung–Siksek.
- Over a quadratic imaginary field F, all we can say is that at least a positive proportion of elliptic curves are modular.
- For more general fields, like $F = \mathbb{Q}(\sqrt[3]{2})$, the situation is hopeless with the current tools.

How do we prove these conjectures? We can assume that ρ takes values in $\mathrm{GL}_2(\overline{\mathbb{Z}}_p)$ and so we can consider its reduction $\overline{\rho}$ (with values in $\mathrm{GL}_2(\overline{\mathbb{F}}_p)$). Then a potential proof can follow these two steps:

(1) Prove that $\overline{\rho} \cong \overline{\rho}_{g,p}$ for some modular form g (residual modularity or Serre's conjecture).

(2) Prove that, if $\overline{\rho} \cong \overline{\rho}_{g,p}$ for a modular form g, then $\rho \cong \rho_{f,p}$ for some (possibly different) modular form f (modularity/automorphy lifting).

This course is about the techniques for this second step. Often step 1 is more difficult than step 2 and, in fact, is based on some kind of induction argument using the latter.

The strategy for the modularity lifting.

- One can construct $\mathbb{Z}_p$–algebras R and $\mathbb{T}$ representing functors related to representations and to eigenforms, respectively. More precisely,

    (i) $\mathrm{Hom}_{\mathbb{Z}_p\text{-Alg}}(\mathbb{T}, \overline{\mathbb{Q}}_p)$ corresponds to the set of systems of Hecke eigenvalues on a certain space M of modular forms (whence ρ should arise) and
    (ii) $\mathrm{Hom}_{\mathbb{Z}_p\text{-Alg}}(R, \overline{\mathbb{Q}}_p)$ corresponds to the set of Galois representations that conjecturally arise from the modular forms in M.
- One constructs a morphism $R \to \mathbb{T}$.
- One has to prove that the morphism $R \to \mathbb{T}$ is an isomorphism or, at least, induces an isomorphism $R^{\mathrm{red}} \cong \mathbb{T}^{\mathrm{red}}$.

1.1 Rough plan of the course

- We want to introduce deformation theory and minimal modularity lifting for $\mathrm{GL}_2(\mathbb{Q})$ (4 or 5 weeks).
- We want to study the theory for $\mathrm{GL}_2(F)$ with F a totally real number field and explain non-minimal modularity lifting (which involves base changes) and maybe higher rank conjugate self-duals (4 or 5 weeks).
- We might study the theory for $\mathrm{GL}_2(F)$ with F a CM field and maybe other topics (2 to 4 weeks).

2 Galois representations valued in Hecke algebras

Fix a prime number p and a finite extension $E/\mathbb{Q}_p$. Let $\mathcal{O}$ denote the ring of integers of E and fix a uniformizer ϖ of E. Write $\mathbb{F} = \mathcal{O}/(\varpi)$ for the residue field, which is a finite field with $q = p^f$ elements. We fix algebraic closures $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ and $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$.

Fix also a weight $k \in \mathbb{Z}_{\geq 2}$ and a level $N \in \mathbb{Z}_{\geq 4}$. We are going to work with the congruence subgroup $\Gamma = \Gamma_1(N)$.

Let S be a finite set of places of $\mathbb{Q}$ containing p, the primes dividing N and the archimedean place ∞. Write $\mathbb{Q}_S$ for the maximal algebraic extension of $\mathbb{Q}$ that is unramified outside S and let $G_{\mathbb{Q},S} = \mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q})$. We fix also an abstract isomorphism $\iota \colon \overline{\mathbb{Q}}_p \to \mathbb{C}$, which induces an isomorphism $S_k(\Gamma, \overline{\mathbb{Q}}_p) \cong S_k(\Gamma, \mathbb{C})$. We are going to obtain an integral structure for this space of cusp forms from the Eichler–Shimura isomorphism. (In this setting, one could use Katz's geometric interpretation of modular forms to obtain an integral structure, but that strategy does not extend well to other settings that we will study later.)

Definition 5. Let $\mathbb{T}^{S,\mathrm{univ}}$ be the $\mathbb{Z}$–algebra generated by the Hecke operators T_ℓ and S_ℓ (formal variables) for all primes $\ell \notin S$. For any commutative ring A, we define

$$\mathbb{T}_A^{S,\mathrm{univ}} = \mathbb{T}^{S,\mathrm{univ}} \otimes_{\mathbb{Z}} A.$$

For a $\mathbb{T}_A^{S,\mathrm{univ}}$–module M, we define

$$\mathbb{T}_A^S(M) = \mathbb{T}^S(M) = \mathrm{Im}\big(\mathbb{T}_A^{S,\mathrm{univ}} \to \mathrm{End}_A(M)\big).$$

Example 6. The algebra $\mathbb{T}_{\mathbb{C}}^{S,\mathrm{univ}}$ acts on the space of cusp forms $S_k(\Gamma, \mathbb{C})$ by double coset operators:

$$\mathrm{T}_\ell = \left[\Gamma\begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix}\Gamma\right] \quad \text{and} \quad \mathrm{S}_\ell = \left[\Gamma\begin{pmatrix} \ell & 0 \\ 0 & \ell \end{pmatrix}\Gamma\right] = \ell^{k-2}\langle \ell \rangle.$$

Since S contains all primes in the level, $S_k(\Gamma, \mathbb{C})$ is a semisimple $\mathbb{T}_{\mathbb{C}}^{S,\mathrm{univ}}$–module (the Petersson inner product shows that each T_ℓ is normal) and so we obtain a decomposition

$$\mathbb{T}^S(S_k(\Gamma, \mathbb{C})) \cong \prod_{\text{eigen.}} \mathbb{C},$$

where the product is over the Hecke eigensystems (i.e., the eigenforms in $S_k(\Gamma, \mathbb{C})$).

This decomposition transforms under ι into

$$\mathbb{T}^S(S_k(\Gamma, \overline{\mathbb{Q}}_p)) \cong \prod_{\text{eigen.}} \overline{\mathbb{Q}}_p.$$

Every eigensystem

$$\lambda\colon \mathbb{T}^S(S_k(\Gamma, \overline{\mathbb{Q}}_p)) \to \overline{\mathbb{Q}}_p$$

(corresponding to an eigenform) gives rise to a Galois representation

$$\rho_\lambda\colon G_{\mathbb{Q},S} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$$

such that, for every prime $\ell \notin S$,

$$\mathrm{CharPoly}(\rho_\lambda(\mathrm{Frob}_\ell)) = X^2 - \lambda(\mathrm{T}_\ell)X + \ell\lambda(\mathrm{S}_\ell).$$

Thus, putting everything together, we obtain

$$\rho = \prod_{\lambda \text{ eigen.}} \rho_\lambda\colon G_{\mathbb{Q},S} \to \mathrm{GL}_2\big(\mathbb{T}^S(S_k(\Gamma, \overline{\mathbb{Q}}_p))\big)$$

with the property that, for every prime $\ell \notin S$,

$$\mathrm{CharPoly}(\rho(\mathrm{Frob}_\ell)) = X^2 - \mathrm{T}_\ell\, X + \ell\, \mathrm{S}_\ell\,.$$

Our goal is to obtain an integral version of this representation. That is, we want to replace $\overline{\mathbb{Q}}_p$ with $\overline{\mathbb{Z}}_p$ (or even $\mathcal{O}$ for a suitable $E/\mathbb{Q}_p$).

Theorem 7 (Eichler–Shimura). *There is an isomorphism of* $\mathbb{T}^{S,\mathrm{univ}}_{\mathbb{C}}$*–modules*

$$M_k(\Gamma, \mathbb{C}) \oplus \overline{S_k(\Gamma, \mathbb{C})} \cong \mathrm{H}^1(\Gamma, \mathrm{Sym}^{k-2}(\mathbb{C}^2)).$$

The action of a double coset operator $[\Gamma\alpha\Gamma]$ *with* $\alpha \in \mathrm{GL}_2(\mathbb{Q})$ *on* $\mathrm{H}^i(\Gamma, \mathrm{Sym}^{k-2}(\mathbb{C}^2))$ *is given by the composition*

$$\begin{tikzcd}
\mathrm{H}^i(\Gamma, \mathrm{Sym}^{k-2}(\mathbb{C}^2)) \arrow[r, "\mathrm{res}"] \arrow[d, dashed, "{[\Gamma\alpha\Gamma]}"] & \mathrm{H}^i(\Gamma \cap \alpha^{-1}\Gamma\alpha, \mathrm{Sym}^{k-2}(\mathbb{C}^2)) \arrow[d, "\alpha_*"] \\
\mathrm{H}^i(\Gamma, \mathrm{Sym}^{k-2}(\mathbb{C}^2)) & \mathrm{H}^i(\alpha\Gamma\alpha^{-1} \cap \Gamma, \mathrm{Sym}^{k-2}(\mathbb{C}^2)) \arrow[l, "\mathrm{cor}"']
\end{tikzcd}$$

(where res *and* cor *denote the obvious restriction and corestriction in group cohomology).*

Remark. This cohomology can also be seen geometrically. Indeed, assuming that $k = 2$ for simplicity, one can identify $\mathrm{H}^1(\Gamma, \mathbb{C}) \cong \mathrm{H}^1(Y(\Gamma), \mathbb{C})$ for $Y(\Gamma) = \Gamma \backslash \mathfrak{H}$ (with integral coefficients this holds only under the assumption that $N \geq 4$; otherwise there might be torsion and the right-hand side has to be modified) and then the action of $[\Gamma\alpha\Gamma]$ is given by

$$\begin{tikzcd}
Y(\Gamma \cap \alpha^{-1}\Gamma\alpha) \arrow[d, "\pi_1"] & Y(\alpha\Gamma\alpha^{-1} \cap \Gamma) \arrow[l, "\alpha"'] \arrow[d, "\pi_2"] \\
Y(\Gamma) & Y(\Gamma)
\end{tikzcd}$$

(or rather, by $\pi_{2,*} \circ \alpha^* \circ \pi_1^*$).

Then $\mathrm{H}^1(\Gamma, \mathrm{Sym}^{k-2}(\mathbb{C}^2)) \cong \mathrm{H}^1(\Gamma, \mathrm{Sym}^{k-2}(\mathbb{Z}^2)) \otimes_{\mathbb{Z}} \mathbb{C}$ has a natural integral structure. In addition, $\mathrm{H}^1(\Gamma, \mathrm{Sym}^{k-2}(\mathbb{Z}^2))$ is a finitely generated abelian group, which implies that $\mathrm{H}^1(\Gamma, \mathrm{Sym}^{k-2}(\mathcal{O}^2)) \cong \mathrm{H}^1(\Gamma, \mathrm{Sym}^{k-2}(\mathbb{Z}^2)) \otimes_{\mathbb{Z}} \mathcal{O}$ is a finitely generated $\mathcal{O}$–module. This is the object that we want to study. All in all, we obtain isomorphisms

$$\begin{tikzcd}
\mathrm{H}^1(\Gamma, \mathrm{Sym}^{k-2}(\mathcal{O}^2)) \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_p \arrow[r, phantom, "\cong"] & \mathrm{H}^1(\Gamma, \mathrm{Sym}^{k-2}(\overline{\mathbb{Q}}_p^2)) \arrow[r, phantom, "\cong"] & \mathrm{H}^1(\Gamma, \mathrm{Sym}^{k-2}(\mathbb{C}^2)) \\
 & S_k(\Gamma, \overline{\mathbb{Q}}_p) \arrow[u, hook] & S_k(\Gamma, \mathbb{C}) \arrow[u, hook]
\end{tikzcd}$$

(where we used the fixed isomorphism $\iota\colon \overline{\mathbb{Q}}_p \cong \mathbb{C}$ and the restriction of the Eichler–Shimura isomorphism).

Choose a Hecke eigenform $g \in S_k(\Gamma, \mathbb{C}) \cong S_k(\Gamma, \overline{\mathbb{Q}}_p)$ and consider the corresponding Hecke eigensystem $\lambda_g\colon \mathbb{T}^S\big(\mathrm{H}^1(\Gamma, \mathrm{Sym}^{k-2}(\overline{\mathbb{Q}}_p^2))\big) \to \overline{\mathbb{Q}}_p$. Enlarging $\mathcal{O}$ if necessary (for this g), we obtain an integral version of λ_g and we can form a "reduction" $\overline{\lambda}_g$ making the diagram

$$\begin{tikzcd}
\lambda_g\colon \mathbb{T}^S\big(\mathrm{H}^1(\Gamma, \mathrm{Sym}^{k-2}(\overline{\mathbb{Q}}_p^2))\big) \arrow[r] & \mathbb{T}^S\big(S_k(\Gamma, \overline{\mathbb{Q}}_p)\big) \arrow[r] & \overline{\mathbb{Q}}_p \\
\mathbb{T}^S\big(\mathrm{H}^1(\Gamma, \mathrm{Sym}^{k-2}(\mathcal{O}^2))\big) \arrow[u] \arrow[rr, dashed, "\lambda_g"] \arrow[rrd, "\overline{\lambda}_g"'] & & \mathcal{O} \arrow[u, hook] \arrow[d, two heads] \\
 & & \mathbb{F}
\end{tikzcd}$$

commutative. Now $\mathfrak{m} = \operatorname{Ker}(\overline{\lambda}_g)$ is a maximal ideal of

$$\mathbb{T}^S(\Gamma, k) = \mathbb{T}^S\big(\mathrm{H}^1(\Gamma, \mathrm{Sym}^{k-2}(\mathscr{O}^2))\big)$$

and we can attach a continuous representation

$$\overline{\rho}_{\mathfrak{m}} \colon G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F})$$

to $\mathfrak{m}$ and $\overline{\lambda}_g$. By construction, this representation satisfies that

$$\mathrm{CharPoly}(\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_\ell)) = X^2 - \overline{\lambda}_g X + \ell\overline{\lambda}_g(\mathrm{S}_\ell) = X^2 - \mathrm{T}_\ell\, X + \ell\, \mathrm{S}_\ell \mod \mathfrak{m}$$

for all primes $\ell \notin S$.

Definition 8. We say that the maximal ideal $\mathfrak{m}$ of $\mathbb{T}^S(\Gamma, k)$ is *non-Eisenstein* if the representation $\overline{\rho}_{\mathfrak{m}}$ is absolutely irreducible.

Proposition 9. *If $\mathfrak{m}$ is a non-Eisenstein maximal ideal of $\mathbb{T}^S(\Gamma, k)$, then the localization $\mathrm{H}^1(\Gamma, \mathrm{Sym}^{k-2}(\mathscr{O}^2))_{\mathfrak{m}}$ is a finite free $\mathscr{O}$–module.*

Since $\mathbb{T}^S(\Gamma, k)_{\mathfrak{m}} \subset \mathrm{End}_{\mathscr{O}}\big(\mathrm{H}^1(\Gamma, \mathrm{Sym}^{k-2}(\mathscr{O}^2))_{\mathfrak{m}}\big)$, we deduce the following:

Corollary 10. *If $\mathfrak{m}$ is a non-Eisenstein maximal ideal of $\mathbb{T}^S(\Gamma, k)$, then $\mathbb{T}^S(\Gamma, k)_{\mathfrak{m}}$ is flat over $\mathscr{O}$ (i.e., torsion-free).*

Proof of proposition 9. We assume that $k = 2$ to simplify the notation. Since we already know that $\mathrm{H}^1(\Gamma, \mathscr{O})_{\mathfrak{m}}$ is finitely generated over $\mathscr{O}$, we just need to show that it is p–torsion-free. Taking cohomology of the short exact sequence

$$0 \longrightarrow \mathscr{O} \xrightarrow{\varpi} \mathscr{O} \longrightarrow \mathbb{F} \longrightarrow 0$$

and localizing at $\mathfrak{m}$, we obtain an exact sequence

$$\mathrm{H}^0(\Gamma, \mathbb{F})_{\mathfrak{m}} \longrightarrow \mathrm{H}^1(\Gamma, \mathscr{O})_{\mathfrak{m}} \xrightarrow{\varpi} \mathrm{H}^1(\Gamma, \mathscr{O})_{\mathfrak{m}}.$$

Thus, it suffices to prove that

$$\mathrm{H}^0(\Gamma, \mathbb{F})_{\mathfrak{m}} = 0.$$

A double coset $[\Gamma\alpha\Gamma]$ with $\alpha \in \mathrm{GL}_2(\mathbb{Q})$ acts on $\mathrm{H}^0(\Gamma, \mathbb{F})$ by

$$\begin{tikzcd}
\mathrm{H}^0(\Gamma, \mathbb{F}) \arrow[r, "\mathrm{res}"] \arrow[d, phantom, "\shortparallel"] & \mathrm{H}^0(\Gamma \cap \alpha^{-1}\Gamma\alpha, \mathbb{F}) \arrow[r, "\alpha"] \arrow[d, phantom, "\shortparallel"] & \mathrm{H}^0(\alpha\Gamma\alpha^{-1} \cap \Gamma, \mathbb{F}) \arrow[r, "\mathrm{cor}"] \arrow[d, phantom, "\shortparallel"] & \mathrm{H}^0(\Gamma, \mathbb{F}) \arrow[d, phantom, "\shortparallel"] \\
\mathbb{F} \arrow[r, "\mathrm{id}_{\mathbb{F}}"] & \mathbb{F} \arrow[r, "\mathrm{id}_{\mathbb{F}}"] & \mathbb{F} \arrow[r, "{[\Gamma:\alpha\Gamma\alpha^{-1}\cap\Gamma]}"] & \mathbb{F}
\end{tikzcd}$$

(i.e., simply by multiplication by $[\Gamma : \alpha\Gamma\alpha^{-1} \cap \Gamma]$). Therefore, for every prime $\ell \notin S$, T_ℓ acts by $\ell + 1$ and S_ℓ acts by 1. That is, if $\mathrm{H}^0(\Gamma, \mathbb{F})_{\mathfrak{m}} \neq 0$, we would have that

$$\mathrm{T}_\ell \equiv 1 + \ell \mod \mathfrak{m} \quad \text{and} \quad \mathrm{S}_\ell \equiv 1 \mod \mathfrak{m}$$

and we would obtain an explicit description of $\mathrm{CharPoly}(\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_\ell))$. But then $\overline{\rho}_{\mathfrak{m}}$ would have to be $1 \oplus \overline{\epsilon}_p$ (where $\overline{\epsilon}_p$ is the cyclotomic character modulo p) by Chebotarev's density theorem, contradicting the fact that $\mathfrak{m}$ is non-Eisenstein. □

Remark. This proof is much more complicated than necessary in this setting (one does not even need to localize at $\mathfrak{m}$ to get the p–torsion-free result), but it will be better for generalizations.

We have

$$\mathbb{T}^S(\Gamma, k)_{\mathfrak{m}} \hookrightarrow \mathbb{T}^S(\Gamma, k)_{\mathfrak{m}} \otimes_{\mathscr{O}} \overline{\mathbb{Q}}_p = \prod_{\text{eigen.}} \overline{\mathbb{Q}}_p,$$

where the product is over the eigensystems lying over $\mathfrak{m}$. Therefore, we have a representation

$$\rho = \prod \rho_i \colon G_{\mathbb{Q},S} \to \mathrm{GL}_2\big(\mathbb{T}^S(\Gamma, k)_{\mathfrak{m}} \otimes_{\mathscr{O}} \overline{\mathbb{Q}}_p\big)$$

satisfying that

$$\mathrm{CharPoly}(\rho(\mathrm{Frob}_\ell)) = X^2 - \mathrm{T}_\ell\, X + \ell\, \mathrm{S}_\ell \in \mathbb{T}^S(\Gamma, k)_{\mathfrak{m}}[X]$$

for all $\ell \notin S$. That is, the values of ρ lie in a much smaller (and integral) subspace corresponding to $\mathfrak{m}$. More precisely, the representation ρ descends to

$$\rho_{\mathfrak{m}} \colon G_{\mathbb{Q},S} \to \mathrm{GL}_2\big(\mathbb{T}^S(\Gamma, k)_{\mathfrak{m}}\big)$$

by the following result:

Theorem 11 (Carayol). *Consider a local ring A with residue field F and let R be an A–algebra (e.g., $A = \mathbb{T}^S(\Gamma, k)_{\mathfrak{m}}$ and $R = A[G_{\mathbb{Q},S}]$). Let A'/A be a semilocal extension with a decomposition*

$$A' = \prod_{i \in I} A'_i$$

where each A'_i is local with maximal ideal $\mathfrak{m}'_i$ and residue field F'_i (e.g., continuing with the A above, $A' = \mathbb{T}^S(\Gamma, k)_{\mathfrak{m}} \otimes_{\mathscr{O}} \overline{\mathbb{Z}}_p$). If there is an A–algebra representation

$$\rho' = \prod_{i \in I} \rho'_i \colon R \otimes_A A' \to \mathrm{M}_n(A') = \prod_{i \in I} \mathrm{M}_n(A'_i)$$

satisfying that

1. $\operatorname{tr}\big(\rho(r \otimes 1)\big) \in A$ *for all* $r \in R$ *and*
2. *the reductions* $\overline{\rho}_i \colon R \otimes_A F'_i \to \mathrm{M}_n(F'_i)$ *are all absolutely irreducible and have equal* $\operatorname{tr}\big(\overline{\rho}'_i(r \otimes 1)\big) \in F$ *for all* $r \in R$ *(i.e.,* $\operatorname{tr}\big(\overline{\rho}'_i(r \otimes 1)\big)$ *is independent of* $i \in I$*),*

then ρ' *is conjugate to the scalar extension* $\cdot \otimes_A A'$ *of a representation* $\rho \colon R \to \mathrm{M}_n(A)$*.*

3 Deformations of Galois representations

Let Γ be a profinite group and let p be a prime number.

Definition 12. We say that Γ satisfies *condition* Φ_p if, for every open subgroup H of Γ, the set $\mathrm{Hom}^{\mathrm{cont}}(H, \mathbb{F}_p)$ is finite or, equivalently, the maximal pro-p quotient $H^{(p)}$ of H is topologically finitely generated.

Example 13. The condition Φ_p is satisfied by

(1) the Galois group $G_{F,S} = \mathrm{Gal}(F_S/F)$, where F is a number field, S is a finite set of places of F and F_S is the maximal extension of F that is unramified outside S, and

(2) the absolute Galois group $G_K = \mathrm{Gal}(\overline{K}/K)$ of a finite extension $K/\mathbb{Q}_\ell$ with ℓ prime.

Let $\mathbb{F}$ be a finite field of characteristic p. Let CNL denote the category of complete noetherian local rings $(A, \mathfrak{m}_A)$ with a fixed isomorphism $A/\mathfrak{m}_A \cong \mathbb{F}$. Let Ar denote the full subcategory of CNL consisting of artinian objects. Given $\Lambda \in \mathrm{Ob}(\mathrm{CNL})$, we define CNL_Λ (resp. Ar_Λ) to be the full subcategory of CNL (resp. Ar) whose objects are Λ–algebras. In particular, the ring of Witt vectors $\mathrm{W}(\mathbb{F})$ is an initial object in CNL and so $\mathrm{CNL} = \mathrm{CNL}_{\mathrm{W}(\mathbb{F})}$.

Fix a continuous homomorphism

$$\overline{\rho}\colon \Gamma \to \mathrm{GL}_n(\mathbb{F}).$$

Definition 14.

(1) A *lift* or *lifting* or *framed deformation* of $\overline{\rho}$ to $A \in \mathrm{Ob}(\mathrm{CNL})$ is a continuous homomorphism

$$\rho\colon \Gamma \to \mathrm{GL}_n(A)$$

such that $\rho \bmod \mathfrak{m}_A = \overline{\rho}$.

(2) We say that two lifts ρ and ρ' to A are *strictly equivalent* if they are conjugate by an element of $1 + \mathrm{M}_n(\mathfrak{m}_A) = \mathrm{Ker}\big(\mathrm{GL}_n(A) \to \mathrm{GL}_n(\mathbb{F})\big)$.

(3) A *deformation* of $\overline{\rho}$ to A is a strict equivalence class of lifts of $\overline{\rho}$ to A.

Remark. By abuse of notation, we will often identify deformations with some representative lift. The reason to consider deformations instead of lifts is that modular forms give rise to Galois representations without a distinguished basis.

Example 15. Let $g \in S_k(\Gamma, \overline{\mathbb{Z}}_p)$ be a Hecke eigenform with Galois representation modulo p

$$\overline{\rho}_g \colon G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F}).$$

Then any Hecke eigenform $f \in S_k(\Gamma, \overline{\mathbb{Z}}_p)$ congruent to g yields a deformation ρ_f of $\overline{\rho}_g$.

3.1 Representability

Definition 16. We define the functors

$$\begin{aligned} D = D_{\overline{\rho}} \colon \mathrm{CNL} &\longrightarrow \mathrm{Set} \\ A &\longmapsto \{\, \text{Deformations of } \overline{\rho} \text{ to } A \,\} \end{aligned}$$

and

$$\begin{aligned} D^{\square} = D^{\square}_{\overline{\rho}} \colon \mathrm{CNL} &\longrightarrow \mathrm{Set} \\ A &\longmapsto \{\, \text{Lifts of } \overline{\rho} \text{ to } A \,\} \end{aligned}$$

Remark. The two functors $D_{\overline{\rho}}$ and $D^{\square}_{\overline{\rho}}$ are continuous in the sense that, for every $A \in \mathrm{Ob}(\mathrm{CNL})$, the natural maps

$$D(A) \to \varprojlim_{i \geq 1} D(A/\mathfrak{m}_A^i) \quad \text{and} \quad D^{\square}(A) \to \varprojlim_{i \geq 1} D^{\square}(A/\mathfrak{m}_A^i)$$

are bijections. Therefore, both functors $D_{\overline{\rho}}$ and $D^{\square}_{\overline{\rho}}$ are completely determined by their restriction to Ar.

We next want to study when these functors are representable (i.e., under what hypotheses we have $D_{\overline{\rho}} \cong \mathrm{Hom}_{\mathrm{CNL}}(R, \cdot\,)$ and $D^{\square}_{\overline{\rho}} \cong \mathrm{Hom}_{\mathrm{CNL}}(R^{\square}, \cdot\,)$ for some $R, R^{\square} \in \mathrm{Ob}(\mathrm{CNL})$).

Proposition 17. *If Γ satisfies the condition Φ_p, then $D^{\square}_{\overline{\rho}}$ is representable.*

Proof. Let $H = \mathrm{Ker}(\overline{\rho})$, which is an open subgroup of Γ. By hypothesis, the maximal pro-p quotient $H^{(p)}$ of H is topologically finitely generated.

Let $N = \mathrm{Ker}\big(H \to H^{(p)}\big)$. Since N is fixed by automorphisms of H, N is a normal subgroup of Γ. By the definition of N and condition Φ_p, the quotient Γ/N is also topologically finitely generated. Fix topological generators $\gamma_1, \ldots, \gamma_g$ of Γ/N. We can define a continuous function

$$\rho \colon \Gamma \to \mathrm{GL}_n\big(\mathrm{W}(\mathbb{F})[\![X_{s,i,j} : 1 \leq s \leq g \text{ and } 1 \leq i,j \leq n]\!]\big)$$

by

$$\gamma_s \mapsto [\overline{\rho}(\gamma_s)] \cdot \big(1 + (X_{s,i,j})_{i,j}\big),$$

where $[\,\cdot\,]$ denotes the Teichmüller lift and 1 denotes the identity matrix. Then $D^{\square}_{\overline{\rho}}$ is represented by the quotient of $W(\mathbb{F})[\![\{\, X_{s,i,j} \,\}_{s,i,j}]\!]$ by the ideal generated by all matrix entries of $\rho(r) - 1$ as r ranges over all relations satisfied by $\gamma_1, \ldots, \gamma_g$. $\square$

Theorem 18 (Mazur). *If* Γ *satisfies the condition* Φ_p *and* $\mathrm{End}_{\mathbb{F}[\Gamma]}(\overline{\rho}) = \mathbb{F}$*, then* $D_{\overline{\rho}}$ *is representable.*

Remark. One way to prove Mazur's theorem is to take the quotient of $D^{\square}_{\overline{\rho}}$ by the free action of the formal group scheme $\widehat{\mathrm{PGL}}_n$.

Next we explain the ingredients that appear in the classical proof of theorem 18. Suppose that a functor $F\colon \mathrm{CNL} \to \mathrm{Set}$ is represented by an object R and consider two morphisms $A \to C$ and $B \to C$ in CNL. Then

$$\begin{aligned} F(A \times_C B) &= \mathrm{Hom}_{\mathrm{CNL}}(R, A \times_C B) \\ &= \mathrm{Hom}_{\mathrm{CNL}}(R, A) \times_{\mathrm{Hom}_{\mathrm{CNL}}(R,C)} \mathrm{Hom}_{\mathrm{CNL}}(R, B) \\ &= F(A) \times_{F(C)} F(B). \end{aligned}$$

Write $\mathbb{F}[\varepsilon] = \mathbb{F}[X]/(X^2)$.

Theorem 19 (Grothendieck). *Let* $F\colon \mathrm{CNL} \to \mathrm{Set}$ *be a continuous functor such that* $F(\mathbb{F})$ *is a singleton. The functor* F *is representable if and only if*

(1) *the restriction of* F *to* Ar *preserves fibre products and*
(2) $\dim_{\mathbb{F}} F(\mathbb{F}[\varepsilon]) < \infty$.

Remark. The structure of $\mathbb{F}$–vector space on $F(\mathbb{F}[\varepsilon])$ is not obvious. Multiplication by an element $\alpha \in \mathbb{F}$ on $F(\mathbb{F}[\varepsilon])$ is obtained by applying F to the $\mathbb{F}$–algebra morphism

$$a + b\varepsilon \mapsto a + \alpha b\varepsilon.$$

For the addition, we use the identification

$$F(\mathbb{F}[\varepsilon]) \times F(\mathbb{F}[\varepsilon]) = F(\mathbb{F}[\varepsilon] \times_{\mathbb{F}} \mathbb{F}[\varepsilon])$$

and apply F to the $\mathbb{F}$–algebra morphism

$$(a + b\varepsilon, a + c\varepsilon) \mapsto a + (b + c)\varepsilon.$$

The condition that $\dim_{\mathbb{F}} F(\mathbb{F}[\varepsilon])$ be finite allows us to obtain noetherianness of the ring representing F.

Condition (1) of theorem 19 can be very hard to check, so we explain an alternative characterization of representability of the functors we are interested in.

We say that a morphism $A \to C$ in the category Ar is *small* if it is surjective and its kernel is a principal ideal annihilated by $\mathfrak{m}_A$. Consider morphisms $\alpha\colon A \to C$ and $\beta\colon B \to C$ in Ar and the natural map

$$F(A \times_C B) \to F(A) \times_{F(C)} F(B).$$

Theorem 20 (Schlessinger). *Let* $F\colon \mathrm{CNL} \to \mathrm{Set}$ *be a continuous functor such that* $F(\mathbb{F})$ *is a singleton. The functor F is representable if and only if*

(1) *for every α and β as above, the natural map*

$$F(A \times_C B) \to F(A) \times_{F(C)} F(B)$$

is surjective whenever α is small,

(2) *for every α as above (and taking $\beta = \alpha$), the natural map*

$$F(A \times_C A) \to F(A) \times_{F(C)} F(A)$$

is bijective whenever α is small,

(3) *taking $C = \mathbb{F}$, α as above and $B = \mathbb{F}[\varepsilon]$, the natural map*

$$F(A \times_{\mathbb{F}} \mathbb{F}[\varepsilon]) \to F(A) \times F(\mathbb{F}[\varepsilon])$$

is bijective and

(4) $\dim_{\mathbb{F}} F(\mathbb{F}[\varepsilon]) < \infty$.

Remark. Grothendieck's criterion looks simpler but it is much more difficult to check in practice than Schlessinger's more technical conditions.

Lemma 21. *If* $\mathrm{End}_{\mathbb{F}[\Gamma]}(\overline{\rho}) = \mathbb{F}$, *then* $\mathrm{End}_{C[\Gamma]}(\rho) = C$ *for every* $C \in \mathrm{Ob}(\mathrm{CNL})$ *and every lift* $\rho\colon \Gamma \to \mathrm{GL}_n(C)$ *of* $\overline{\rho}$ *to* C.

Idea of the proof. The lemma can be proved by reducing to the artinian case and using an induction argument on the length of C. □

Now we are in a position to prove (most of) theorem 18, that we recall here:

Theorem 18 (Mazur). *If Γ satisfies the condition Φ_p and $\mathrm{End}_{\mathbb{F}[\Gamma]}(\overline{\rho}) = \mathbb{F}$, then $D_{\overline{\rho}}$ is representable.*

Proof. We use Schlessinger's criterion (i.e., theorem 20).

(1) Take lifts ρ_A and ρ_B of $\overline{\rho}$ to A and B, respectively, such that $\alpha \circ \rho_A$ and $\beta \circ \rho_B$ are $(1 + \mathrm{M}_n(\mathfrak{m}_C))$–conjugate. Thus, we may take $g \in 1 + \mathrm{M}_n(\mathfrak{m}_C)$ such that $g(\alpha \circ \rho_A)g^{-1} = \beta \circ \rho_B$. Since α is surjective, we can lift g to $h \in 1 + \mathrm{M}_n(\mathfrak{m}_A)$. Then $(h\rho_A h^{-1}, \rho_B)$ defines a lift of $\overline{\rho}$ to $A \times_C B$ and is a preimage of (ρ_A, ρ_B).

(2) We are going to use lemma 21. Let $\alpha\colon A \to C$ be a small morphism in the category Ar. By the previous part, we only have to prove that the natural map

$$\Phi\colon D_{\overline{\rho}}(A \times_C A) \to D_{\overline{\rho}}(A) \times_{D_{\overline{\rho}}(C)} D_{\overline{\rho}}(A)$$

is injective. Take $\rho, r \in D^{\square}_{\overline{\rho}}(A \times_C A)$ such that $\Phi(\rho) = \Phi(r)$ (regarded as deformations). Write (ρ_1, ρ_2) (resp. (r_1, r_2)) for the image of ρ (resp. r) in $D^{\square}_{\overline{\rho}}(A) \times_{D^{\square}_{\overline{\rho}}(C)} D^{\square}_{\overline{\rho}}(A)$. By assumption, we can express $\rho_i = g_i r_i g_i^{-1}$ for some $g_i \in 1 + \mathrm{M}_n(\mathfrak{m}_A)$. But $\alpha \circ \rho_1 = \alpha \circ \rho_2$ and $\alpha \circ r_1 = \alpha \circ r_2$ as lifts (not just as deformations). Therefore,

$$\begin{aligned}\alpha \circ \rho_1 &= \alpha(g_1)(\alpha \circ r_1)\alpha(g_1)^{-1} = \alpha(g_1)(\alpha \circ r_2)\alpha(g_1)^{-1} \\ &= \alpha(g_1 g_2^{-1})(\alpha \circ \rho_2)\alpha(g_1 g_2^{-1})^{-1} = \alpha(g_1 g_2^{-1})(\alpha \circ \rho_1)\alpha(g_1 g_2^{-1})^{-1},\end{aligned}$$

which means that $\alpha(g_1 g_2^{-1})$ commutes with $\alpha \circ \rho_1$. By lemma 21, we deduce that $\alpha(g_1 g_2^{-1}) \in \mathrm{C} \cap \mathrm{M}_n(1 + \mathfrak{m}_C) = 1 + \mathfrak{m}_C$. We can take a lift $a_1 \in 1 + \mathfrak{m}_A$ of $\alpha(g_1 g_2^{-1})$ and, replacing g_1 with $a_1^{-1} g_1$, we may assume that $\alpha(g_1) = \alpha(g_2)$. All in all, we obtain $g = (g_1, g_2) \in 1 + \mathrm{M}_n(\mathfrak{m}_{A \times_C A})$ such that $\rho = grg^{-1}$, which means that ρ and r define the same deformation.

(3) We skip the proof of this part.

(4) We are going to check that $D_{\overline{\rho}}(\mathbb{F}[\varepsilon])$ is finite later (see corollary 24). □

Lemma 22. *Suppose that Γ satisfies the condition Φ_p and that $\mathrm{End}_{\mathbb{F}[\Gamma]}(\overline{\rho}) = \mathbb{F}$. Let R be the ring that represents the functor $D_{\overline{\rho}}$: CNL $\to$ Set. For every $\Lambda \in$ CNL, the restriction of $D_{\overline{\rho}}$ to CNL_Λ is represented by $R \,\widehat{\otimes}_{\mathrm{W}(\mathbb{F})} \Lambda$.*

Remark. If $R^{\square}$ represents the functor $D^{\square}_{\overline{\rho}}$, then there is a universal object $\rho^{\square} \in D^{\square}_{\overline{\rho}}$ corresponding to $\mathrm{id}_{R^{\square}}$. Thus, for every $A \in \mathrm{Ob}(\mathrm{CNL})$ and every $\rho \in D^{\square}_{\overline{\rho}}$, there exists a unique $\alpha\colon R^{\square} \to A$ such that $\rho = \alpha \circ \rho^{\square}$. Sometimes the existence of the universal object is useful.

3.2 The tangent space

Let $\mathrm{ad}(\overline{\rho}) = \mathrm{M}_n(\mathbb{F})$ with adjoint Γ–action. That is, given $\sigma \in \Gamma$ and $X \in \mathrm{ad}(\overline{\rho})$,

$$\sigma \cdot X = \overline{\rho}(\sigma) X \overline{\rho}(\sigma)^{-1}.$$

Remark. We may view $\mathrm{ad}(\overline{\rho}) = \mathfrak{gl}_n$ and that is the right way to interpret this object for generalizations. That is, if we replace GL_n with another group scheme, then the role of $\mathrm{ad}(\overline{\rho})$ is played by a Lie algebra over $\mathbb{F}$.

Take a lift $\rho \colon \Gamma \to \mathrm{GL}_n(\mathbb{F}[\varepsilon])$ of $\overline{\rho}$. For every $\sigma \in \Gamma$, we can express

$$\rho(\sigma) = \big(1 + \varepsilon c(\sigma)\big)\overline{\rho}(\sigma) \quad \text{with } c(\sigma) \in \mathrm{M}_n(\mathbb{F}).$$

For $\sigma, \tau \in \Gamma$, we rewrite the relation $\rho(\sigma\tau) = \rho(\sigma)\rho(\tau)$ as

$$\big(1 + \varepsilon c(\sigma\tau)\big)\overline{\rho}(\sigma\tau) = \big(1 + \varepsilon c(\sigma)\big)\overline{\rho}(\sigma)\big(1 + \varepsilon c(\tau)\big)\overline{\rho}(\tau),$$

whence

$$c(\sigma\tau)\overline{\rho}(\sigma\tau) = c(\sigma)\overline{\rho}(\sigma)\overline{\rho}(\tau) + \overline{\rho}(\sigma)c(\tau)\overline{\rho}(\tau)$$

or, equivalently,

$$c(\sigma\tau) = c(\sigma) + \overline{\rho}(\sigma)c(\tau)\overline{\rho}(\sigma)^{-1}.$$

That is, $c \in \mathrm{Z}^1(\Gamma, \mathrm{ad}(\overline{\rho}))$.

In this way, we obtain a bijection $D^{\square}_{\overline{\rho}}(\mathbb{F}[\varepsilon]) \cong \mathrm{Z}^1(\Gamma, \mathrm{ad}(\overline{\rho}))$. One can check that the $\mathbb{F}$–vector space structures on $D^{\square}_{\overline{\rho}}(\mathbb{F}[\varepsilon])$ and on $\mathrm{Z}^1(\Gamma, \mathrm{ad}(\overline{\rho}))$ agree. If $R^{\square}$ represents $D^{\square}_{\overline{\rho}}$, then this also agrees with $\mathrm{Hom}_{\mathbb{F}}\big(\mathfrak{m}_{R^{\square}}/(\mathfrak{m}^2_{R^{\square}}, p), \mathbb{F}\big)$.

Two lifts $\rho_1 = (1 + \varepsilon c_1)\overline{\rho}$ and $\rho_2 = (1 + \varepsilon c_2)\overline{\rho}$ of $\overline{\rho}$ to $\mathbb{F}[\varepsilon]$ define the same deformation if and only if there exists some $X \in \mathrm{M}_n(\mathbb{F})$ with the property that $\rho_1 = (1 + \varepsilon X)\rho_2(1 - \varepsilon X)$. But this is equivalent to

$$c_1\overline{\rho} = X\overline{\rho} + c_2\overline{\rho} - \overline{\rho}X.$$

Therefore, ρ_1 and ρ_2 define the same deformation if and only if there exists some $X \in \mathrm{M}_n(\mathbb{F})$

$$c_1(\sigma) = c_2(\sigma) + X - \overline{\rho}(\sigma)X\overline{\rho}(\sigma)^{-1} = c_2(\sigma) - (\sigma - 1)X \quad \text{for all } \sigma \in \Gamma,$$

which happens precisely when c_1 and c_2 define the same class in $\mathrm{H}^1(\Gamma, \mathrm{ad}(\overline{\rho}))$.

Proposition 23. *There are isomorphisms of* $\mathbb{F}$*–vector spaces*

$$D^{\square}_{\overline{\rho}}(\mathbb{F}[\varepsilon]) \cong Z^1(\Gamma, \mathrm{ad}(\overline{\rho})) \quad \text{and} \quad D_{\overline{\rho}}(\mathbb{F}[\varepsilon]) \cong \mathrm{H}^1(\Gamma, \mathrm{ad}(\overline{\rho}))$$

Corollary 24. *If* Γ *satisfies the condition* Φ_p*, then* $D_{\overline{\rho}}(\mathbb{F}[\varepsilon])$ *is finite-dimensional over* $\mathbb{F}$.

Proof. Let $H = \mathrm{Ker}(\overline{\rho})$. In the inflation-restriction exact sequence

$$0 \longrightarrow \mathrm{H}^1(\Gamma/H, \mathrm{ad}(\overline{\rho})) \longrightarrow \mathrm{H}^1(\Gamma, \mathrm{ad}(\overline{\rho})) \longrightarrow \mathrm{H}^1(H, \mathrm{ad}(\overline{\rho})),$$

we see that $\mathrm{H}^1(\Gamma/H, \mathrm{ad}(\overline{\rho}))$ is finite because Γ/H is a finite group and the condition Φ_p implies that $\mathrm{H}^1(H, \mathrm{ad}(\overline{\rho})) \cong \mathrm{Hom}^{\mathrm{cont}}(H, \mathbb{F}^{n^2})$ is finite. In conclusion, $\mathrm{H}^1(\Gamma, \mathrm{ad}(\overline{\rho}))$ is finite too. $\square$

Remark. Assume that Γ satisfies the condition Φ_p and that $\mathrm{End}_{\mathbb{F}[\Gamma]}(\overline{\rho}) = \mathbb{F}$ and consider the ring R that represents $D_{\overline{\rho}}$. One can find a presentation of the form

$$R \cong \mathrm{W}(\mathbb{F})[\![X_1, \ldots, X_g]\!]/(f_1, \ldots, f_r),$$

where $g = \dim_{\mathbb{F}} \mathrm{H}^1(\Gamma, \mathrm{ad}(\overline{\rho}))$ and $r = \dim_{\mathbb{F}} \mathrm{H}^2(\Gamma, \mathrm{ad}(\overline{\rho}))$.

Conjecture 25 (Mazur). *Take* $\Gamma = G_{F,S}$ *for a number field* F *and a finite set* S *of places of* F *containing the archimedean places and the primes over* p*. If* $\overline{\rho}$ *is absolutely irreducible and* R *is the ring representing* $D_{\overline{\rho}}$*, then*

$$\dim(R) = 1 + h_1 - h_2, \quad \text{where } h_i = \dim_{\mathbb{F}} \mathrm{H}^i(G_{F,S}, \mathrm{ad}(\overline{\rho})).$$

Remark. For $n = 1$ (i.e., in the case of deformations of characters), this conjecture is equivalent to Leopoldt's conjecture.

3.3 Deformation conditions

Fix $\overline{\rho}\colon \Gamma \to \mathrm{GL}_n(\mathbb{F})$ as before. We want to study subfunctors of $D^{\square}_{\overline{\rho}}$ or $D_{\overline{\rho}}$ (in particular, subfunctors with arithmetic properties). Fix $\Lambda \in \mathrm{Ob}(\mathrm{CNL})$. We often take Λ to be the ring of integers $\mathscr{O}$ of a finite totally ramified extension of $\mathrm{W}(\mathbb{F})[p^{-1}]$.

Example 26 (fixed determinant deformations). Fix a continuous morphism

$$\psi\colon \Gamma \to \mathscr{O}^{\times} \quad \text{such that} \quad \psi \bmod \mathfrak{m}_{\mathscr{O}} = \det(\overline{\rho}).$$

Let $D^{\square,\psi}_{\overline{\rho}}\colon \mathrm{CNL}_{\mathscr{O}} \to \mathrm{Set}$ be the subfunctor of $D^{\square}_{\overline{\rho}}$ of lifts $\rho\colon \Gamma \to \mathrm{GL}_n(A)$ with $\det(\rho) = \psi$ (to be precise, in the right hand side we should compose ψ with the structure morphism $\mathscr{O} \to A$). This condition is invariant under conjugation and so defines also a subfunctor $D_{\overline{\rho}}^{\psi}$ of $D_{\overline{\rho}}$ of deformations with determinant ψ.

Proposition 27.

(1) *The functor $D_{\overline{\rho}}^{\square,\psi}$ is represented by a quotient of the universal lifting ring $R^{\square}$ (which represents $D_{\overline{\rho}}^{\square}$).*

(2) *If $\mathrm{End}_{\mathbb{F}[\Gamma]}(\overline{\rho}) = \mathbb{F}$, then $D_{\overline{\rho}}^{\psi}$ is represented by a quotient of the universal deformation ring R^{univ} (which represents $D_{\overline{\rho}}$).*

Proof. Let $\rho^{\square}\colon \Gamma \to \mathrm{GL}_n(R^{\square})$ be the universal lift of $\overline{\rho}$. Let J be the ideal of $R^{\square}$ generated by

$$\{\, \det(\rho^{\square}(\sigma)) - \psi(\sigma) : \sigma \in \Gamma \,\}.$$

For every lift $\rho\colon \Gamma \to \mathrm{GL}_n(A)$ of $\overline{\rho}$ to $A \in \mathrm{Ob}(\mathrm{CNL}_{\mathscr{O}})$, there exists a unique morphism $\phi\colon R^{\square} \to A$ in $\mathrm{CNL}_{\mathscr{O}}$ with the property that $\rho = \phi \circ \rho^{\square}$. It is easy to see that $\det(\rho) = \psi$ if and only if $\phi(J) = 0$. Therefore, $R^{\square,\psi} = R^{\square}/J$ represents $D_{\overline{\rho}}^{\square,\psi}$. The proof for deformations is essentially the same. □

Let $\mathrm{ad}^0(\overline{\rho})$ denote the subspace of matrices in $\mathrm{ad}(\overline{\rho}) = \mathrm{M}_n(\mathbb{F})$ (i.e., with adjoint Γ–action) that have trace 0.

Proposition 28.

(1) $D_{\overline{\rho}}^{\square,\psi}(\mathbb{F}[\varepsilon]) \cong \mathrm{Z}^1(\Gamma, \mathrm{ad}^0(\overline{\rho}))$.

(2) $D_{\overline{\rho}}^{\psi}(\mathbb{F}[\varepsilon]) \cong \mathrm{Im}\big(\mathrm{H}^1(\Gamma, \mathrm{ad}^0(\overline{\rho})) \to \mathrm{H}^1(\Gamma, \mathrm{ad}(\overline{\rho}))\big)$ *(and the right-hand side is $\cong \mathrm{H}^1(\Gamma, \mathrm{ad}^0(\overline{\rho}))$ if $p \nmid n$).*

Proof. Take a lift $\rho\colon \Gamma \to \mathrm{GL}_n(\mathbb{F}[\varepsilon])$. Write $\rho = (1 + \varepsilon c)\overline{\rho}$ with $c \in \mathrm{Z}^1(\Gamma, \mathrm{ad}(\overline{\rho}))$. We can check that $\det(\rho) = \psi = \det(\overline{\rho})$ if and only if $1 + \varepsilon\,\mathrm{tr}(c) = 1$, which happens precisely when $c \in \mathrm{Z}^1(\Gamma, \mathrm{ad}^0(\overline{\rho}))$.

The statement for deformations now follows using that coboundaries have coefficients in $\mathrm{ad}^0(\overline{\rho})$. □

Remark. We can identify $\mathrm{ad}(\overline{\rho}) \cong \mathfrak{gl}_n$ and $\mathrm{ad}^0(\overline{\rho}) \cong \mathfrak{sl}_n$.

Definition 29. A *deformation condition* or *deformation problem* $\mathscr{D}$ on CNL_{Λ} is a collection of lifts ρ of $\overline{\rho}$ to objects A of CNL_{Λ} satisfying the following properties:

(1) $(\mathbb{F}, \overline{\rho}) \in \mathscr{D}$;

(2) if $(A, \rho) \in \mathscr{D}$ and $\phi\colon A \to B$ is a morphism in CNL_{Λ}, then $(B, \phi \circ \rho) \in \mathscr{D}$;

(3) if $(A, \rho_A), (B, \rho_B) \in \mathscr{D}$ and we have two morphisms $A \to C$ and $B \to C$ in Ar_{Λ}, then $(A \times_C B, \rho_A \times \rho_B) \in \mathscr{D}$;

(4) if $(A_i, \rho_i)_{i \in I}$ is an inverse system in $\mathscr{D}$ and $\varprojlim A_i \in \mathrm{Ob}(\mathrm{CNL}_\Lambda)$, then

$$\left(\varprojlim_{i \in I} A_i, \varprojlim_{i \in I} \rho_i\right) \in \mathscr{D};$$

(5) $\mathscr{D}$ is stable under strict equivalence, and

(6) if $\phi\colon A \to B$ is an injective morphism in CNL_Λ and (A, ρ) is a lift with the property that $(B, \phi \circ \rho) \in \mathscr{D}$, then $(A, \rho) \in \mathscr{D}$.

Proposition 30. *Let* $R^\square$ *be the universal lifting ring on* CNL_Λ *and let* $R^\square \twoheadrightarrow R$ *be a quotient morphism in* CNL_Λ *with the following property: for every lift* $\rho\colon \Gamma \to \mathrm{GL}_n(A)$ *of* $\overline{\rho}$ *to* $A \in \mathrm{Ob}(\mathrm{CNL}_\Lambda)$ *and every* $g \in 1 + \mathrm{M}_n(\mathfrak{m}_A)$*, the morphism* $R^\square \to A$ *corresponding to* ρ *factors through* R *if and only if the morphism corresponding to* $g\rho g^{-1}$ *does. The collection of lifts whose associated morphisms* $R^\square \to \cdot$ *factor through* R *form a deformation problem. Moreover, every deformation problem arises in this way.*

Proof. The first claim is easy to verify. For the second claim, let $\mathscr{D}$ be a deformation problem. Let $\mathscr{I}$ be the set of all ideals I of $R^\square$ such that $(R^\square/I, \rho_I) \in \mathscr{D}$, where ρ_I is the composition of $\rho^\square$ with the canonical morphism $\mathrm{GL}_n(R^\square) \to \mathrm{GL}_n(R^\square/I)$. From the definition of a deformation problem:

- $\mathscr{I} \neq \varnothing$ by condition (1) (as $\mathfrak{m}_{R^\square} \in \mathscr{I}$);
- a lift (A, ρ) is in $\mathscr{D}$ if and only if $\mathrm{Ker}(R^\square \to A) \in \mathscr{I}$ by conditions (2) and (6) (where the morphism $R^\square \to A$ classifies ρ);
- $\mathscr{I}$ is closed under nested intersections by condition (4) and the previous property;
- $\mathscr{I}$ is closed under finite intersections by conditions (3) and (4), and
- $\mathscr{I}$ contains a minimal element J that is contained in every $I \in \mathscr{I}$ by the previous properties and Zorn's lemma.

All in all, we can use $R = R^\square/J$ with the quotient map $R^\square \twoheadrightarrow R$ to recover $\mathscr{D}$. □

Consider a quotient map $R^\square \twoheadrightarrow R = R^\square/J$ corresponding to a deformation problem $\mathscr{D}$. There is a subspace $\mathscr{L}_\mathscr{D} \subset \mathrm{Z}^1(\Gamma, \mathrm{ad}(\overline{\rho}))$ given by the image of the map that makes the diagram

$$\begin{tikzcd}
\mathrm{Hom}_\mathbb{F}(\mathfrak{m}_R/(\mathfrak{m}_R^2, \mathfrak{m}_\Lambda), \mathbb{F}) \arrow[r, phantom, "\cong"] \arrow[d, dashed] & \mathrm{Hom}_\mathbb{F}(\mathfrak{m}_{R^\square}/(\mathfrak{m}_{R^\square}^2, J, \mathfrak{m}_\Lambda), \mathbb{F}) \arrow[d, hook] \\
\mathrm{Z}^1(\Gamma, \mathrm{ad}(\overline{\rho})) \arrow[r, phantom, "\cong"] & \mathrm{Hom}_\mathbb{F}(\mathfrak{m}_{R^\square}/(\mathfrak{m}_{R^\square}^2, \mathfrak{m}_\Lambda), \mathbb{F})
\end{tikzcd}$$

commutative. Let $L_\mathscr{D}$ denote the image of $\mathscr{L}_\mathscr{D}$ in $\mathrm{H}^1(\Gamma, \mathrm{ad}(\overline{\rho}))$, so that we obtain a surjective map $\mathscr{L}_{\mathscr{D}} \twoheadrightarrow L_{\mathscr{D}}$ and

$$\dim_{\mathbb{F}}(\mathscr{L}_{\mathscr{D}}) = \dim_{\mathbb{F}}(L_{\mathscr{D}}) + n^2 - \dim_{\mathbb{F}}\big(\mathrm{H}^0(\Gamma, \mathrm{ad}(\overline{\rho}))\big).$$

Example 31. Let $\overline{\rho}\colon \Gamma \to \mathrm{GL}_2(\mathbb{F})$ be a representation of the form

$$\overline{\rho} = \begin{pmatrix} \overline{\chi}_1 & * \\ 0 & \overline{\chi}_2 \end{pmatrix}$$

and suppose that there is a normal subgroup I of Γ such that

$$\overline{\rho}|_I \neq 1 \quad \text{and} \quad \overline{\chi}_1|_I = 1.$$

Let $\psi\colon I \to \mathscr{O}^\times$ be a lift of $\overline{\chi}_2|_I$. The collection of lifts ρ of $\overline{\rho}$ that are conjugate to a representation of the form

$$\begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$$

with $\chi_1|_I = 1$ and $\chi_2|_I = \psi$ is a deformation problem.

Example 32. Let $\overline{\rho}\colon \Gamma \to \mathrm{GL}_2(\mathbb{F})$ be the trivial representation. The condition from example 31 (that lifts be conjugate to upper triangular representations with a fixed diagonal on I) is not a deformation problem. In particular, if $\Gamma = \widehat{\mathbb{Z}} = \langle \gamma \rangle$, we can consider $\rho_1, \rho_2\colon \Gamma \to \mathrm{GL}_2(\mathbb{F}[\varepsilon])$ defined by

$$\rho_1(\gamma) = \begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \rho_2(\gamma) = \begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \end{pmatrix}$$

and it is impossible to conjugate $\rho_1 \times \rho_2\colon \Gamma \to \mathrm{GL}_2(\mathbb{F}[\varepsilon] \times_{\mathbb{F}} \mathbb{F}[\varepsilon])$ to make it upper triangular.

3.4 Examples of deformation conditions

Let Γ be a profinite group satisfying the condition Φ_p and let I be a normal subgroup of Γ. Let $\mathscr{O}$ denote the ring of integers of some finite extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}$. We consider a continuous homomorphism $\overline{\rho}\colon \Gamma \to \mathrm{GL}_2(\mathbb{F})$ (for the examples, we want $n = 2$).

Example 33 (ordinary deformations). Suppose that we can express

$$\overline{\rho} = \begin{pmatrix} \overline{\chi}_1 & * \\ 0 & \overline{\chi}_2 \end{pmatrix}$$

and that $\overline{\chi}_1|_I = 1$ but $\overline{\rho}|_I \neq 1$. Fix a continuous character $\psi\colon I \to \mathcal{O}^\times$. We consider the functor $D^{\mathrm{ord}}\colon \mathrm{CNL}_{\mathcal{O}} \to \mathrm{Set}$ which sends A to the set of lifts ρ of $\overline{\rho}$ to A that are strictly equivalent to a representation of the form

$$\begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix} \quad \text{with } \chi_1|_I = 1 \text{ and } \chi_2|_I = \psi.$$

We claim that D^{ord} is a deformation problem. Indeed, recall that one way to think about deformation problems is as quotients of the universal lifting ring $R^\square$ that has a certain conjugacy invariance property (see proposition 30). But observe that, if D^{ord} were represented by a quotient of $R^\square$, then it would automatically satisfy the condition in proposition 30. Therefore, it suffices to prove that D^{ord} is represented by a ring $R^{\mathrm{ord}} \in \mathrm{Ob}(\mathrm{CNL}_{\mathcal{O}})$ that is a quotient of $R^\square$.

First suppose that D^{ord} is represented by some $R^{\mathrm{ord}} \in \mathrm{Ob}(\mathrm{CNL}_{\mathcal{O}})$ and let us see that in this case R^{ord} must be a quotient of $R^\square$. Indeed, the inclusion $D^{\mathrm{ord}}(\mathbb{F}[\varepsilon]) \subset D^\square(\mathbb{F}[\varepsilon])$ can be expressed as

$$\mathrm{Hom}_{\mathbb{F}}\big(\mathfrak{m}_{R^{\mathrm{ord}}}/(\mathfrak{m}_{R^{\mathrm{ord}}}^2, \mathfrak{m}_{\mathcal{O}}), \mathbb{F}\big) \hookrightarrow \mathrm{Hom}_{\mathbb{F}}\big(\mathfrak{m}_{R^\square}/(\mathfrak{m}_{R^\square}^2, \mathfrak{m}_{\mathcal{O}}), \mathbb{F}\big)$$

and taking duals we obtain

$$\mathfrak{m}_{R^\square}/(\mathfrak{m}_{R^\square}^2, \mathfrak{m}_{\mathcal{O}}) \twoheadrightarrow \mathfrak{m}_{R^{\mathrm{ord}}}/(\mathfrak{m}_{R^{\mathrm{ord}}}^2, \mathfrak{m}_{\mathcal{O}}).$$

By Nakayama's lemma, we conclude that the natural map $R^\square \to R^{\mathrm{ord}}$ corresponding to the universal lift in $D^{\mathrm{ord}}(R^{\mathrm{ord}}) \subset D^\square(R^{\mathrm{ord}})$ must be surjective.

It only remains to prove that D^{ord} is represented by some ring in $\mathrm{CNL}_{\mathcal{O}}$. To do so, consider the subfunctor $D^{\mathrm{Bor}}\colon \mathrm{CNL}_{\mathcal{O}} \to \mathrm{Set}$ which sends A to the set of lifts ρ of $\overline{\rho}$ to A of the form

$$\rho = \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix} \quad \text{with } \chi_1|_I = 1 \text{ and } \chi_2|_I = \psi.$$

Consider also the functor $L\colon \mathrm{CNL}_{\mathcal{O}} \to \mathrm{Set}$ which sends A to the set of lifts ρ of $\overline{\rho}$ to A of the form

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \quad \text{with } x \in \mathfrak{m}_A.$$

There is a natural transformation $\phi\colon L \times D^{\mathrm{Bor}} \to D^{\mathrm{ord}}$ given by

$$(u, \rho) \mapsto u\rho u^{-1}.$$

We claim that ϕ is an isomorphism. But one can prove that D^{Bor} is represented by a ring $R^{\mathrm{Bor}} \in \mathrm{Ob}(\mathrm{CNL}_{\mathscr{O}})$ (the proof is similar to that of proposition 17) and L is represented by $\mathscr{O}[\![z]\!]$. Therefore, the last claim implies that D^{ord} is represented by

$$R^{\mathrm{ord}} = R^{\mathrm{Bor}} \mathbin{\widehat{\otimes}_{\mathscr{O}}} \mathscr{O}[\![z]\!] \cong R^{\mathrm{Bor}}[\![z]\!].$$

Next we want to prove that, for every $A \in \mathrm{Ob}(\mathrm{CNL}_{\mathscr{O}})$, the map

$$\begin{aligned} \phi_A \colon L(A) \times D^{\mathrm{Bor}}(A) &\longrightarrow D^{\mathrm{ord}}(A) \\ (u, \rho) &\longmapsto u\rho u^{-1} \end{aligned}$$

is a bijection. For the surjectivity, observe that every $g \in 1 + \mathrm{M}_2(\mathfrak{m}_A)$ can be expressed as

$$g = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \qquad \text{with } x, b, 1-a, 1-d \in \mathfrak{m}_A.$$

Thus, if $\rho \in D^{\mathrm{ord}}(A)$ satisfies that

$$g\rho g^{-1} = \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix} \in D^{\mathrm{Bor}}(A),$$

then

$$\left(\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \rho \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^{-1} \right) \in L(A) \times D^{\mathrm{Bor}}(A)$$

is a preimage of ρ. As for the injectivity, suppose that $u_1\rho_1 u_1^{-1} = u_2 \rho_2 u_2^{-1}$. Then $u\rho_1 u^{-1} = \rho_2$, where $u = u_2^{-1} u_1 \in L(A)$. To prove that $u = 1$, it suffices to prove that u is upper triangular. This is a consequence of the following more general fact.

Given $\rho \in D^{\mathrm{Bor}}(A)$ and $g \in 1 + \mathrm{M}_2(\mathfrak{m}_A)$, if $g\rho g^{-1} \in D^{\mathrm{Bor}}(A)$, then g must be upper triangular. Indeed, we can reduce to the case in which A is artinian and then argue by induction on the length of A. Write

$$\rho(\sigma) = \begin{pmatrix} \chi_1(\sigma) & b(\sigma) \\ 0 & \chi_2(\sigma) \end{pmatrix} \qquad \text{for all } \sigma \in \Gamma.$$

Using the decomposition

$$g = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \qquad \text{with } x, b, 1-a, 1-d \in \mathfrak{m}_A,$$

we may conjugate ρ by the second matrix and assume that

$$g = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}.$$

Moreover, by induction, we may assume that $\mathfrak{m}_A^{i+1} = 0$ and that $x \in \mathfrak{m}_A^i$. Then we compute for $\sigma \in I$

$$\begin{aligned} g\rho(\sigma)g^{-1} &= \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \begin{pmatrix} 1 & b(\sigma) \\ 0 & \psi(\sigma) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 - b(\sigma)x & * \\ (1-\psi(\sigma))x & * \end{pmatrix} = \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}, \end{aligned}$$

where the last equality follows from the assumption that $g\rho g^{-1} \in D^{\mathrm{Bor}}(A)$. Since $\overline{\rho}|_I \neq 1$, we can find $\sigma \in I$ such that

$$\text{either } \overline{\rho}(\sigma) = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \text{ with } \alpha \neq 0 \text{ or } \overline{\rho}(\sigma) = \begin{pmatrix} 1 & * \\ 0 & \beta \end{pmatrix} \text{ with } \beta \neq 1.$$

That is, either $b(\sigma) \in A^\times$ or $1 - \psi(\sigma) \in A^\times$. In both cases, $x = 0$ and so g is upper triangular.

A particular case of interest is the following: one might take $\Gamma = G_K$ for some finite extension $K/\mathbb{Q}_p$, $I = I_K$ (the inertia subgroup) and $\psi = \epsilon_p^{1-k}$ for some $k \geq 2$. This is the kind of situation that we encounter when working with representations associated with p–ordinary eigenforms of weight k.

Another variant can be obtained by setting $\Lambda = \mathscr{O}[\![\mathscr{O}_K^\times(p)]\!]$, where $\mathscr{O}_K^\times(p)$ is the maximal pro-p quotient of $\mathscr{O}_K^\times$. Then one might consider $D_\Lambda^{\mathrm{ord}} \colon \mathrm{CNL}_\Lambda \to \mathrm{Set}$ defined analogously to the D^{ord} from above but replacing $\psi \colon I_K \to \mathscr{O}^\times$ with the universal character $\Psi \colon I_K \to \Lambda^\times$ provided by the local class field theory isomorphism $I_K^{\mathrm{ab}} \cong \mathscr{O}_K^\times$. In this way, one can consider deformation problems associated with Hida families.

3.4.1 Minimally ramified lifts

Other cases of interest come up from taking $\Gamma = G_K$ for some finite extension $K/\mathbb{Q}_\ell$ with $\ell \neq p$ and $I = I_K$.

Example 34. Suppose that

$$1 \neq \overline{\rho}(I_K) \subseteq \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}) \right\}.$$

Taking $\psi = 1$, we obtain the deformation problem $D^{\min}\colon \mathrm{CNL}_{\mathscr{O}} \to \mathrm{Set}$ that sends A to the set of lifts ρ of $\overline{\rho}$ to A such that $\rho(I_K)$ is strictly equivalent to a subgroup of

$$\left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(A) \right\},$$

called *minimally ramified lifts* of $\overline{\rho}$.

Example 35. Suppose that

$$\overline{\rho} = \begin{pmatrix} \overline{\chi}_1 & 0 \\ 0 & \overline{\chi}_2 \end{pmatrix} \quad \text{with } \overline{\chi}_1|_{I_K} = 1 \text{ but } \overline{\chi}_2|_{I_K} \neq 1.$$

There is a deformation problem $D^{\min}\colon \mathrm{CNL}_{\mathscr{O}} \to \mathrm{Set}$ that sends A to the set of lift ρ of $\overline{\rho}$ to A which are strictly equivalent to

$$\begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix} \quad \text{with } \chi_1|_{I_K} = 1 \text{ and } \chi_2|_{I_K} = I_K \xrightarrow{\overline{\chi}_2} \mathbb{F}^\times \to \mathscr{O}^\times \to A^\times,$$

called *minimally ramified lifts* of $\overline{\rho}$. To see that $D^{\min}$ is indeed a deformation problem, one can use example 33 with ψ the composition of the Teichmüller character and $\overline{\chi}_2|_{I_K}$ together with some facts about the structure of G_K (especially the tame and wild ramification).

More generally, if $\overline{\rho}\colon G_K \to \mathrm{GL}_2(\mathbb{F})$ satisfies that $\overline{\rho}(I_K)$ has order prime to p, then there is a deformation condition $D^{\min}\colon \mathrm{CNL}_{\mathscr{O}} \to \mathrm{Set}$ that sends A to the set of lifts ρ of $\overline{\rho}$ to A such that the reduction modulo $\mathfrak{m}_A$ induces an isomorphism $\rho(I_K) \cong \overline{\rho}(I_K)$, called *minimally ramified lifts* of $\overline{\rho}$.

3.5 A computation of a local deformation ring

Take a finite extension K of $\mathbb{Q}_p$ and $\Gamma = G_K$. Fix a continuous homomorphism

$$\overline{\rho}\colon G_K \to \mathrm{GL}_2(\mathbb{F})$$

of the form

$$\overline{\rho} = \begin{pmatrix} \overline{\chi}_1 & * \\ 0 & \overline{\chi}_2 \end{pmatrix}$$

such that $\overline{\rho}|_{I_K} \neq 1$ but $\overline{\chi}_1|_{I_K} = 1$. Choose some continuous character

$$\psi\colon I_K \to \mathscr{O}^\times$$

and let $D^{\text{ord}} \colon \mathrm{CNL}_{\mathscr{O}} \to \mathrm{Set}$ be the functor that sends A to the set of lifts ρ of $\overline{\rho}$ to A that are strictly equivalent to a representation of the form

$$\begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix} \text{ with } \chi_1|_{I_K} = 1 \text{ and } \chi_2|_{I_K} = \psi.$$

This functor is represented by a ring R^{ord} in $\mathrm{CNL}_{\mathscr{O}}$ (cf. example 33).

Assume further that $\overline{\chi}_1\overline{\chi}_2^{-1} \neq 1$ or $\overline{\epsilon}_p$. Under these assumptions, we want to prove that $R^{\text{ord}} \cong \mathscr{O}[\![x_1, \dots, x_g]\!]$, where $g = 4 + [K : \mathbb{Q}_p]$.

3.5.1 Two preliminary results

Let ℓ be a prime number and consider a finite extension $L/\mathbb{Q}_\ell$. Let V be a finite $\mathbb{F}$–vector space with a continuous G_L–action. Write V^* for the dual representation to V and $V^*(1) = V^* \otimes_{\mathbb{F}} \overline{\epsilon}_p$.

Theorem 36 (local Tate duality). *For $i \in \{ 0, 1, 2 \}$, we have canonical isomorphisms*

$$\mathrm{H}^i(G_L, V) \cong \mathrm{H}^{2-i}(G_L, V^*(1))^*$$

Theorem 37 (local Euler characteristic). *The Euler characteristic of V is*

$$\sum_{i=0}^{2} (-1)^i \dim_{\mathbb{F}} \mathrm{H}^i(G_L, V) = \begin{cases} 0 & \text{if } \ell \neq p, \\ -[L : \mathbb{Q}_\ell] \dim_{\mathbb{F}}(V) & \text{if } \ell = p. \end{cases}$$

Combining these results, we can compute the dimensions of all the cohomology spaces as long as we can find the dimension of H^0, which is comparatively easier.

Also, when $V = \mathrm{ad}(\overline{\rho})$, the pairing

$$(X, Y) \mapsto \mathrm{tr}(XY)$$

is perfect and Γ–equivariant. In particular, $(\mathrm{ad}(\overline{\rho}))^*(1) = \mathrm{ad}(\rho)(1)$.

3.5.2 The form of R^{ord}

Proposition 38. *Under our assumptions for $\overline{\rho}$, the functor D^{ord} is formally smooth in the sense that, for every $A \in \mathrm{Ob}(\mathrm{Ar}_{\mathscr{O}})$ and every ideal I of A such that $I^2 = 0$, the canonical map*

$$D^{\text{ord}}(A) \to D^{\text{ord}}(A/I)$$

is surjective.

Proof. Arguing by induction on the length of I, we can reduce the proof to the case in which $I = (f)$, $\mathfrak{m}_A I = 0$ and $I \cong \mathbb{F}$ as $\mathscr{O}$–modules.

Fix $\rho' \in D^{\mathrm{ord}}(A/I)$ and assume (up to taking a strict equivalent representation) that

$$\rho' = \begin{pmatrix} \chi_1' & b' \\ 0 & \chi_2' \end{pmatrix} \text{ with } \chi_1'|_{I_K} = 1 \text{ and } \chi_2'|_{I_K} = \psi$$

and $b' \in \mathrm{Z}^1(G_K, (A/I)(\chi_1'(\chi_2')^{-1}))$. We can lift χ_i' to $\chi_i \colon G_K \to A^\times$ by lifting $\chi_i'(\mathrm{Frob}_K)$ (as the action on I_K is determined by the definition of D^{ord}). Then, we just need to lift b' to a cocycle $b \in \mathrm{Z}^1(G_K, A(\chi_1\chi_2^{-1}))$ and

$$\rho = \begin{pmatrix} \chi_1 & b \\ 0 & \chi_2 \end{pmatrix} \in D^{\mathrm{ord}}(A)$$

will be a preimage of ρ'.

We can lift any coboundary easily, so it suffices to show that the natural map

$$\mathrm{H}^1(G_K, A(\chi_1\chi_2^{-1})) \to \mathrm{H}^1(G_K, (A/I)(\chi_1\chi_2^{-1}))$$

is surjective. But the cokernel of this map injects into

$$\mathrm{H}^2(G_K, I(\chi_1\chi_2^{-1})) \cong \mathrm{H}^2(G_K, \mathbb{F}(\overline{\chi}_1\overline{\chi}_2^{-1})) \cong \mathrm{H}^0(G_K, \mathbb{F}(\overline{\chi}_1^{-1}\overline{\chi}_2\overline{\epsilon}_p)) = 0$$

as $\overline{\chi}_1\overline{\chi}_2^{-1} \neq \overline{\epsilon}_p$. (For the first isomorphism, we use the assumption that $I \cong \mathbb{F}$; for the second isomorphism, we use local Tate duality.) □

Using general results of commutative algebra applied to $\mathrm{CNL}_{\mathscr{O}}$ (namely, the fact that a ring that represents a formally smooth functor is formally smooth itself and the structure of formally smooth algebras), this proposition implies the following:

Corollary 39. *The ring R^{ord} is of the form*

$$R^{\mathrm{ord}} \cong \mathscr{O}[\![x_1, \ldots, x_g]\!]$$

for some $g \in \mathbb{Z}_{\geq 1}$.

It remains to compute the value of g.

3.5.3 Computing g

Observe that

$$g = \dim_{\mathbb{F}}\big(\mathfrak{m}_{R^{\mathrm{ord}}}/(\mathfrak{m}_{R^{\mathrm{ord}}}^2, \mathfrak{m}_O)\big) = \dim_{\mathbb{F}}\big(\mathfrak{m}_{R^{\mathrm{ord}}}/(\mathfrak{m}_{R^{\mathrm{ord}}}^2, \mathfrak{m}_O)\big)^* = \dim_{\mathbb{F}} D^{\mathrm{ord}}(\mathbb{F}[\varepsilon]).$$

But D^{ord} is a subfunctor of $D^{\square}$ and we can define

$$\begin{tikzcd}
D^{\mathrm{ord}}(\mathbb{F}[\varepsilon]) \arrow[r, phantom, "\subseteq"] \arrow[d, dashed, "\wr"] & D^{\square}(\mathbb{F}[\varepsilon]) \arrow[d, "\wr"] \\
\mathrm{Z}^1_{\mathrm{ord}}(G_K, \mathrm{ad}(\overline{\rho})) \arrow[r, phantom, "\subseteq"] \arrow[d, dashed, two heads] & \mathrm{Z}^1(G_K, \mathrm{ad}(\overline{\rho})) \arrow[d, two heads] \\
\mathrm{H}^1_{\mathrm{ord}}(G_K, \mathrm{ad}(\overline{\rho})) \arrow[r, phantom, "\subseteq"] & \mathrm{H}^1(G_K, \mathrm{ad}(\overline{\rho}))
\end{tikzcd}$$

and so

$$\begin{aligned}
g &= \dim_{\mathbb{F}} \mathrm{H}^1_{\mathrm{ord}}(G_K, \mathrm{ad}(\overline{\rho})) + \dim_{\mathbb{F}} \mathrm{B}^1(G_K, \mathrm{ad}(\overline{\rho})) \\
&= \dim_{\mathbb{F}} \mathrm{H}^1_{\mathrm{ord}}(G_K, \mathrm{ad}(\overline{\rho})) + 4 - \dim_{\mathbb{F}} \mathrm{H}^0(G_K, \mathrm{ad}(\overline{\rho})) \\
&= \dim_{\mathbb{F}} \mathrm{H}^1_{\mathrm{ord}}(G_K, \mathrm{ad}(\overline{\rho})) + \begin{cases} 3 & \text{if } \overline{\rho} \text{ is non-split,} \\ 2 & \text{if } \overline{\rho} \text{ is split.} \end{cases}
\end{aligned}$$

It remains to compute $\dim_{\mathbb{F}} \mathrm{H}^1_{\mathrm{ord}}(G_K, \mathrm{ad}(\overline{\rho}))$.

Let $\mathfrak{b}$ be the space of upper triangular matrices in $\mathrm{ad}(\overline{\rho})$ and let $\mathfrak{n}$ be the space of nilpotent matrices in $\mathfrak{b}$. Both of these subsets are stable under the (adjoint) action of G_K. Observe that, under the trace pairing,

$$\mathfrak{n}^* \cong \mathrm{ad}(\overline{\rho})/\mathfrak{b}.$$

Consider the composition

$$\phi\colon\ \mathrm{H}^1(G_K, \mathfrak{b}) \longrightarrow \mathrm{H}^1(G_K, \mathfrak{b}/\mathfrak{n}) \xrightarrow{\mathrm{res}} \mathrm{H}^1(I_K, \mathfrak{b}/\mathfrak{n}).$$

Proposition 40. *In the situation above,*

$$\mathrm{H}^1_{\mathrm{ord}}(G_K, \mathrm{ad}(\overline{\rho})) = \mathrm{Im}\big(\mathrm{Ker}(\phi) \subseteq \mathrm{H}^1(G_K, \mathfrak{b}) \to \mathrm{H}^1(G_K, \mathrm{ad}(\overline{\rho}))\big).$$

Proof. Left as an exercise. The idea is that D^{ord} is defined by requiring that representations be (conjugate to) upper triangular, so the cocycles in $\mathrm{H}^1_{\mathrm{ord}}(G_K, \mathrm{ad}(\overline{\rho}))$

should come from $\mathrm{H}^1(G_K, \mathfrak{b})$. On the other hand, the constraints on the restriction to I_K correspond to mapping to 0 in $\mathrm{H}^1(I_K, \mathfrak{b}/\mathfrak{n})$. $\square$

Now observe that

$$\mathrm{H}^0(G_K, \mathrm{ad}(\overline{\rho})/\mathfrak{b}) = \mathrm{H}^0(G_K, \mathbb{F}(\overline{\chi}_1^{-1}\overline{\chi}_2)) = 0$$

because $\overline{\chi}_1 \neq \overline{\chi}_2$, whence we obtain

$$\mathrm{H}^1(G_K, \mathfrak{b}) \hookrightarrow \mathrm{H}^1(G_K, \mathrm{ad}(\overline{\rho}))$$

from the long exact sequence of cohomology. Therefore,

$$\mathrm{H}^1_{\mathrm{ord}}(G_K, \mathrm{ad}(\overline{\rho})) \cong \mathrm{Ker}\big(\phi\colon\ \mathrm{H}^1(G_K, \mathfrak{b}) \to \mathrm{H}^1(I_K, \mathfrak{b}/\mathfrak{n})\big).$$

Taking cohomology of the short exact sequence

$$0 \longrightarrow \mathfrak{n} \longrightarrow \mathfrak{b} \longrightarrow \mathfrak{b}/\mathfrak{n} \longrightarrow 0$$

we obtain

$$\begin{tikzcd}
\mathrm{H}^1(G_K, \mathfrak{b}) \arrow[r] & \mathrm{H}^1(G_K, \mathfrak{b}/\mathfrak{n}) \arrow[r] \arrow[d, equal, "\wr"] & \mathrm{H}^2(G_K, \mathfrak{n}) \arrow[d, equal, "\wr"] \\
 & \mathrm{Hom}^{\mathrm{cont}}(G_K, \mathfrak{b}/\mathfrak{n}) \arrow[d, equal, "\wr"] & \mathrm{H}^0(G_K, (\mathrm{ad}(\overline{\rho})/\mathfrak{b})(1)) \arrow[d, equal, "\wr"] \\
 & \mathrm{Hom}^{\mathrm{cont}}(K^\times, \mathbb{F})^2 & \mathrm{H}^0(G_K, \mathbb{F}(\overline{\chi}_1^{-1}\overline{\chi}_2\overline{\epsilon}_p)) = 0
\end{tikzcd}$$

(where in the last column we have used local Tate duality and that $\overline{\chi}_1\overline{\chi}_2^{-1} \neq \overline{\epsilon}_p$). All in all, ϕ factors as

$$\begin{tikzcd}
\phi\colon\ \mathrm{H}^1(G_K, \mathfrak{b}) \arrow[r, two heads] & \mathrm{H}^1(G_K, \mathfrak{b}/\mathfrak{n}) \arrow[r, "\mathrm{res}"] \arrow[d, equal, "\wr"] & \mathrm{H}^1(I_K, \mathfrak{b}/\mathfrak{n}) \arrow[d, equal, "\wr"] \\
 & \mathrm{Hom}^{\mathrm{cont}}(\pi_K^{\mathbb{Z}} \times \mathscr{O}_K^\times, \mathbb{F})^2 \arrow[r, dashed, "\mathrm{res}"] & \mathrm{Hom}^{\mathrm{cont}}(\mathscr{O}_K^\times, \mathbb{F})^2
\end{tikzcd}$$

and so

$$\begin{aligned}
\dim_{\mathbb{F}} \mathrm{Ker}(\phi) &= \dim_{\mathbb{F}} \mathrm{Ker}\big(\mathrm{res}\colon\ \mathrm{H}^1(G_K, \mathfrak{b}/\mathfrak{n}) \to \mathrm{H}^1(I_K, \mathfrak{b}/\mathfrak{n})\big) + \\
&\qquad + \dim_{\mathbb{F}} \mathrm{Ker}\big(\mathrm{H}^1(G_K, \mathfrak{b}) \to \mathrm{H}^1(G_K, \mathfrak{b}/\mathfrak{n})\big) \\
&= 2 + \dim_{\mathbb{F}} \mathrm{Im}\big(\mathrm{H}^1(G_K, \mathfrak{n}) \to \mathrm{H}^1(G_K, \mathfrak{b})\big) \\
&= 2 + \dim_{\mathbb{F}} \mathrm{H}^1(G_K, \mathfrak{n}) - \dim_{\mathbb{F}} \mathrm{Ker}\big(\mathrm{H}^1(G_K, \mathfrak{n}) \to \mathrm{H}^1(G_K, \mathfrak{b})\big)
\end{aligned}$$

$$
\begin{aligned}
&= 2 + \dim_{\mathbb{F}} \mathrm{H}^1(G_K, \mathfrak{n}) - \dim_{\mathbb{F}} \mathrm{H}^0(G_K, \mathfrak{b}/\mathfrak{n}) + \\
&\qquad + \dim_{\mathbb{F}} \mathrm{Im}\big(\mathrm{H}^0(G_K, \mathfrak{b}) \to \mathrm{H}^0(G_K, \mathfrak{b}/\mathfrak{n})\big) \\
&= \dim_{\mathbb{F}} \mathrm{H}^1(G_K, \mathfrak{n}) + \dim_{\mathbb{F}} \mathrm{Im}\big(\mathrm{H}^0(G_K, \mathfrak{b}) \to \mathrm{H}^0(G_K, \mathfrak{b}/\mathfrak{n})\big) \\
&= \dim_{\mathbb{F}} \mathrm{H}^1(G_K, \mathfrak{n}) + \begin{cases} 1 & \text{if } \overline{\rho} \text{ is non-split,} \\ 2 & \text{if } \overline{\rho} \text{ is split,} \end{cases}
\end{aligned}
$$

where we have used the exact sequence

$$\begin{tikzcd}
\mathrm{H}^0(G_K,\mathfrak{n}) \arrow[r] \arrow[d, phantom, "\shortparallel"] & \mathrm{H}^0(G_K,\mathfrak{b}) \arrow[r] \arrow[d, phantom, "\cong"] & \mathrm{H}^0(G_K,\mathfrak{b}/\mathfrak{n}) \arrow[r] \arrow[d, phantom, "\cong"] & \mathrm{H}^1(G_K,\mathfrak{n}) \arrow[r] & \mathrm{H}^1(G_K,\mathfrak{b}) \arrow[r] & \mathrm{H}^1(G_K,\mathfrak{b}/\mathfrak{n}) \\
0 & \mathbb{F} \text{ or } \mathbb{F}^2 & \mathbb{F}^2 & & &
\end{tikzcd}$$

(the dimension of the second term depends on whether $\overline{\rho}$ is non-split or split).

We can evaluate the dimension of the remaining term using the Euler characteristic formula:

$$\dim_{\mathbb{F}} \mathrm{H}^1(G_K, \mathfrak{n}) = [K : \mathbb{Q}_p] + \dim_{\mathbb{F}} \mathrm{H}^0(G_K, \mathfrak{n}) + \dim_{\mathbb{F}} \mathrm{H}^2(G_K, \mathfrak{n}) = [K : \mathbb{Q}_p].$$

Combining everything, we finally conclude that

$$g = \dim_{\mathbb{F}} \mathrm{Z}^1_{\mathrm{ord}}(G_K, \mathrm{ad}(\overline{\rho})) = 4 + [K : \mathbb{Q}_p].$$

3.5.4 Variants

Proposition 41. *Let L be a finite extension of $\mathbb{Q}_\ell$ with $\ell \neq p$. Consider a continuous homomorphism*

$$\overline{\rho} \colon G_L \to \mathrm{GL}_2(\mathbb{F})$$

satisfying that

(1) *either*

$$1 \neq \overline{\rho}(I_L) \subseteq \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}) \right\}$$

(2) *or $\overline{\rho} = \overline{\chi}_1 \oplus \overline{\chi}_2$ with $\overline{\chi}_1|_{I_L} = 1$ but $\overline{\chi}_2|_{I_L} \neq 1$.*

The minimal deformation problem $D^{\min} \colon \mathrm{CNL}_{\mathscr{O}} \to \mathrm{Set}$ introduced either in example 34 or in example 35 is represented by $R^{\min} \cong \mathscr{O}[\![x_1, x_2, x_3, x_4]\!]$.

Proof. Left as an exercise. It is similar to the previous case that we worked out, but somewhat simpler. □

3.6 Global deformation problems

Fix a number field F and a prime number p. Fix a finite set S of finite places of F containing all primes above p and let F_S be the maximal extension of F unramified outside S and outside the archimedean places. Write $G_{F,S} = \mathrm{Gal}(F_S/F)$.

Let $\mathscr{O}$ be the ring of integers of a finite extension of $\mathbb{Q}_p$ and let $\mathbb{F}$ be its residue field. Fix a continuous homomorphism $\overline{\rho}\colon G_{F,S} \to \mathrm{GL}_n(\mathbb{F})$ (for some $n \in \mathbb{Z}_{\geq 1}$). We assume that $p \nmid 2n$.

We consider the deformation functor $D_{\overline{\rho}}\colon \mathrm{CNL}_{\mathscr{O}} \to \mathrm{Set}$, which is represented by a ring $R_{\overline{\rho}}^{\mathrm{univ}}$ in $\mathrm{CNL}_{\mathscr{O}}$ if $\mathrm{End}_{\mathbb{F}[G_{F,S}]}(\overline{\rho}) = \mathbb{F}$ (see theorem 18).

Next we want to impose additional conditions. Given a place v of F, there is a localization map $D_{\overline{\rho}} \to D_{\overline{\rho}|_{G_{F_v}}}$ given by restriction from G_F to G_{F_v}. Fix a continuous character $\psi\colon G_{F,S} \to \mathscr{O}^\times$ and, for each $v \in S$, a deformation problem $D_v \subseteq D^{\square,\psi}_{\overline{\rho}|_{G_{F_v}}}$.

We will refer to the tuple

$$\mathscr{S} = (\overline{\rho}, S, \psi, \mathscr{O}, (D_v)_{v\in S})$$

as a *global deformation problem*.

Definition 42. We say that a lift ρ of $\overline{\rho}$ to $A \in \mathrm{Ob}(\mathrm{CNL}_{\mathscr{O}})$ is *of type $\mathscr{S}$* if

(1) ρ is unramified outside S,

(2) $\det(\rho) = \psi$ and

(3) $\rho|_{G_{F_v}} \in D_v(A)$ for every $v \in S$.

Similarly, we say that a deformation is *of type $\mathscr{S}$* if one lift (or, equivalently, all lifts) in the corresponding equivalence class is of type $\mathscr{S}$.

Thus, we can define a functor $D_{\mathscr{S}}\colon \mathrm{CNL}_{\mathscr{O}} \to \mathrm{Set}$ that sends A to the set of deformations of $\overline{\rho}$ to A of type $\mathscr{S}$.

Proposition 43. *Let $\mathscr{S}$ be as above. If $\mathrm{End}_{\mathbb{F}[G_{F,S}]}(\overline{\rho}) = \mathbb{F}$, then $D_{\mathscr{S}}$ is represented by a quotient $R_{\mathscr{S}}$ of $R_{\overline{\rho}}^{\mathrm{univ}}$.*

Proof. By proposition 27, the deformation problem obtained after fixing determinants is still representable by a quotient $R_{\overline{\rho}}^{\psi}$ of $R_{\overline{\rho}}^{\mathrm{univ}}$.

Choose a lift ρ in the class of the universal $R_{\overline{\rho}}^{\psi}$–deformation. For every $v \in S$, the restriction $\rho|_{G_{F_v}}$ corresponds to a morphism $R^{\square}_{\overline{\rho}|_{G_{F_v}}} \to R_{\overline{\rho}}^{\psi}$.

On the other hand, let R_v be the quotient of $R^{\square}_{\overline{\rho}|_{G_{F_v}}}$ representing D_v (see proposition 30). Set

$$R_S^{\square} = \bigotimes_{v\in S,\mathscr{O}} R^{\square}_{\overline{\rho}|_{G_{F_v}}} \quad \text{and} \quad R_{\mathscr{S}}^{\mathrm{loc}} = \bigotimes_{v\in S,\mathscr{O}} R_v.$$

Then, the functor $D_{\mathscr{S}}$ is represented by

$$R_{\mathscr{S}} = R^{\psi}_{\overline{\rho}} \mathbin{\widehat{\otimes}}_{R^{\square}_{S}} R^{\mathrm{loc}}_{\mathscr{S}}.$$

This quotient of $R^{\psi}_{\overline{\rho}}$ is independent of the choice of the lift ρ in the class of the universal $R^{\psi}_{\overline{\rho}}$–deformation, as the quotients

$$R^{\square}_{\overline{\rho}|_{G_{F_v}}} \twoheadrightarrow R_v$$

are invariant under strict equivalence classes. □

Remark. With the notation in the proof, observe that $R_{\mathscr{S}}$ is an algebra over $R^{\mathrm{loc}}_{\mathscr{S}}$ but not canonically (because of the choice of ρ). It is useful to have a variant of $R_{\mathscr{S}}$ that is canonically an algebra over $R^{\mathrm{loc}}_{\mathscr{S}}$.

Fix a subset $T \subseteq S$.

Definition 44. A *T–framed lift of $\overline{\rho}$ to $A \in \mathrm{Ob}(\mathrm{CNL}_{\mathscr{O}})$* is a tuple $\big(\rho, (\beta_v)_{v\in T}\big)$, where ρ is a lift of $\overline{\rho}$ to A and $\beta_v \in 1 + \mathrm{M}_n(\mathfrak{m}_A)$ for all $v \in T$. We say that $\big(\rho, (\beta_v)_{v\in T}\big)$ is *of type $\mathscr{S}$* if ρ is.

Definition 45. We say that two T–framed lifts $\big(\rho, (\beta_v)_{v\in T}\big)$ and $\big(\rho', (\beta'_v)_{v\in T}\big)$ of $\overline{\rho}$ to $A \in \mathrm{Ob}(\mathrm{CNL}_{\mathscr{O}})$ are *strictly equivalent* if there exists $g \in 1 + \mathrm{M}_n(\mathfrak{m}_A)$ such that $\rho' = g\rho g^{-1}$ and $\beta'_v = g\beta_v$ for all $v \in T$. A *T–framed deformation of $\overline{\rho}$ to $A \in \mathrm{Ob}(\mathrm{CNL}_{\mathscr{O}})$* is a strict equivalence class of T–framed lifts of $\overline{\rho}$ to A. We say that a T–framed deformation is *of type $\mathscr{S}$* if one lift (equivalently, all lifts) in the corresponding equivalence class is of type $\mathscr{S}$.

Proposition 46.

1. *If $\mathrm{End}_{\mathbb{F}[G_{F,S}]}(\overline{\rho}) = \mathbb{F}$ or $T \neq \emptyset$, then the functor $D^{T}_{\mathscr{S}} \colon \mathrm{CNL}_{\mathscr{O}} \to \mathrm{Set}$ that sends A to the set of T–framed deformations of $\overline{\rho}$ to A of type $\mathscr{S}$ is represented by a ring $R^{T}_{\mathscr{S}} \in \mathrm{Ob}(\mathrm{CNL}_{\mathscr{O}})$.*
2. *If $\mathrm{End}_{\mathbb{F}[G_{F,S}]}(\overline{\rho}) = \mathbb{F}$ and $T \neq \emptyset$, the choice of a lift in the universal deformation of type $\mathscr{S}$ gives rise to an isomorphism*

$$R^{T}_{\mathscr{S}} \cong R_{\mathscr{S}}[\![x_1, \ldots, x_{n^2|T|-1}]\!].$$

Proof. Left as an exercise. The idea for the second part is that $n^2|T|$ is the dimension of the space of choices of $(\beta_v)_{v\in T}$ and the -1 comes from scaling each β_v by the same element in $1 + \mathfrak{m}_A$, which stabilizes ρ. □

3.6.1 Tangent spaces

Assume for simplicity that $\mathrm{End}_{\mathbb{F}[G_{F,S}]}(\overline{\rho}) = \mathbb{F}$, so that $D^T_{\mathscr{S}}$ is represented by a ring $R^T_{\mathscr{S}}$. Observe that, for every T–framed lift $\big(\rho, (\beta_v)_{v\in T}\big)$ in the universal T–framed deformation and every $v \in T$, we have a local lift

$$\beta_v^{-1}\rho|_{G_{F_v}}\beta_v \colon G_{F_v} \to \mathrm{GL}_n(R^T_{\mathscr{S}})$$

that is independent of the choice in the strict equivalence class. Thus, we get a canonical morphism $R_v \to R^T_{\mathscr{S}}$ for every $v \in T$. Putting these together, we obtain a morphism

$$R^{T-\mathrm{loc}}_{\mathscr{S}} \to R^T_{\mathscr{S}} \quad \text{with } R^{T-\mathrm{loc}}_{\mathscr{S}} = \bigotimes_{v\in T,\mathscr{O}} R_v.$$

Next, we want to describe the relative tangent space

$$\mathfrak{m}_{\mathscr{S}}/(\mathfrak{m}^2_{\mathscr{S}}, \mathfrak{m}_{T-\mathrm{loc}}),$$

where $\mathfrak{m}_{\mathscr{S}}$ and $\mathfrak{m}_{T-\mathrm{loc}}$ are the maximal ideal of $R^T_{\mathscr{S}}$ and of $R^{T-\mathrm{loc}}_{\mathscr{S}}$, respectively.

Let M be an $\mathbb{F}[G_{F,S}]$–module of finite dimension over $\mathbb{F}$. We write $C^\bullet(F_S/F, M)$ for the complex of inhomogeneous cochains computing the $G_{F,S}$–cohomology with coefficients in M and $C^\bullet(F_v, M)$ for the complex of inhomogeneous cochains computing the G_{F_v}–cohomology with coefficients in M. We want to consider these complexes for $M = \mathrm{ad}(\overline{\rho})$ or $\mathrm{ad}^0(\overline{\rho})$.

Recall that, for every $v \in S$, we have a deformation problem $D_v \subseteq D^{\square,\psi}_{\overline{\rho}|_{G_{F_v}}}$ which has a corresponding subspace

$$L_v \subseteq \mathrm{H}^1(F_v, \mathrm{ad}^0(\overline{\rho})).$$

Namely, $D_v(\mathbb{F}[\varepsilon])$ corresponds to a subspace $\mathscr{L}_v \subseteq \mathrm{Z}^1(F_v, \mathrm{ad}^0(\overline{\rho}))$ whose image under $\mathrm{Z}^1(F_v, \mathrm{ad}^0(\overline{\rho})) \twoheadrightarrow \mathrm{H}^1(F_v, \mathrm{ad}^0(\overline{\rho}))$ is L_v. Define a complex $C^\bullet_{\mathscr{S},T}(\mathrm{ad}^0(\overline{\rho}))$ as follows:

$$C^i_{\mathscr{S},T}(\mathrm{ad}^0(\overline{\rho})) = \begin{cases} C^0(F_S/F, \mathrm{ad}(\overline{\rho})) & \text{if } i = 0, \\ C^1(F_S/F, \mathrm{ad}^0(\overline{\rho})) \oplus \left(\bigoplus_{v \in T} C^0(F_v, \mathrm{ad}(\overline{\rho}))\right) & \text{if } i = 1, \\ C^2(F_S/F, \mathrm{ad}^0(\overline{\rho})) \oplus \left(\bigoplus_{v \in T} C^1(F_v, \mathrm{ad}^0(\overline{\rho}))\right) \oplus & \\ \qquad \oplus \left(\bigoplus_{v \in S \setminus T} C^1(F_v, \mathrm{ad}^0(\overline{\rho}))/\mathscr{L}_v\right) & \text{if } i = 2, \\ C^i(F_S/F, \mathrm{ad}^0(\overline{\rho})) \oplus \left(\bigoplus_{v \in S} C^{i-1}(F_v, \mathrm{ad}^0(\overline{\rho}))\right) & \text{if } i > 2. \end{cases}$$

with boundary map

$$\big(f, (g_v)_{v \in S}\big) \longmapsto \big(\partial f, (f|_{G_{F_v}} - \partial g_v)_{v \in S}\big).$$

We write $\mathrm{H}^i_{\mathscr{S},T}(\mathrm{ad}^0(\overline{\rho}))$ for the corresponding cohomology groups. Also, write $h^i_{\mathscr{S},T}(\mathrm{ad}^0(\overline{\rho}))$ for the dimension of $\mathrm{H}^i_{\mathscr{S},T}(\mathrm{ad}^0(\overline{\rho}))$ (and define $h^i(F_S/F, \mathrm{ad}^0(\overline{\rho}))$ and $h^i(F_v, \mathrm{ad}^0(\overline{\rho}))$ similarly).

Since we assumed that $p \nmid n$, there is a $G_{F,S}$–equivariant decomposition

$$\mathrm{ad}(\overline{\rho}) = \mathrm{ad}^0(\overline{\rho}) \oplus \mathbb{F}$$

and the pairing

$$(X, Y) \mapsto \mathrm{tr}(XY)$$

on $\mathrm{ad}^0(\overline{\rho})$ is perfect and defines an isomorphism $(\mathrm{ad}^0(\overline{\rho}))^* \cong \mathrm{ad}^0(\overline{\rho})$.

Proposition 47. *There is a canonical isomorphism*

$$\mathrm{Hom}_{\mathbb{F}}\big(\mathfrak{m}_{\mathscr{S}}/(\mathfrak{m}^2_{\mathscr{S}}, \mathfrak{m}_{T-\mathrm{loc}}), \mathbb{F}\big) \cong \mathrm{H}^1_{\mathscr{S},T}(\mathrm{ad}^0(\overline{\rho})).$$

Proof. Take a T–framed lift $(\rho, (\beta_v)_{v \in T})$ of $\overline{\rho}$ to $\mathbb{F}[\varepsilon]$. For this lift to arise from an element of $\mathrm{Hom}_{\mathbb{F}}\big(\mathfrak{m}_{\mathscr{S}}/(\mathfrak{m}^2_{\mathscr{S}}, \mathfrak{m}_{T-\mathrm{loc}}), \mathbb{F}\big)$, we want it to be of type $\mathscr{S}$ and to have trivial restriction at each $v \in T$. More precisely, we want

(i) $\det(\rho) = \psi = \det(\overline{\rho})$,

(ii) $\rho|_{G_{F_v}} = \beta_v \overline{\rho}|_{G_{F_v}} \beta_v^{-1}$ for every $v \in T$ and

(iii) $\rho|_{G_{F_v}} \in D_v(\mathbb{F}[\varepsilon])$ for every $v \in S \setminus T$.

We can express $\rho = (1 + \varepsilon\phi)\overline{\rho}$ with $\phi \in \mathrm{Z}^1(F_S/F, \mathrm{ad}(\overline{\rho}))$ and $\beta_v = 1 + \varepsilon\alpha_v$ with $\alpha_v \in \mathrm{ad}(\overline{\rho}) = \mathrm{C}^0(F_v, \mathrm{ad}(\overline{\rho}))$ for every $v \in T$. The previous conditions can be reformulated as follows:

(i) $\phi \in \mathrm{Z}^1(F_S/F, \mathrm{ad}^0(\overline{\rho}))$,

(ii) for every $v \in T$,

$$\phi|_{G_{F_v}} = \big(\sigma \mapsto \alpha_v - \sigma\alpha_v\sigma^{-1}\big) = \partial\alpha_v,$$

and

(iii) $\phi|_{G_{F_v}} \in \mathscr{L}_v$ for all $v \in S \setminus T$.

But by construction of $C^{\bullet}_{\mathscr{S},T}(\mathrm{ad}^0(\overline{\rho}))$,

$$\partial\big((\phi, (\alpha_v)_{v\in T})\big) = \big(\partial\phi, (\phi|_{G_{F_v}} - \partial\alpha_v)_{v\in T}\big).$$

Therefore, the conditions (i), (ii) and (iii) hold if and only if $\partial\big((\phi, (\alpha_v)_{v\in T})\big) = 0$.

Furthermore, two such cocycles $(\phi, (\alpha_v)_{v\in T})$ and $(\phi', (\alpha'_v)_{v\in T})$ give strictly equivalent lifts if and only if there exists $g = 1 + \varepsilon\alpha$ satisfying that

$$\begin{cases} \phi' = \phi + \partial\alpha, \\ \alpha'_v = \alpha_v + \alpha \qquad \text{for every } v \in T. \end{cases}$$

This happens precisely when

$$\big(\phi' - \phi, (\alpha'_v - \alpha_v)_{v\in T}\big) = \partial\alpha \quad \text{in } C^{\bullet}_{\mathscr{S},T}(\mathrm{ad}^0(\overline{\rho})). \qquad \square$$

Since the complex $C^{\bullet}_{\mathscr{S},T}(\mathrm{ad}^0(\overline{\rho}))$ is defined *almost* as a cone of two other complexes, a diagram chase gives an exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathrm{H}^0_{\mathscr{S},T}(\mathrm{ad}^0(\overline{\rho})) \longrightarrow \mathrm{H}^0(F_S/F, \mathrm{ad}(\overline{\rho})) \longrightarrow \bigoplus_{v\in T} \mathrm{H}^0(F_v, \mathrm{ad}(\overline{\rho})) - \\ &\to \mathrm{H}^1_{\mathscr{S},T}(\mathrm{ad}^0(\overline{\rho})) \longrightarrow \mathrm{H}^1(F_S/F, \mathrm{ad}^0(\overline{\rho})) - \\ &\to \Big(\bigoplus_{v\in T} \mathrm{H}^1(F_v, \mathrm{ad}^0(\overline{\rho}))\Big) \oplus \Big(\bigoplus_{v\in S\setminus T} \mathrm{H}^1(F_v, \mathrm{ad}^0(\overline{\rho}))/L_v\Big) - \\ &\to \mathrm{H}^2_{\mathscr{S},T}(\mathrm{ad}^0(\overline{\rho})) \longrightarrow \mathrm{H}^2(F_S/F, \mathrm{ad}^0(\overline{\rho})) - \\ &\to \bigoplus_{v\in S} \mathrm{H}^2(F_v, \mathrm{ad}^0(\overline{\rho})) \longrightarrow \mathrm{H}^3_{\mathscr{S},T}(\mathrm{ad}^0(\overline{\rho})) \longrightarrow 0 \end{aligned}$$

If we wanted to allow $p = 2$, then the local terms would be more complicated because the archimedean places could appear in the corresponding sequence.

We want to give some formula for $h^1_{\mathscr{S},T}(\mathrm{ad}^0(\overline{\rho}))$. First, we can use the previous exact sequence to compare Euler characteristics:

$$\begin{aligned}\chi_{\mathscr{S},T}(\mathrm{ad}^0(\overline{\rho})) = 1 - |T| + \chi(F_S/F, \mathrm{ad}^0(\overline{\rho})) - \\ - \sum_{v \in S} \chi(F_v, \mathrm{ad}^0(\overline{\rho})) + \sum_{v \in S \setminus T} \big(h^0(F_v, \mathrm{ad}^0(\overline{\rho})) - \dim_{\mathbb{F}}(L_v)\big).\end{aligned}$$

Here, the term $1 - |T|$ comes from the fact that the H^0 groups have coefficients in $\mathrm{ad}(\overline{\rho}) \cong \mathrm{ad}^0(\overline{\rho}) \oplus \mathbb{F}$ instead of in $\mathrm{ad}^0(\overline{\rho})$.

We will be able to obtain formulae for the Euler characteristics $\chi(F_v, \mathrm{ad}^0(\overline{\rho}))$ using theorems 36 and 37. Analogously, for $\chi(F_S/F, \mathrm{ad}^0(\overline{\rho}))$, we can use the following general results:

Theorem 48 (Poitou–Tate). *Let M be a finite $\mathbb{F}$–vector space endowed with a continuous $G_{F,S}$–action and let M^* denote the dual representation. There is an exact sequence*

$$\begin{aligned} 0 \longrightarrow \mathrm{H}^0(F_S/F, M) \longrightarrow \bigoplus_{v \in S \text{ or } v | \infty} \mathrm{H}^0(F_v, M) \longrightarrow \mathrm{H}^2(F_S/F, M^*(1))^* - \\ \to \mathrm{H}^1(F_S/F, M) \longrightarrow \bigoplus_{v \in S} \mathrm{H}^1(F_v, M) \longrightarrow \mathrm{H}^1(F_S/F, M^*(1))^* \\ \to \mathrm{H}^2(F_S/F, M) \longrightarrow \bigoplus_{v \in S} \mathrm{H}^2(F_v, M) \longrightarrow \mathrm{H}^0(F_S/F, M^*(1))^* \longrightarrow 0. \end{aligned}$$

(In general, we should include the infinite places everywhere, but with the assumption $p \neq 2$ only the corresponding H^0 groups are non-trivial.)

Theorem 49 (global Euler characteristic). *With the same notation as in theorem 48, we have*

$$\chi(F_S/F, M) = -[F : \mathbb{Q}] \dim_{\mathbb{F}}(M) + \sum_{v | \infty} h^0(F_v, M).$$

We will apply the previous theorems to $M = \mathrm{ad}^0(\overline{\rho})$, noting that $M^* = M$ in this case.

Let $L_v^\perp$ be the orthogonal complement of L_v in $\mathrm{H}^1(F_v, \mathrm{ad}^0(\overline{\rho})(1))$ under local Tate duality (see theorem 36) and define

$$\mathrm{H}^1_{\mathscr{S}^\perp, T}(\mathrm{ad}^0(\overline{\rho})(1)) = \mathrm{Ker}\Big(\mathrm{H}^1(F_S/F, \mathrm{ad}^0(\overline{\rho})(1)) \to \bigoplus_{v \in S \setminus T} \mathrm{H}^1(F_v, \mathrm{ad}^0(\overline{\rho})(1))/L_v^\perp\Big).$$

By theorem 48, we obtain an exact sequence

$$\mathrm{H}^1(F_S/F, \mathrm{ad}^0(\overline{\rho})) \longrightarrow \Big(\bigoplus_{v \in T} \mathrm{H}^1(F_v, \mathrm{ad}^0(\overline{\rho}))\Big) \oplus \Big(\bigoplus_{v \in S \setminus T} \mathrm{H}^1(F_v, \mathrm{ad}^0(\overline{\rho}))/L_v\Big) -$$

$$\to \mathrm{H}^1_{\mathscr{S}^\perp,T}(\mathrm{ad}^0(\overline{\rho})(1))^* \longrightarrow \mathrm{H}^2(F_S/F, \mathrm{ad}^0(\overline{\rho})) \longrightarrow \bigoplus_{v\in S} \mathrm{H}^2(F_v, \mathrm{ad}^0(\overline{\rho})) -$$
$$\to \mathrm{H}^0(F_S/F, \mathrm{ad}^0(\overline{\rho})(1)) \longrightarrow 0$$

that we can compare to the original exact sequence to deduce that

$$h^2_{\mathscr{S},T}(\mathrm{ad}^0(\overline{\rho})) = h^1_{\mathscr{S}^\perp,T}(\mathrm{ad}^0(\overline{\rho})(1))$$

and

$$h^3_{\mathscr{S},T}(\mathrm{ad}^0(\overline{\rho})) = h^0(F_S/F, \mathrm{ad}^0(\overline{\rho})(1)).$$

Now we can use the local and the global Euler characteristic formulae (see theorems 37 and 49) and the fact that S contains all primes above p to compute

$$\chi_{\mathscr{S},T}(\mathrm{ad}^0(\overline{\rho})) = 1 - |T| + \sum_{v|\infty} h^0(F_v, \mathrm{ad}^0(\overline{\rho})) + \sum_{v\in S\setminus T} \big(h^0(F_v, \mathrm{ad}^0(\overline{\rho})) - \dim_{\mathbb{F}}(L_v)\big).$$

In conclusion, knowing $h^2_{\mathscr{S},T}$, $h^3_{\mathscr{S},T}$ and $\chi_{\mathscr{S},T}$, we can compute $h^0_{\mathscr{S},T}$ (easy) and obtain the following result:

Theorem 50 (Greenberg–Wiles formula). *In the situation above, we have*

$$\begin{aligned} h^1_{\mathscr{S},T}(\mathrm{ad}^0(\overline{\rho})) = h^1_{\mathscr{S}^\perp,T}(\mathrm{ad}^0(\overline{\rho})(1)) &+ \sum_{v\in S\setminus T} \big(\dim_{\mathbb{F}}(L_v) - h^0(F_v, \mathrm{ad}^0(\overline{\rho}))\big) - \\ &- \sum_{v|\infty} h^0(F_v, \mathrm{ad}^0(\overline{\rho})) - h^0(F_S/F, \mathrm{ad}^0(\overline{\rho})(1)) + \\ &+ \begin{cases} |T| - 1 & \text{if } T \neq \emptyset, \\ 0 & \text{if } T = \emptyset. \end{cases} \end{aligned}$$

3.7 Taylor–Wiles primes

Fix again a global deformation problem

$$\mathscr{S} = (\overline{\rho}, S, \psi, \mathscr{O}, (D_v)_{v\in S}),$$

where $\overline{\rho}\colon G_{F,S} \to \mathrm{GL}_2(\mathbb{F})$ (with the same notation as in section 3.6). Assume, for simplicity, that $p > 2$.

Definition 51. A *Taylor–Wiles prime (for $\mathscr{S}$)* is a (finite) prime v of F such that

(1) $v \notin S$,

(2) $q_v = \mathrm{N}(v) \equiv 1 \bmod p$ and

(3) $\overline{\rho}(\mathrm{Frob}_v)$ has distinct $\mathbb{F}$–rational eigenvalues.

We say that a Taylor–Wiles prime v *has level* $N \in \mathbb{Z}_{\geq 1}$ if, moreover,

$$q_v \equiv 1 \mod p^N.$$

Remarks.

(1) Up to enlarging $\mathbb{F}$, we can (and do) assume that all eigenvalues of all the elements in $\overline{\rho}(G_{F,S})$ are defined over $\mathbb{F}$.

(2) In ranks higher than 2, the generalization of condition (3) varies depending on the context.

Proposition 52. *Let v be a Taylor–Wiles prime. For every $A \in \mathrm{Ob}(\mathrm{CNL}_{\mathscr{O}})$, every lift $\rho\colon G_{F_v} \to \mathrm{GL}_2(A)$ of $\overline{\rho}|_{G_{F_v}}$ is strictly equivalent to a diagonal lift*

$$\begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix}.$$

Proof. We can reduce to the case where A is artinian by taking the limit of quotients by powers of $\mathfrak{m}_A$. Then, fix a lift $\phi \in G_{F_v}$ of $\mathrm{Frob}_v \in G_{F_v}/I_{F_v}$. Since $\overline{\rho}(\mathrm{Frob}_v)$ has distinct eigenvalues in $\mathbb{F}^\times$, we can find a basis for ρ such that

$$\rho(\phi) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \quad \text{for some } \alpha, \beta \in A.$$

Since $\overline{\rho}(I_{F_v}) = 1$, we have $\rho(I_{F_v}) \subseteq 1 + \mathrm{M}_2(\mathfrak{m}_A)$. In particular, $\rho(I_{F_v})$ is a pro-p group and $\rho|_{I_{F_v}}$ must factor through the tame inertia quotient. Fix a topological generator t for the tame inertia. It suffices to prove that, in the basis fixed above, $\rho(t)$ is also diagonal.

We argue by induction on the length of A. The base case is trivial because $\overline{\rho}$ is unramified outside S. For the inductive step, we can assume that $\mathfrak{m}_A^{n+1} = 0$ and

$$\rho(t) = 1 + X \in 1 + \mathrm{M}_2(\mathfrak{m}_A) \quad \text{with } X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } b, c \in \mathfrak{m}_A^n.$$

It is easy to check that X^k is diagonal for all $k \in \mathbb{Z}_{\geq 2}$. But we know that $\phi^{-1} t \phi = t^{q_v}$ and we can compute

$$\begin{aligned} 0 &= \rho(\phi)^{-1}\rho(t)\rho(\phi) - \rho(t)^{q_v} \\ &= \left[1 + \begin{pmatrix} a & \alpha^{-1}\beta b \\ \alpha\beta^{-1}c & d \end{pmatrix}\right] - \left[1 + q_v \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}\right] \end{aligned}$$

$$= \begin{pmatrix} 0 & (\alpha^{-1}\beta - 1)b \\ (\alpha\beta^{-1} - 1)c & 0 \end{pmatrix} + \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix},$$

where in the last equality we used that $(q_v - 1)b = 0 = (q_v - 1)c$ because, by the assumptions on v, $p \mid (q_v - 1)$ and so $(q_v - 1) \in \mathfrak{m}_A$. But $(\alpha\beta^{-1} - 1)$ and $(\alpha^{-1}\beta - 1)$ are units in A, as $\alpha \not\equiv \beta \bmod \mathfrak{m}_A$. Therefore, $b = c = 0$. $\square$

Let v be a Taylor–Wiles prime for $\mathscr{S}$ and let $R_v^{\square,\psi}$ denote the universal lifting ring for $\overline{\rho}|_{G_{F_v}}$ with fixed determinant ψ. Let ρ^{ψ} be the universal lift corresponding to the identity on $R_v^{\square,\psi}$. By proposition 52, the lift ρ^{ψ} is strictly equivalent to a lift of the form

$$\begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix} \quad \text{with } \chi_1, \chi_2 \colon G_{F_v} \to (R_v^{\square,\psi})^{\times} \text{ such that } \chi_1\chi_2 = \psi.$$

In particular, since ψ is unramified at v,

$$\chi_1|_{I_{F_v}} = \chi_2^{-1}|_{I_{F_v}}.$$

But $\overline{\rho}$ is unramified at v, so $\chi_1|_{I_{F_v}}$ must be a pro-p character of

$$I_{F_v}^{\mathrm{ab}} \cong \mathscr{O}_{F_v}^{\times} \cong \kappa_v^{\times} \times \mathbb{Z}_s^d \times G_v,$$

where κ_v is the residue field of F at v, s is the characteristic of κ_v and G_v is a finite s–group. Therefore, $\chi_1|_{I_{F_v}}$ must factor through $\kappa_v^{\times}$ (as $p \nmid v$). Let Δ_v be the maximal p–power quotient of $\kappa_v^{\times}$ and write $\mathscr{O}[\Delta_v]$ for the associated group algebra over $\mathscr{O}$ and $\mathfrak{a}_v$ for the augmentation ideal of $\mathscr{O}[\Delta_v]$. The character $\chi_1|_{I_{F_v}}$ determines an $\mathscr{O}[\Delta_v]$–algebra structure on $R_v^{\square,\psi}$. Moreover, there exists a natural surjection

$$R_v^{\square,\psi} \twoheadrightarrow R_v^{\mathrm{ur},\psi},$$

where $R_v^{\mathrm{ur},\psi}$ represents the lifts of $\overline{\rho}|_{G_{F_v}}$ that are unramified at v and with determinant ψ, whose kernel is precisely $\mathfrak{a}_v R_v^{\square,\psi}$ because a lift $\rho \colon G_{F,S} \to \mathrm{GL}_2(A)$ is unramified if and only if the corresponding map $\phi \colon R_v^{\square,\psi} \to A$ satisfies that the composition

$$\begin{tikzcd} I_{F_v} \arrow[r, "\chi_1"] \arrow[dr, dashed] & \left(R_v^{\square,\psi}\right)^{\times} \arrow[r, "\phi"] & A^{\times} \\ & \Delta_v \arrow[u, dashed] & \end{tikzcd}$$

is trivial (or, equivalently, ϕ factors through $R_v^{\square,\psi}/\mathfrak{a}_v R_v^{\square,\psi}$). All in all, we obtain an isomorphism

$$R_v^{\square,\psi}/\mathfrak{a}_v R_v^{\square,\psi} \cong R_v^{\mathrm{ur},\psi}.$$

Let Q be a finite set of Taylor–Wiles primes. Define

$$\Delta_Q = \prod_{v \in Q} \Delta_v$$

and consider the group algebra $\mathscr{O}[\Delta_Q]$ and its augmentation ideal $\mathfrak{a}_Q$. We define the global deformation problem

$$\mathscr{S}_Q = \left(\overline{\rho}, S \cup Q, \psi, \mathscr{O}, (D_v)_{v \in S} \cup (D_v^{\psi})_{v \in Q}\right),$$

where, for each $v \in Q$, D_v^{ψ} is the deformation condition of all lifts of $\overline{\rho}|_{G_{F_v}}$ with determinant $\psi|_{G_{F_v}}$. (The idea is that sometimes we want to enlarge S.) Assume that $\mathrm{End}_{\mathbb{F}[G_{F,S}]}(\overline{\rho}) = \mathbb{F}$, so that there exist universal rings $R_{\mathscr{S}_Q}$ and $R_{\mathscr{S}}$ representing the deformation problems $\mathscr{S}_Q$ and $\mathscr{S}$. For every $T \subseteq S$, we also have rings $R_{\mathscr{S}_Q}^T$ and $R_{\mathscr{S}}^T$ representing the corresponding T–framed problems. Observe that, applying the construction of the previous paragraph to each $v \in Q$, the ring $R_{\mathscr{S}_Q}^T$ has the structure of an $\mathscr{O}[\Delta_Q]$–algebra and the natural surjection $R_{\mathscr{S}_Q}^T \twoheadrightarrow R_{\mathscr{S}}^T$ has kernel $\mathfrak{a}_Q R_{\mathscr{S}_Q}^T$.

Recall that, for every (possibly empty) $T \subseteq S$, the tangent space of $R_{\mathscr{S}}^T$ (relative to $R_{\mathscr{S}}^{T-\mathrm{loc}}$) is given by a cohomology group $\mathrm{H}^1_{\mathscr{S},T}(\mathrm{ad}^0(\overline{\rho}))$ whose dimension is

$$\begin{aligned} h^1_{\mathscr{S},T}(\mathrm{ad}^0(\overline{\rho})) = h^1_{\mathscr{S}^{\perp},T}(\mathrm{ad}^0(\overline{\rho})(1)) &+ \sum_{v \in S \setminus T} \left(\dim_{\mathbb{F}}(L_v) - h^0(F_v, \mathrm{ad}^0(\overline{\rho}))\right) - \\ &- \sum_{v \mid \infty} h^0(F_v, \mathrm{ad}^0(\overline{\rho})) - h^0(F_S/F, \mathrm{ad}^0(\overline{\rho})(1)) + \\ &+ \begin{cases} |T| - 1 & \text{if } T \neq \emptyset, \\ 0 & \text{if } T = \emptyset, \end{cases} \end{aligned}$$

by theorem 50 (see section 3.6.1 for all the notation). Now add the following assumptions:

(1) $\overline{\rho}|_{G_{F(\zeta_p)}}$ is absolutely irreducible, which implies that there are no non-scalar $G_{F,S}$–equivariant morphisms $\overline{\rho} \to \overline{\rho}(1)$ and so $\mathrm{H}^0(F_S/F, \mathrm{ad}^0(\overline{\rho})(1)) = 0$;

(2) F is totally real and $\det(\overline{\rho}(c_v)) = -1$ for all places $v \mid \infty$, where c_v denotes the complex conjugation at v, which implies that $h^0(F_v, \mathrm{ad}^0(\overline{\rho})) = 1$;

(3)
  - for every $v \in S \setminus T$ such that $v \mid p$,

$$\dim_{\mathbb{F}}(L_v) - h^0(F_v, \mathrm{ad}^0(\overline{\rho})) = [F_v : \mathbb{Q}_p]$$

(e.g., this holds if

$$\overline{\rho}|_{G_{F_v}} \cong \begin{pmatrix} \overline{\chi}_1 & * \\ 0 & \overline{\chi}_2 \end{pmatrix} \quad \text{with } \overline{\chi}_1|_{I_{F_v}} = 1 \text{ and } \overline{\chi}_2|_{I_{F_v}} \neq 1$$

and $D_v = D_v^{\mathrm{ord},\psi}$ is analogous to example 33 adding the condition of fixed determinant ψ);

- for every $v \in T$ such that $v \mid p$, the ring R_v is $\mathscr{O}$–flat of relative dimension $3 + [F_v : \mathbb{Q}_p]$ over $\mathscr{O}$ (so that $\dim(R_v) = 4 + [F_v : \mathbb{Q}_p]$);

(4)
- for every $v \in S \setminus T$ such that $v \nmid p$,

$$\dim_{\mathbb{F}}(L_v) - h^0(F_v, \mathrm{ad}^0(\overline{\rho})) = 0$$

(e.g., this holds if

$$\overline{\rho}|_{I_{F_v}} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \neq 1$$

or

$$\overline{\rho}|_{G_{F_v}} = \begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix} \quad \text{with } \chi_1|_{I_{F_v}} = 1 \text{ and } \chi_2|_{I_{F_v}} \neq 1$$

and $D_v = D_v^{\min,\psi}$ is analogous to examples 34 and 35, respectively, adding the condition of fixed determinant ψ), and

- for every $v \in T$ such that $v \nmid p$, the ring R_v is $\mathscr{O}$–flat of relative dimension 3 over $\mathscr{O}$ (so that $\dim(R_v) = 4$).

Remark. In applications, the conditions appearing in (3) and (4) for the places $v \in T$ are essentially always true, while the conditions for the places $v \in S \setminus T$ hold if and only if D_v (equivalently R_v) is formally smooth over $\mathscr{O}$.

Under these additional assumptions, the formula becomes simpler:

$$h^1_{\mathscr{S},T}(\mathrm{ad}^0(\overline{\rho})) = h^1_{\mathscr{S}^\perp,T}(\mathrm{ad}^0(\overline{\rho})(1)) - \sum_{\substack{v \in T \\ v \mid p}} [F_v : \mathbb{Q}_p] + \begin{cases} |T| - 1 & \text{if } T \neq \emptyset, \\ 0 & \text{if } T = \emptyset. \end{cases}$$

In particular, if $T = \emptyset$, then

$$h^1_{\mathscr{S}}(\mathrm{ad}^0(\overline{\rho})) = h^1_{\mathscr{S}^\perp}(\mathrm{ad}^0(\overline{\rho})(1)).$$

Similarly, if T contains all the primes of F lying over p,

$$h^1_{\mathscr{S},T}(\mathrm{ad}^0(\overline{\rho})) = |T| - 1 - [F : \mathbb{Q}] + h^1_{\mathscr{S}^\perp,T}(\mathrm{ad}^0(\overline{\rho})(1)).$$

One can show that $\dim(R_{\mathscr{S}}^{T-\mathrm{loc}}) = 1 + 3|T| + [F : \mathbb{Q}]$ and so

$$\dim(R_{\mathscr{S}}^{T-\mathrm{loc}}) + h^1_{\mathscr{S},T}(\mathrm{ad}^0(\overline{\rho})) = h^1_{\mathscr{S}^\perp,T}(\mathrm{ad}^0(\overline{\rho})(1)) + 4|T|.$$

(One should interpret this $4|T|$ as the sum of $4|T| - 1$, the relative dimension of $R^T_{\mathscr{S}}$ over $R_{\mathscr{S}}$, and 1, coming from $\mathscr{O}$.)

Now let Q be a finite set of Taylor–Wiles primes for $\mathscr{S}$. Recall that we defined a global deformation problem

$$\mathscr{S}_Q = \big(\overline{\rho}, S \cup Q, \psi, (D_v)_{v \in S} \cup (D_v^\psi)_{v \in Q}\big).$$

We want to study how the formulae for $h^1_{\mathscr{S},T}(\mathrm{ad}^0(\overline{\rho}))$ in the two cases $T = \varnothing$ or $T \supseteq \{\, v \mid p \,\}$ changes if we replace $\mathscr{S}$ with $\mathscr{S}_Q$.

- Observe that $h^1_{\mathscr{S}^\perp,T}(\mathrm{ad}^0(\overline{\rho})(1))$ gets replaced by $h^1_{\mathscr{S}_Q^\perp,T}(\mathrm{ad}^0(\overline{\rho})(1))$. But, for $v \in Q$, D_v^ψ contains all lifts with determinant $\psi|_{G_{F_v}}$ and so

  $$L_v = \mathrm{H}^1(F_v, \mathrm{ad}^0(\overline{\rho})) \quad \text{and} \quad L_v^\perp = 0.$$

  Therefore,

  $$\begin{aligned}\mathrm{H}^1_{\mathscr{S}_Q^\perp,T}(\mathrm{ad}^0(\overline{\rho})) = \mathrm{Ker}\Bigg(&\mathrm{H}^1(F_{S\cup Q}/F, \mathrm{ad}^0(\overline{\rho})(1)) \longrightarrow \\ &\to \Big(\bigoplus_{v \in S\setminus T} \mathrm{H}^1(F_v, \mathrm{ad}^0(\overline{\rho})(1))/L_v^\perp\Big) \oplus \Big(\bigoplus_{v \in Q} \mathrm{H}^1(F_v, \mathrm{ad}^0(\overline{\rho})(1))\Big)\Bigg) \\ = \mathrm{Ker}\Bigg(&\mathrm{H}^1_{\mathscr{S}^\perp,T}(\mathrm{ad}^0(\overline{\rho})(1)) \longrightarrow \bigoplus_{v \in Q} \mathrm{H}^1(F_v, \mathrm{ad}^0(\overline{\rho})(1))\Bigg).\end{aligned}$$

- We have to add

  $$\begin{aligned}\sum_{v \in Q} \big(\dim_{\mathbb{F}}(L_v) - h^0(F_v, \mathrm{ad}^0(\overline{\rho}))\big) &= \sum_{v \in Q} \big(h^1(F_v, \mathrm{ad}^0(\overline{\rho})) - h^0(F_v, \mathrm{ad}^0(\overline{\rho}))\big) \\ &= \sum_{v \in Q} h^2(F_v, \mathrm{ad}^0(\overline{\rho})) \\ &= \sum_{v \in Q} h^0(F_v, \mathrm{ad}^0(\overline{\rho})(1)) \\ &= \sum_{v \in Q} h^0(F_v, \mathrm{ad}^0(\overline{\rho})) = |Q|,\end{aligned}$$

  where we have used theorems 36 and 37 and then at the end the defining properties of Taylor–Wiles primes, namely that $q_v \equiv 1 \bmod p$ and that $\overline{\rho}(\mathrm{Frob}_v)$ has distinct eigenvalues.

Our next goal is to get $h^1_{\mathscr{S}_Q^\perp,T}(\mathrm{ad}^0(\overline{\rho})(1)) = 0$ by adding (at least) $h^1_{\mathscr{S}^\perp,T}(\mathrm{ad}^0(\overline{\rho})(1))$ Taylor–Wiles primes. But this number (of primes) depends only on the original data. In that case, we will be able to obtain a formula for $h^1_{\mathscr{S}_Q,T}(\mathrm{ad}^0(\overline{\rho}))$ depending only on $\mathscr{S}$.

Definition 53. Let Γ be a subgroup of $\mathrm{GL}_2(\mathbb{F})$ acting absolutely irreducibly on $\mathbb{F}^2$ and such that all the eigenvalues of elements of Γ are defined over $\mathbb{F}$. Let ad^0 denote the space of matrices of trace 0 in $\mathrm{M}_2(\mathbb{F})$ with adjoint Γ–action. We say that Γ is *adequate* or *big* or *enormous* if

(1) Γ has no quotient of order p,
(2) $\mathrm{H}^0(\Gamma, \mathrm{ad}^0) = 0 = \mathrm{H}^1(\Gamma, \mathrm{ad}^0)$ and
(3) for every simple $\mathbb{F}[\Gamma]$–submodule W of ad^0, there exists $\gamma \in \Gamma$ with distinct eigenvalues such that $W^\gamma \neq 0$ (i.e., there are non-trivial elements of W that are invariant under γ).

Remark. In rank 2, the notions of adequate, big and enormous coincide. However, in higher ranks, the definition above is only the definition of enormous. The definition of big is obtained by replacing "with distinct eigenvalues" with "semisimple with an eigenvalue of multiplicity 1" in condition (3); the definition of adequate is obtained by replacing "with distinct eigenvalues" with "semisimple" and $W^\gamma \neq 0$ with something more technical in condition (3).

Theorem 54. *With the assumptions that Γ acts absolutely irreducibly on $\mathbb{F}^2$ and that $p > 2$, the group Γ is enormous unless*

- *either $p = 3$ and the image of Γ in $\mathrm{PGL}_2(\overline{\mathbb{F}}_3)$ is conjugate to $\mathrm{PSL}_2(\mathbb{F}_3)$*
- *or $p = 5$ and the image of Γ in $\mathrm{PGL}_2(\overline{\mathbb{F}}_5)$ is conjugate to $\mathrm{PSL}_2(\mathbb{F}_5)$.*

Remark. The first case fails because $\mathrm{PSL}_2(\mathbb{F}_3) \cong A_4$ has a quotient of order 3. The second case fails because $\mathrm{H}^1(\Gamma, \mathrm{ad}^0) \neq 0$. This last case is more delicate and so must often be avoided.

Proposition 55. *Suppose that $\Gamma = \overline{\rho}(G_{F(\zeta_p)})$ is enormous. Let $q = h^1_{\mathscr{S}^\perp,T}(\mathrm{ad}^0(\overline{\rho})(1))$. For every $N \in \mathbb{Z}_{\geq 1}$, we can find a set Q_N of Taylor–Wiles primes of level N such that*

(1) *$|Q_N| = q$ and*
(2) *$\mathrm{H}^1_{\mathscr{S}_{Q_N}^\perp,T}(\mathrm{ad}^0(\overline{\rho})(1)) = 0$.*

Proof. Fix $N \in \mathbb{Z}_{\geq 1}$. Take Taylor–Wiles primes $Q = \{ v_1, \dots, v_{j-1} \}$ of level N for some $j \in \{ 1, \dots, q \}$ such that

$$h^1_{\mathscr{S}_Q^\perp, T}(\mathrm{ad}^0(\overline{\rho})(1)) = q - (j-1).$$

We show how to find another Taylor–Wiles prime v_j of level N such that, setting $Q' = Q \cup v_j$,

$$h^1_{\mathscr{S}_{Q'}^\perp, T}(\mathrm{ad}^0(\overline{\rho})(1)) = q - j.$$

Thus, iterating this construction, we obtain the proposition.

Fix a class $[\kappa] \in \mathrm{H}^1_{\mathscr{S}_Q^\perp, T}(\mathrm{ad}^0(\overline{\rho})(1)) \setminus \{ 0 \}$, where κ is a 1–cocycle. It suffices to show that there exist infinitely many primes v of F such that

(a) $v \notin S$,

(b) $q_v \equiv 1 \bmod p^N$,

(c) $\overline{\rho}(\mathrm{Frob}_v)$ has distinct eigenvalues (defined in $\mathbb{F}$) and

(d) the restriction map res_v induces an isomorphism

$$\mathbb{F}[\kappa] \cong \mathrm{H}^1(F_v^{\mathrm{ur}}/F_v, \mathrm{ad}^0(\overline{\rho})(1)).$$

If v satisfies the first 3 conditions, then

$$\mathrm{H}^1(F_v^{\mathrm{ur}}/F_v, \mathrm{ad}^0(\overline{\rho})(1)) \cong \mathrm{ad}^0(\overline{\rho})/(\mathrm{Frob}_v - 1)\mathrm{ad}^0(\overline{\rho})$$

via $[\phi] \mapsto \phi(\mathrm{Frob}_v)$ and the right-hand side has dimension 1. (In particular, we could get rid of the Tate twist on the right-hand side thanks to condition (b).) Therefore, we can replace condition (d) with

(d′) $\mathrm{res}_v(\kappa) \notin (\mathrm{Frob}_v - 1)\mathrm{ad}^0(\overline{\rho})$.

By Chebotarev's density theorem, it suffices to show that there is $\sigma \in G_{F,S}$ such that

(i) $\sigma \in G_{F(\zeta_p^N)}$,

(ii) $\overline{\rho}(\sigma)$ has distinct eigenvalues and

(iii) $\kappa(\sigma) \notin (\sigma - 1)\mathrm{ad}^0(\overline{\rho})$.

Indeed, these conditions are open (in the sense that they will be true for an open neighbourhood of σ) and then there will be a positive density of primes v satisfying the conditions from above.

Let L_1 be the extension of $F_1 = F(\zeta_p)$ cut out by $\overline{\rho}|_{G_{F(\zeta_p)}}$. Consider also the extensions $F_N = F(\zeta_{p^N})$ and $L_N = L_1 \cdot F_N$ (where $\cdot$ means the compositum). We obtain a diagram

$$\begin{tikzcd}
 & L_N \arrow[dl, dash] \arrow[dr, dash, "\Gamma"] & \\
L_1 \arrow[dr, dash, "\Gamma"'] & & F_N \arrow[dl, dash] \\
 & F_1 \arrow[d, dash] & \\
 & F &
\end{tikzcd}$$

using the first property in definition 53 to see that $L_1 \cap F_N = F_1$. We claim that $\mathrm{H}^1(L_N/F, \mathrm{ad}^0(\overline{\rho})(1)) = 0$. Indeed, in the inflation-restriction exact sequence

$$0 \longrightarrow \mathrm{H}^1\big(F_N/F, (\mathrm{ad}^0(\overline{\rho})(1))^{\mathrm{Gal}(L_N/F_N)}\big) \longrightarrow \mathrm{H}^1(L_N/F, \mathrm{ad}^0(\overline{\rho})(1)) \longrightarrow \mathrm{H}^1(L_N/F_N, \mathrm{ad}^0(\overline{\rho})(1))$$

we see that

$$(\mathrm{ad}^0(\overline{\rho})(1))^{\mathrm{Gal}(L_N/F_N)} = \mathrm{H}^0(\Gamma, \mathrm{ad}^0(\overline{\rho})) = 0$$

and

$$\mathrm{H}^1(L_N/F_N, \mathrm{ad}^0(\overline{\rho})(1)) = \mathrm{H}^1(\Gamma, \mathrm{ad}^0(\overline{\rho})) = 0$$

by the second property in definition 53, whence the claim follows. Therefore, using the claim and the inflation-restriction exact sequence for $F_S/L_N/F$, the restriction morphism

$$\mathrm{H}^1(F_S/F, \mathrm{ad}^0(\overline{\rho})(1)) \longrightarrow \mathrm{H}^1(F_S/L_N, \mathrm{ad}^0(\overline{\rho})(1))^{\mathrm{Gal}(L_N/F)}$$

is injective. In particular,

$$0 \neq \mathrm{res}([\kappa]) \in \mathrm{H}^1(F_S/L_N, \mathrm{ad}^0(\overline{\rho})(1))^{\mathrm{Gal}(F_S/L_N)} \subseteq \mathrm{Hom}_\Gamma(\mathrm{Gal}(F_S/L_N), \mathrm{ad}^0(\overline{\rho})).$$

Let W be a non-zero irreducible subrepresentation of Γ in the $\mathbb{F}$–span of $\kappa(\mathrm{Gal}(F_S/L_N))$ inside $\mathrm{ad}^0(\overline{\rho})$. By the third property in definition 53, we can find $\sigma_0 \in \mathrm{Gal}(L_N/F_N)$ such that $\overline{\rho}(\sigma_0)$ has distinct eigenvalues and $W^{\sigma_0} \neq 0$. That is, the element σ_0 satisfies conditions (i) and (ii). If $\kappa(\sigma_0) \notin (\sigma_0 - 1)\mathrm{ad}^0(\overline{\rho})$, which is condition (iii), we can take $\sigma = \sigma_0$ and we are done. Now assume that $\kappa(\sigma_0) \in (\sigma_0 - 1)\mathrm{ad}^0(\overline{\rho})$. After conjugation if necessary, we can assume that

$$\overline{\rho}(\sigma_0) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \quad \text{with } \alpha \neq \beta.$$

Therefore,

$$(\sigma_0 - 1)\mathrm{ad}^0(\overline{\rho}) = \left\{ \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \right\},$$

which has no non-zero $\overline{\rho}(\sigma_0)$–invariant vectors. Since $W^{\sigma_0} \neq 0$, we deduce that

$$W \not\subseteq (\sigma_0 - 1)\mathrm{ad}^0(\overline{\rho}),$$

which implies that $\kappa(\mathrm{Gal}(F_S/L_N)) \not\subseteq (\sigma_0 - 1)\mathrm{ad}^0(\overline{\rho})$ and so there exists some $\tau \in \mathrm{Gal}(F_S/L_N)$ such that $\kappa(\tau) \notin (\sigma_0 - 1)\mathrm{ad}^0(\overline{\rho})$. We can take $\sigma = \tau\sigma_0$, which satisfies that

$$\sigma \in G_{F_N} \quad \text{and} \quad \overline{\rho}(\sigma) = \overline{\rho}(\sigma_0)$$

and

$$\kappa(\sigma) = \kappa(\tau) + \tau\kappa(\sigma_0) = \kappa(\tau) + \kappa(\sigma_0) \notin (\sigma_0 - 1)\mathrm{ad}^0(\overline{\rho}).$$

The result follows because $(\sigma_0 - 1)\mathrm{ad}^0(\overline{\rho}) = (\sigma - 1)\mathrm{ad}^0(\overline{\rho})$. □

If we further assume that the deformation problems D_v for $v \in S$ are nice like in the cases

(1) $T = \emptyset$ or

(2) T containing all the primes of F lying above p,

then we obtain the following result:

Corollary 56. *There exists $q \in \mathbb{Z}_{\geq 0}$ such that, for every $N \in \mathbb{Z}_{\geq 1}$, there exist a set Q_N consisting of exactly q Taylor–Wiles primes of level N and a surjection*

$$R_{\mathscr{S}}^{T-\mathrm{loc}}[\![x_1, \dots, x_g]\!] \twoheadrightarrow R_{\mathscr{S}_{Q_N}}^{T},$$

where

(1) *if $T = \emptyset$ (so that $R_{\mathscr{S}}^{T-\mathrm{loc}} = \mathscr{O}$), then $g = q$, and*

(2) *if T contains all the places above p, then*

$$\dim(R_{\mathscr{S}}^{T-\mathrm{loc}}) + g = q + 4|T|.$$

Definition 57. A *Taylor–Wiles datum (for $\mathscr{S}$ and T)* is a pair $(Q, (\alpha_v)_{v\in Q})$, where Q is a set of Taylor–Wiles primes and α_v is an eigenvalue of $\overline{\rho}(\mathrm{Frob}_v)$ for each $v \in Q$.

We saw after proposition 52 that, if

$$\rho^{\mathrm{univ}} \colon G_{F,S} \to \mathrm{GL}_2(R_{\mathscr{S}_Q})$$

is the universal deformation of type $\mathscr{S}_Q$, then for every $v \in Q$

$$\rho^{\mathrm{univ}}|_{G_{F_v}} \cong \chi_1 \oplus \chi_2$$

and the composition $\mathscr{O}_{F_v}^\times \to R_{\mathscr{S}}^\times$ of χ_i with (the restriction of) the local Artin reciprocity map factors through the maximal quotient Δ_v of $\kappa_v^\times$ of p–power order. The choice of an eigenvalue α_v of $\overline{\rho}(\mathrm{Frob}_v)$ determines an ordering of the characters χ_1 and χ_2 by, say, requiring that $\overline{\chi}_1(\mathrm{Frob}_v) = \alpha_v$. (Here, we are using that the two eigenvalues are distinct.)

Thus, a Taylor–Wiles datum gives rise to a morphism of $\mathscr{O}$–algebras

$$\mathscr{O}[\Delta_Q] \to R_{\mathscr{S}_Q}^T,$$

by means of which we regard $R_{\mathscr{S}_Q}^T$ as an $\mathscr{O}[\Delta_Q]$–algebra. There is a natural surjective morphism $R_{\mathscr{S}_Q}^T \twoheadrightarrow R_{\mathscr{S}}^T$ (corresponding to the forgetful functor) and we saw in the paragraphs after proposition 52 that its kernel is $\mathfrak{a}_Q R_{\mathscr{S}_Q}^T$, where $\mathfrak{a}_Q$ is the augmentation ideal of $\mathscr{O}[\Delta_Q]$. That is, we obtain a short exact sequence

$$0 \longrightarrow \mathfrak{a}_Q R_{\mathscr{S}_Q}^T \longrightarrow R_{\mathscr{S}_Q}^T \longrightarrow R_{\mathscr{S}}^T \longrightarrow 0.$$

Take a Taylor–Wiles datum with Q as in corollary 56. Write $q = |Q|$ and define $S_\infty = \mathscr{O}[\![y_1, \dots, y_q]\!]$ and $\mathfrak{a}_\infty = (y_1, \dots, y_q)S_\infty$.

(1) Suppose that $T = \emptyset$. Since each Δ_v (for $v \in Q$) is cyclic of p–power order, we have a diagram

$$\begin{tikzcd}
 & \mathscr{O}[\![\mathbb{Z}_p^q]\!] \arrow[dddl, dashed] \arrow[dd, two heads] & \cong \mathscr{O}[\![y_1, \dots, y_q]\!] = S_\infty & \supset \mathfrak{a}_\infty \arrow[dd, two heads] \\
 & & 1 + y_i \arrow[d, mapsto] & \\
 & \mathscr{O}[\Delta_q] \arrow[d] & (\text{gen. of } \Delta_{v_i}) & \mathfrak{a}_Q \\
\mathscr{O}[\![x_1, \dots, x_g]\!] \arrow[r, two heads] & R_{\mathscr{S}_Q} & &
\end{tikzcd}$$

which shows that $R_{\mathscr{S}_Q}/\mathfrak{a}_\infty \cong R_{\mathscr{S}}$. Moreover, we know that $g = q$. Later we will use this fact together with the two maps

$$\mathscr{O}[\![x_1, \dots, x_g]\!] \twoheadrightarrow R_{\mathscr{S}_Q} \leftarrow \mathscr{O}[\![y_1, \dots, y_q]\!].$$

(2) Suppose that T contains all primes of F lying over p. Fix an isomorphism

$R^T_{\mathscr{S}_Q} \cong R_{\mathscr{S}_Q}\llbracket z_1, \ldots, z_{4|T|-1}\rrbracket \cong R_{\mathscr{S}_Q} \widehat{\otimes}_{\mathscr{O}} \mathcal{T}$, where $\mathcal{T} = \mathscr{O}\llbracket z_1, \ldots, z_{4|T|-1}\rrbracket$. In this case, we have a similar diagram

$$\begin{tikzcd}
 & \mathcal{T}\llbracket \mathbb{Z}_p^q \rrbracket \arrow[ddl, dashed] \arrow[d, two heads] & \cong \mathcal{T}\llbracket y_1, \ldots, y_q \rrbracket \cong \mathcal{T} \widehat{\otimes}_{\mathscr{O}} S_\infty \supset & \mathcal{T} \widehat{\otimes}_{\mathscr{O}} \mathfrak{a}_\infty \arrow[d, two heads] \\
 & \mathcal{T}[\Delta_Q] \arrow[d] & & \mathcal{T} \widehat{\otimes}_{\mathscr{O}} \mathfrak{a}_Q \\
R^{T-\mathrm{loc}}_{\mathscr{S}}\llbracket x_1, \ldots, x_g \rrbracket \arrow[r, two heads] & R^T_{\mathscr{S}_Q} & &
\end{tikzcd}$$

which shows that $R^T_{\mathscr{S}_Q}/\mathfrak{a}_\infty \cong R_{\mathscr{S}}$. Moreover, we know that

$$\dim\left(R^{T-\mathrm{loc}}_{\mathscr{S}}\llbracket x_1, \ldots, x_g \rrbracket\right) = \dim\left(\mathcal{T} \widehat{\otimes}_{\mathscr{O}} S_\infty\right).$$

Later we will use this fact together with the two maps

$$R^{T-\mathrm{loc}}_{\mathscr{S}}\llbracket x_1, \ldots, x_g \rrbracket \twoheadrightarrow R_{\mathscr{S}_Q} \leftarrow \mathscr{O}\llbracket z_1, \ldots, z_{4|T|-1}, y_1, \ldots, y_q \rrbracket.$$

4 Modularity lifting

Fix a finite extension $E/\mathbb{Q}_p$ with ring of integers $\mathscr{O}$ and residue field $\mathbb{F}$. We always assume that $p > 2$. We consider a continuous homomorphism

$$\overline{\rho}\colon G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F}),$$

where S is a finite set of prime numbers containing p, $\mathbb{Q}_S$ is the maximal extension of $\mathbb{Q}$ unramified outside S and $G_{\mathbb{Q},S} = \mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q})$. We fix a Taylor–Wiles datum of level N

$$\left(Q, (\alpha_v)_{v \in Q}\right)$$

for $\overline{\rho}$, which consists of a finite set Q of prime numbers v not in S and such that

(1) $v \equiv 1 \bmod p^N$,

(2) $\overline{\rho}(\mathrm{Frob}_v)$ has $\mathbb{F}$–rational eigenvalues α_v and β_v satisfying that $\alpha_v \neq \beta_v$.

Then $(\alpha_v)_{v \in Q}$ is a choice of one eigenvalue for each $v \in Q$ in the last property. Associated with such data, there is a theory on the Galois side that we have studied in section 3, and a theory on the automorphic side that we will study next.

Fix a torsion-free subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ such that $\Gamma_1(M) \subseteq \Gamma \subseteq \Gamma_0(M)$ for some $M \in \mathbb{Z}_{\geq 1}$. Assume that $\overline{\rho}$ arises from a Hecke eigenform $g \in S_2(\Gamma, \mathscr{O})$. To simplify the notation, we write $S_2(\Gamma) = S_2(\Gamma, \mathscr{O})$.

Remark. We have the following analogy with Hida theory. In Hida theory, one considers $S_2(\Gamma \cap \Gamma_1(p^{N+1}))$ and take (co)invariants under

$$\big(\Gamma \cap \Gamma_0(p^{N+1})\big) / \big(\Gamma \cap \Gamma_1(p^{N+1})\big) \cong (\mathbb{Z}/p^{N+1}\mathbb{Z})^\times \cong (\mathbb{Z}/p\mathbb{Z})^\times \times (\mathbb{Z}/p^N\mathbb{Z})$$

to obtain $S_2(\Gamma \cap \Gamma_0(p^{N+1}))$. Then one applies Hida's idempotent operator to pass from $S_2(\Gamma \cap \Gamma_1(p^{N+1}))^{\mathrm{ord}}$ to

$$S_2(\Gamma \cap \Gamma_0(p^{N+1}))^{\mathrm{ord}} \cong S_2(\Gamma \cap \Gamma_0(p))^{\mathrm{ord}}.$$

Fixing a tame character, we can build a module over $\Lambda = \mathscr{O}[\![\mathbb{Z}_p]\!]$ which, after modding out by the augmentation ideal, recovers $S_2(\Gamma \cap \Gamma_0(p))^{\mathrm{ord}}$.

Taylor and Wiles defined

$$\Gamma_1(Q) = \Gamma \cap \Gamma_1\Big(\prod_{v \in Q} v\Big) \quad \text{and} \quad \Gamma_0(Q) = \Gamma \cap \Gamma_0\Big(\prod_{v \in Q} v\Big)$$

and considered the subgroup Γ_Q of $\Gamma_0(Q)$ which contains $\Gamma_1(Q)$ and such that $\Gamma_0(Q)/\Gamma_Q$ is the maximal p–power quotient of

$$\Gamma_0(Q)/\Gamma_1(Q) \cong \prod_{v \in Q} (\mathbb{Z}/v\mathbb{Z})^\times.$$

In particular, $\Gamma_0(Q)/\Gamma_Q \cong \Delta_Q$ (the same Δ_Q that appeared in section 3.7). They consider $S_2(\Gamma_Q)$ and take (co)invariants under Δ_Q (which projects onto $(\mathbb{Z}/p^N\mathbb{Z})^q$, where $q = |Q|$) to obtain $S_2(\Gamma_0(Q))$ (by the property of Taylor–Wiles primes regarded modulo p^N). After localizing at appropriate maximal ideals $\mathfrak{m}$ and $\mathfrak{m}_Q$ defined using the condition on the Frobenius-eigenvalues of Taylor–Wiles primes, one passes from $S_2(\Gamma_Q)_{\mathfrak{m}_Q}$ to

$$S_2(\Gamma_0(Q))_{\mathfrak{m}_Q} \cong S_2(\Gamma)_{\mathfrak{m}}.$$

We use this to build a module over $S_\infty = \mathscr{O}[\![\mathbb{Z}_p^q]\!]$ which, after modding out by the augmentation ideal, recovers $S_2(\Gamma)_{\mathfrak{m}}$.

4.1 Taylor–Wiles primes and Hecke algebras

Let $\mathbb{T}^S(\Gamma)$ be the subalgebra of $\mathrm{End}_{\mathscr{O}}\big(\mathrm{H}^1(\Gamma, \mathscr{O})\big) = \mathrm{End}_{\mathscr{O}}\big(\mathrm{H}^1(Y, \mathscr{O})\big)$ (where we write $Y = Y(\Gamma)$) generated by the Hecke operators T_ℓ and $\mathrm{S}_\ell = \langle \ell \rangle$ for all primes

$\ell \notin S$. Our representation $\overline{\rho}$ arising from the eigenform g gives rise to a maximal ideal $\mathfrak{m}$ of $\mathbb{T}^S(\Gamma)$ (the kernel of the corresponding Hecke eigensystem $\overline{\lambda}_g$; cf. section 2). We assume that $\overline{\rho}$ is absolutely irreducible, which means that $\mathfrak{m}$ is non-Eisenstein. We showed in proposition 9 that $\mathrm{H}^i(\Gamma, \mathbb{F})_{\mathfrak{m}} = 0$ unless $i = 1$. Consequently:

1. The $\mathscr{O}$–module $\mathrm{H}^1(\Gamma, \mathscr{O})_{\mathfrak{m}}$ is free. More generally, if R is an $\mathscr{O}$–algebra, then $\mathrm{H}^1(\Gamma, R)_{\mathfrak{m}}$ is a free R–module.
2. On $\mathscr{O}$–modules, the functor
$$R \mapsto \mathrm{H}^1(\Gamma, R)_{\mathfrak{m}}$$
is exact.
3. In particular, we have a Hecke–equivariant isomorphism
$$\mathrm{Hom}_{\mathscr{O}}\big(\mathrm{H}^1(Y, \mathscr{O})_{\mathfrak{m}}, \mathscr{O}\big) \cong \mathrm{H}_1(Y, \mathscr{O})_{\mathfrak{m}}.$$

Recall from section 2 that we have a Galois representation
$$\rho_{\mathfrak{m}} \colon G_{\mathbb{Q},S} \to \mathrm{GL}_2\big(\mathbb{T}^S(\Gamma)_{\mathfrak{m}}\big)$$
with the property that, for every prime $\ell \notin S$,
$$\mathrm{CharPoly}(\rho_{\mathfrak{m}}(\mathrm{Frob}_\ell)) = X^2 - \mathrm{T}_\ell\, X + \ell\, \mathrm{S}_\ell\, .$$
In particular, for $v \in Q$, we have
$$\mathrm{CharPoly}(\overline{\rho}(\mathrm{Frob}_v)) = X^2 - \mathrm{T}_v\, X + v\, \mathrm{S}_v$$
and this is $\equiv (X - \alpha_v)(X - \beta_v) \mod \mathfrak{m}$. Theorem 11 says that, for a representation $\rho \colon G_{\mathbb{Q},S} \to \mathrm{GL}_2(R)$ with R a complete local ring with residue field $\mathbb{F}$, if the residual representation is absolutely irreducible, then ρ can be conjugated to take values in the subring generated by the traces on a dense subset of $G_{\mathbb{Q},S}$. We can apply this to “remove” finitely many more primes apart from those in S (for example, those in Q) and still recover $\mathbb{T}^S(\Gamma)_{\mathfrak{m}}$ from the characteristic polynomials of $\rho_{\mathfrak{m}}$.

To make this more precise, by abuse of notation, let $\mathfrak{m}$ also denote $\mathfrak{m} \cap \mathbb{T}^{S\cup Q}(\Gamma)$. The inclusion $\mathbb{T}^{S\cup Q}(\Gamma) \subseteq \mathbb{T}^S(\Gamma)$ induces an isomorphism $\mathbb{T}^{S\cup Q}(\Gamma)_{\mathfrak{m}} \cong \mathbb{T}^S(\Gamma)_{\mathfrak{m}}$. Thus, by Hensel’s lemma, we can find $A_v \in \mathbb{T}^{S\cup Q}(\Gamma)_{\mathfrak{m}}$ lifting our fixed root α_v for every $v \in Q$. Define

$$\mathbb{T}_Q^{S \cup Q}(\Gamma_0(Q)) = \mathbb{T}^{S \cup Q}(\Gamma_0(Q))[\mathrm{U}_v : v \in Q]$$

and let $\mathfrak{m}_Q = (\mathfrak{m}, (\mathrm{U}_v - A_v)_{v \in Q}) \subseteq \mathbb{T}_Q^{S \cup Q}(\Gamma_0(Q))$.

Proposition 58. *The ideal* $\mathfrak{m}_Q$ *of* $\mathbb{T}_Q^{S \cup Q}(\Gamma_0(Q))$ *is maximal and the natural map*

$$\mathrm{H}_1(Y_0(Q), \mathcal{O}) \to \mathrm{H}_1(Y, \mathcal{O})$$

induces an isomorphism

$$\mathrm{H}_1(Y_0(Q), \mathcal{O})_{\mathfrak{m}_Q} \cong \mathrm{H}_1(Y, \mathcal{O})_{\mathfrak{m}}$$

that is equivariant for the Hecke operators at primes outside $S \cup Q$.

Define $\mathbb{T}_Q^{S \cup Q}(\Gamma_Q)$ to be the $\mathcal{O}[\Delta_Q]$–subalgebra of $\mathrm{End}_{\mathcal{O}}\big(\mathrm{H}_1(Y_Q, \mathcal{O})\big)$ (where $Y_Q = Y(\Gamma_Q)$) generated by the Hecke operators T_ℓ and S_ℓ for primes $\ell \notin S \cup Q$ and U_v for primes $v \in Q$. We consider the ideal

$$\mathfrak{m}_Q = \big(\mathfrak{m}, \mathfrak{a}_Q, (\mathrm{U}_v - A_v)_{v \in Q}\big) \subseteq \mathbb{T}_Q^{S \cup Q}(\Gamma_Q),$$

where $\mathfrak{a}_Q$ is the augmentation ideal of $\mathcal{O}[\Delta_Q]$. We have a natural map

$$\mathrm{H}_1(Y_Q, \mathcal{O})_{\mathfrak{m}_Q} \to \mathrm{H}_1(Y_0(Q), \mathcal{O})_{\mathfrak{m}_Q}.$$

But, by Shapiro's lemma, we can identify

$$\mathrm{H}_1(Y_Q, \mathcal{O})_{\mathfrak{m}_Q} \cong \mathrm{H}_1(Y_0(Q), \mathcal{O}[\Delta_Q])_{\mathfrak{m}_Q}$$

and the latter is a free $\mathcal{O}[\Delta_Q]$–module with coinvariants

$$\cong \mathrm{H}_1(Y_0(Q), \mathcal{O})_{\mathfrak{m}_Q} \cong \mathrm{H}_1(Y, \mathcal{O})_{\mathfrak{m}}$$

(where we used proposition 58). Therefore, the natural map above on homology spaces has a particularly simple form.

Recall that we can view $\mathbb{T}^S(\Gamma)$ as the image of $\mathbb{T}^{S,\mathrm{univ}} = \mathcal{O}[\mathrm{T}_\ell, \mathrm{S}_\ell : \ell \notin S]$ in $\mathrm{End}_{\mathcal{O}}\big(\mathrm{H}^1(Y, \mathcal{O})\big)$. We defined $\mathfrak{m}$ to be the maximal ideal of $\mathbb{T}^S(\Gamma)$ arising from $\overline{\rho}$, but sometimes we also want to view it as a maximal ideal of $\mathbb{T}^{S,\mathrm{univ}}$ (its pullback under the natural map). Apart from that, we used the symbol $\mathfrak{m}$ to mean the maximal ideal $\mathfrak{m} \cap \mathbb{T}^{S \cup Q}(\Gamma)$ too and we get analogous results.

Lemma 59. *Let* $\mathfrak{m}^Q = \mathfrak{m} \cap \mathbb{T}^{S\cup Q,\mathrm{univ}}$*. The natural inclusion*

$$\mathbb{T}^{S\cup Q}(\Gamma)_{\mathfrak{m}^Q} \to \mathbb{T}^S(\Gamma)_{\mathfrak{m}}$$

is an isomorphism.

Proof. By Nakayama's lemma, it suffices to prove that

$$\mathbb{T}^S(\Gamma)_{\mathfrak{m}}/(\mathfrak{m}^Q) \cong \mathbb{F}$$

(i.e., surjectivity modulo $\mathfrak{m}^Q$). But this reduces us to proving that, for every $v \in Q$, the operators T_v and S_v act modulo $\mathfrak{m}^Q$ as multiplication by an element of $\mathbb{F}$.

Since $\overline{\rho}$ is absolutely irreducible, we have Galois representations

- $\rho_{\mathfrak{m}} \colon G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{T}(\Gamma)_{\mathfrak{m}})$ characterized by

  $$\mathrm{CharPoly}(\rho_{\mathfrak{m}}(\mathrm{Frob}_\ell)) = X^2 - \mathrm{T}_\ell X + \ell \mathrm{S}_\ell$$

  for all $\ell \notin S$ and
- $\rho_{\mathfrak{m}^Q} \colon G_{\mathbb{Q},S\cup Q} \to \mathrm{GL}_2(\mathbb{T}(\Gamma)_{\mathfrak{m}^Q})$ characterized by

  $$\mathrm{CharPoly}(\rho_{\mathfrak{m}^Q}(\mathrm{Frob}_\ell)) = X^2 - \mathrm{T}_\ell X + \ell \mathrm{S}_\ell$$

  for all $\ell \notin S \cup Q$.

Consider the Galois representation $\rho = \rho_{\mathfrak{m}} \bmod \mathfrak{m}^Q$. For $\ell \notin S \cup Q$, we can compute

$$\begin{aligned} \mathrm{tr}\big(\rho(\mathrm{Frob}_\ell)\big) &= \big(\mathrm{tr}\big(\rho_{\mathfrak{m}}(\mathrm{Frob}_\ell)\big) \mod \mathfrak{m}^Q\big) = (\mathrm{T}_\ell \mod \mathfrak{m}^Q) \\ &= \big(\mathrm{tr}\big(\rho_{\mathfrak{m}^Q}(\mathrm{Frob}_\ell)\big) \mod \mathfrak{m}^Q\big) = \mathrm{tr}\big(\overline{\rho}(\mathrm{Frob}_\ell)\big) \in \mathbb{F}. \end{aligned}$$

By continuity, we deduce that $\mathrm{tr}(\rho)$ is $\mathbb{F}$–valued. In particular, for $v \in Q$,

$$(\mathrm{T}_v \mod \mathfrak{m}^Q) = \mathrm{tr}\big(\rho(\mathrm{Frob}_v)\big) \in \mathbb{F}.$$

The same argument with det in place of tr shows that $(\mathrm{S}_v \mod \mathfrak{m}^Q) \in \mathbb{F}$ too. □

Remark. Lemma 59 shows that

$$\mathrm{H}^i(Y, \mathscr{O})_{\mathfrak{m}^Q} = \mathrm{H}^i(Y, \mathscr{O})_{\mathfrak{m}}$$

(and similarly using $\mathbb{F}$–coefficients instead of $\mathscr{O}$–coefficients). For this reason, we will just write $\mathfrak{m} = \mathfrak{m}^Q$ in what follows.

Define $\mathbb{T}_Q^{S\cup Q,\mathrm{univ}} = \mathbb{T}^{S\cup Q,\mathrm{univ}}[\mathrm{U}_v : v \in Q]$. For each $v \in Q$, choose a lift $\widetilde{\alpha}_v \in \mathscr{O}$ of the eigenvalue $\alpha_v \in \mathbb{F}$ and define

$$\mathfrak{m}_Q = \big(\mathfrak{m}, (\mathrm{U}_v - \widetilde{\alpha}_v)_{v\in Q}\big) \subseteq \mathbb{T}_Q^{S\cup Q,\mathrm{univ}}.$$

By the theory of oldforms, the ideal $\mathfrak{m}_Q$ is in the support of $\mathrm{H}^1(Y_0(Q), \mathbb{F})$ (we prove it later) and so also of $\mathrm{H}_1(Y_0(Q), \mathbb{F}) \cong \mathrm{Hom}_{\mathbb{F}}\big(\mathrm{H}^1(Y_0(Q), \mathbb{F}), \mathbb{F}\big)$. Observe that the natural map

$$\mathbb{T}^{S\cup Q}(\Gamma_0(Q)) \to \mathbb{T}_Q^{S\cup Q}(\Gamma_0(Q))$$

is a morphism of finite $\mathscr{O}$–algebras. In addition, $\mathbb{T}^{S\cup Q}(\Gamma_0(Q))_{\mathfrak{m}}$ is a complete local ring and $\mathbb{T}_Q^{S\cup Q}(\Gamma_0(Q))_{\mathfrak{m}}$ is a complete semilocal ring, one of whose (local) direct summands is $\mathbb{T}_Q^{S\cup Q}(\Gamma_0(Q))_{\mathfrak{m}_Q}$. Therefore, $\mathrm{H}_1(Y_0(Q), \mathscr{O})_{\mathfrak{m}_Q}$ is a direct summand of $\mathrm{H}_1(Y_0(Q), \mathscr{O})_{\mathfrak{m}}$. The same is true if we replace $\Gamma_0(Q)$ and $Y_0(Q)$ with Γ_Q and Y_Q (or if we use cohomology instead of homology).

We could prove proposition 58 using that, since

$$\mathrm{H}_i(Y, \mathbb{F})_{\mathfrak{m}} = \mathrm{Hom}_{\mathbb{F}}\big(\mathrm{H}^i(Y, \mathbb{F})_{\mathfrak{m}}, \mathbb{F}\big) = 0 \quad \text{if } i \neq 1,$$

the properties recalled before for H^i also hold for H_i (also if we replace Γ with $\Gamma_0(Q)$ or Γ_Q). Instead, we prove the following analogue directly with cohomology groups:

Proposition 60. *The natural map* $\mathrm{H}^1(Y, \mathscr{O}) \to \mathrm{H}^1(Y_0(Q), \mathscr{O})$ *induces an isomorphism*

$$\mathrm{H}^1(Y, \mathscr{O})_{\mathfrak{m}} \cong \mathrm{H}^1(Y_0(Q), \mathscr{O})_{\mathfrak{m}_Q}$$

of $\mathbb{T}^{S\cup Q,\mathrm{univ}}$*–modules whose inverse is given by the trace map up to units in* $\mathscr{O}^{\times}$*.*

To prove proposition 60, it suffices to prove that

$$\mathrm{H}^1(Y, \mathbb{F})_{\mathfrak{m}} \cong \mathrm{H}^1(Y_0(Q), \mathbb{F})_{\mathfrak{m}_Q}.$$

Indeed, both $\mathrm{H}^1(Y, \mathscr{O})_{\mathfrak{m}}$ and $\mathrm{H}^1(Y_0(Q), \mathscr{O})_{\mathfrak{m}_Q}$ are finite free $\mathscr{O}$–modules satisfying that

$$\mathrm{H}^1(Y, \mathscr{O})_{\mathfrak{m}} \otimes_{\mathscr{O}} \mathbb{F} \cong \mathrm{H}^1(Y, \mathbb{F})_{\mathfrak{m}}$$

and

$$\mathrm{H}^1(Y_0(Q), \mathscr{O})_{\mathfrak{m}_Q} \otimes_{\mathscr{O}} \mathbb{F} \cong \mathrm{H}^1(Y_0(Q), \mathbb{F})_{\mathfrak{m}_Q}.$$

Thus, we can apply the snake lemma to the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^1(Y, \mathscr{O})_{\mathfrak{m}} \arrow[r, "\varpi"] \arrow[d] & \mathrm{H}^1(Y, \mathscr{O})_{\mathfrak{m}} \arrow[r] \arrow[d] & \mathrm{H}^1(Y, \mathbb{F})_{\mathfrak{m}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{H}^1(Y_0(Q), \mathscr{O})_{\mathfrak{m}_Q} \arrow[r, "\varpi"] & \mathrm{H}^1(Y_0(Q), \mathscr{O})_{\mathfrak{m}_Q} \arrow[r] & \mathrm{H}^1(Y_0(Q), \mathbb{F})_{\mathfrak{m}_Q} \arrow[r] & 0
\end{tikzcd}$$

and apply Nakayama's lemma.

Arguing by induction on $|Q|$, we may assume that $Q = \{ v \}$. Consider

$$K = \mathrm{GL}_2(\mathbb{Z}_v) \quad \text{and} \quad I = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K : c \equiv 0 \mod v \right\}.$$

For an open compact subset U of K, let $\mathcal{H}_U$ denote the convolution algebra of compactly supported U–biinvariant functions $f\colon \mathrm{GL}_2(\mathbb{Q}_v) \to \mathbb{F}$, which is generated by double coset operators $[UgU]$ for $g \in \mathrm{GL}_2(\mathbb{Q}_v)$. Its identity element is the characteristic function $[U]$ of U.

Let $M = \mathrm{H}^1(Y_0(Q), \mathbb{F})_{\mathfrak{m}}$ and $N = \mathrm{H}^1(Y, \mathbb{F})_{\mathfrak{m}}$. By definition, M is a $\mathcal{H}_I$–module and N is a $\mathcal{H}_K$–module. We will define some isomorphisms that depend on the choice of a square root $v^{1/2} \in \mathbb{F}$. Since $v \equiv 1 \mod p$, we can choose $v^{1/2} = 1$.

Observe that $\mathcal{H}_K = \mathbb{F}[\mathrm{T}_v, \mathrm{S}_v]$, where

$$\mathrm{T}_v = \left[K \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} K \right] \quad \text{and} \quad \mathrm{S}_v = \left[K \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} K \right].$$

Let T be the diagonal torus in GL_2 and let $X_*(T)$ be the group of cocharacters of T. We can express $X_*(T) = \mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2$, where

$$\lambda_1(t) = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \lambda_2(t) = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}.$$

Let $W = \{ 1, w_0 \}$ be the Weyl group of GL_2. We will use the Satake isomorphism

$$\mathbb{F}[X_*(T)]^W \cong \mathcal{H}_K$$

given by

$$\lambda_1 + \lambda_2 \mapsto v^{1/2}\,\mathrm{T}_v = \mathrm{T}_v \quad \text{and} \quad \lambda_1\lambda_2 \mapsto v\,\mathrm{S}_v = \mathrm{S}_v$$

(where we used that $v = 1$ in $\mathbb{F}$). There is an analogous description of $\mathcal{H}_I$:

Lemma 61. *Since $v \equiv 1 \bmod p$, we have an isomorphism*

$$\mathcal{H}_I \cong \mathbb{F}[X_*(T) \rtimes W]$$

characterized as follows:

- *$\lambda \in X_*(T)_+ = \{ a\lambda_1 + b\lambda_2 : a \geq b \}$ is mapped to $[I\lambda(v)I] \in \mathcal{H}_I$ and*
- *$w \in W$ is mapped to $[I\widetilde{w}I] \in \mathcal{H}_I$, where $\widetilde{w} \in N(T)$ is a lift of w.*

Under this isomorphism, the centre $Z(\mathcal{H}_I)$ of $\mathcal{H}_I$ corresponds to $\mathbb{F}[X_*(T)]^W$ and the composition

$$\begin{aligned} \mathbb{F}[X_*(T)]^W \cong Z(\mathcal{H}_I) &\longrightarrow \mathcal{H}_K \\ f &\longmapsto [K]f \end{aligned}$$

is the Satake isomorphism described above.

Remark. Lemma 61 follows from the Bernstein presentation or from the Iwahori–Matsumoto presentation of $\mathcal{H}_I$.

Lemma 62. *The inclusion $N \hookrightarrow M$ is split by*

$$x \mapsto [K]x.$$

Proof. Geometrically, the morphism

$$\begin{aligned} \mathrm{H}^1(Y_0(Q), \mathbb{F})_{\mathfrak{m}} &\longrightarrow \mathrm{H}^1(Y, \mathbb{F})_{\mathfrak{m}} \\ x &\longmapsto [K]x \end{aligned}$$

is the trace map (equivalently, in terms of group cohomology, it is the corestriction). Therefore, the composition

$$N \hookrightarrow M \xrightarrow{[K]} N$$

is multiplication by $[K : I] = v + 1 \in \mathbb{F}^\times$, as $v = 1 \in \mathbb{F}$ and $p > 2$. □

By the Bruhat decomposition, in $\mathcal{H}_I$ we can express

$$[K] = [I] + [Iw_0I] = 1 + w_0 = \sum_{w \in W} w.$$

Since $|W| = 2$ is invertible in $\mathbb{F}$, we deduce that

$$M^W = \Big(\sum_{w \in W} w \Big) M.$$

But $N \subseteq M^W$ and, by lemma 62,

$$N = M^W = [K]M.$$

We know that the operators T_v and S_v act on N as multiplication by $\alpha_v + \beta_v$ and $\alpha_v \beta_v$, respectively. Consider the maximal ideal

$$\mathfrak{n} = (\lambda_1 + \lambda_2 - \alpha_v - \beta_v, \lambda_1 \lambda_2 - \alpha_v \beta_v) \subset \mathbb{F}[X_*(T)]^W \cong \mathbb{F}[\mathrm{T}_v, \mathrm{S}_v].$$

By the explicit description of the Satake isomorphism, we see that $N_{\mathfrak{n}} = N$. On the other hand, since $\alpha_v \neq \beta_v$, there are exactly two maximal ideals $\mathfrak{m}_\alpha, \mathfrak{m}_\beta \subset \mathbb{F}[X_*(T)]$ above $\mathfrak{n}$, namely

$$\mathfrak{m}_\alpha = (\lambda_1 - \alpha_v, \lambda_2 - \beta_v) \quad \text{and} \quad \mathfrak{m}_\beta = (\lambda_1 - \beta_v, \lambda_2 - \alpha_v).$$

But note that λ_1 corresponds to U_v under the isomorphism of lemma 61, whence

$$M_{\mathfrak{m}_\alpha} = \mathrm{H}^1(Y_0(Q), \mathbb{F})_{\mathfrak{m}_Q}.$$

Thus, we have to prove that the composition $N \to M \to M_{\mathfrak{m}_\alpha}$ is an isomorphism. Since $\mathfrak{n} = \mathfrak{m}_\alpha \cap \mathbb{F}[X_*(T)]^W$, it suffices to show that the composition of natural morphisms

$$N = N_{\mathfrak{n}} \longrightarrow M_{\mathfrak{n}} \longrightarrow M_{\mathfrak{m}_\alpha}$$

is an isomorphism.

Lemma 63. *In the situation from above, we have a decomposition* $M_{\mathfrak{n}} \cong M_{\mathfrak{m}_\alpha} \oplus M_{\mathfrak{m}_\beta}$ *and* $w_0 \in W$ *maps* $M_{\mathfrak{m}_\alpha}$ *isomorphically onto* $M_{\mathfrak{m}_\beta}$ *and vice versa.*

Proof. Since M is finite-dimensional over $\mathbb{F}$, the action of $\mathbb{F}[X_*(T)]_{\mathfrak{n}}$ on $M_{\mathfrak{n}}$ factors through an artinian quotient A of $\mathbb{F}[X_*(T)]$. Since $\mathfrak{m}_\alpha$ and $\mathfrak{m}_\beta$ are the two (distinct) maximal ideals of A, there is a canonical decomposition $A \cong A_{\mathfrak{m}_\alpha} \times A_{\mathfrak{m}_\beta}$, which in turn induces a decomposition $M_{\mathfrak{n}} \cong M_{\mathfrak{m}_\alpha} \oplus M_{\mathfrak{m}_\beta}$. It is straight-forward from the definitions that the Weyl group W permutes $\mathfrak{m}_\alpha$ and $\mathfrak{m}_\beta$, whence the second claim follows. $\square$

Now, using the isomorphisms

$$N = M^W = [K]M \quad \text{and} \quad N_{\mathfrak{n}} = M_{\mathfrak{n}}^W = [K]M_{\mathfrak{n}}$$

together with $N_\mathfrak{n} = N$, we can conclude that the compositions

$$N \hookrightarrow M_\mathfrak{n} = M_{\mathfrak{m}_\alpha} \oplus M_{\mathfrak{m}_\beta} \twoheadrightarrow M_{\mathfrak{m}_\alpha}$$

and

$$M_{\mathfrak{m}_\alpha} \hookrightarrow M_{\mathfrak{m}_\alpha} \oplus M_{\mathfrak{m}_\beta} = M_\mathfrak{n} \xrightarrow{[K]} N$$

are isomorphisms and are inverse to each other up to multiplication by $\mathbb{F}^\times$. This concludes the proof of proposition 60.

Proposition 64. *The homology group* $\mathrm{H}_1(Y_Q, \mathscr{O})_{\mathfrak{m}_Q}$ *is a free* $\mathscr{O}[\Delta_Q]$*–module and the natural map* $\mathrm{H}_1(Y_Q, \mathscr{O})_{\mathfrak{m}_Q} \to \mathrm{H}_1(Y_0(Q), \mathscr{O})_{\mathfrak{m}_Q}$ *induces an isomorphism*

$$\frac{\mathrm{H}_1(Y_Q, \mathscr{O})_{\mathfrak{m}_Q}}{\mathfrak{a}_Q \, \mathrm{H}_1(Y_Q, \mathscr{O})_{\mathfrak{m}_Q}} \cong \mathrm{H}_1(Y_0(Q), \mathscr{O})_{\mathfrak{m}_Q}.$$

Remark. Proposition 64 is also true if we localize only at $\mathfrak{m} = \mathfrak{m} \cap \mathbb{T}^{S \cup Q, \mathrm{univ}}$, and the further localization at $\mathfrak{m}_Q$ gives only a direct summand of this more general result.

Combining proposition 64 with proposition 58, we obtain the main result of this subsection:

Proposition 65. *The homology group* $\mathrm{H}_1(Y_Q, \mathscr{O})_{\mathfrak{m}_Q}$ *is a free* $\mathscr{O}[\Delta_Q]$*–module and the natural map* $\mathrm{H}_1(Y_Q, \mathscr{O})_{\mathfrak{m}_Q} \to \mathrm{H}_1(Y, \mathscr{O})_{\mathfrak{m}}$ *induces an isomorphism*

$$\frac{\mathrm{H}_1(Y_Q, \mathscr{O})_{\mathfrak{m}_Q}}{\mathfrak{a}_Q \, \mathrm{H}_1(Y_Q, \mathscr{O})_{\mathfrak{m}_Q}} \cong \mathrm{H}_1(Y, \mathscr{O})_{\mathfrak{m}}.$$

To prove proposition 64, we are going to use again the key fact that

$$\mathrm{H}_i(Y_Q, \mathbb{F})_\mathfrak{m} \cong \mathrm{Hom}_\mathbb{F}\big(\mathrm{H}^i(Y_Q, \mathbb{F})_\mathfrak{m}, \mathbb{F}\big) = 0 \quad \text{if } i \neq 1$$

and so

$$\mathrm{H}_i(Y_Q, \mathscr{O})_\mathfrak{m} = \begin{cases} 0 & \text{if } i \neq 1, \\ \mathscr{O} & \text{if } i = 1. \end{cases}$$

In what follows we switch to group homology (instead of singular homology). Since Γ_Q is a normal subgroup of $\Gamma_0(Q)$ with quotient Δ_Q, there is a Hochschild–Serre spectral sequence

$$\mathrm{H}_i(\Delta_Q, \mathrm{H}_j(\Gamma_Q, \mathscr{O})) \implies \mathrm{H}_{i+j}(\Gamma_0(Q), \mathscr{O}).$$

After localizing at $\mathfrak{m}$, only the column $j = 1$ of the second page of the spectral sequence is non-trivial. In particular,

$$\mathrm{H}_0(\Delta_Q, \mathrm{H}_1(\Gamma_Q, \mathcal{O})_{\mathfrak{m}}) \cong \mathrm{H}_0(\Gamma_0(Q), \mathcal{O})_{\mathfrak{m}}.$$

But, by definition,

$$\mathrm{H}_0(\Delta_Q, \mathrm{H}_1(\Gamma_Q, \mathcal{O})_{\mathfrak{m}}) = \frac{\mathrm{H}_1(\Gamma_Q, \mathcal{O})_{\mathfrak{m}}}{\mathfrak{a}_Q \mathrm{H}_1(\Gamma_Q, \mathcal{O})_{\mathfrak{m}}},$$

which means that we have proved the second claim of proposition 64. It remains to prove that $\mathrm{H}_1(\Gamma_Q, \mathcal{O})_{\mathfrak{m}}$ is a free $\mathcal{O}[\Delta_Q]$–module.

Since $\mathcal{O}[\Delta_Q]$ is a local ring, a finitely generated $\mathcal{O}[\Delta_Q]$–module M is free if and only if M is flat or, equivalently, $\mathrm{Tor}_1^{\mathcal{O}[\Delta_Q]}(M, \mathbb{F}) = 0$. Using again that, after localizing the Hochschild–Serre spectral sequence at $\mathfrak{m}$, only the column $j = 1$ of the second page is non-trivial, we deduce that

$$0 = \mathrm{H}_2(\Gamma_0(Q), \mathcal{O})_{\mathfrak{m}} = \mathrm{H}_1(\Delta_Q, \mathrm{H}_1(\Gamma_Q, \mathcal{O})_{\mathfrak{m}}) = \mathrm{Tor}_1^{\mathcal{O}[\Delta_Q]}(\mathrm{H}_1(\Gamma_Q, \mathcal{O})_{\mathfrak{m}}, \mathcal{O}).$$

But, if we tensor the short exact sequence

$$0 \longrightarrow \mathcal{O} \xrightarrow{\varpi} \mathcal{O} \longrightarrow \mathbb{F} \longrightarrow 0$$

with $\mathrm{H}_1(\Gamma_Q, \mathcal{O})_{\mathfrak{m}}$ over $\mathcal{O}[\Delta_Q]$, we obtain an exact sequence

$$\begin{aligned} 0 = \mathrm{Tor}_1^{\mathcal{O}[\Delta_Q]}(\mathrm{H}_1(\Gamma_Q, \mathcal{O})_{\mathfrak{m}}, \mathcal{O}) \longrightarrow \mathrm{Tor}_1^{\mathcal{O}[\Delta_Q]}(\mathrm{H}_1(\Gamma_Q, \mathcal{O})_{\mathfrak{m}}, \mathbb{F}) \,— \\ \to \mathrm{H}_1(\Gamma_Q, \mathcal{O})_{\mathfrak{m}} \otimes_{\mathcal{O}[\Delta_Q]} \mathcal{O} \xrightarrow{\varpi} \mathrm{H}_1(\Gamma_Q, \mathcal{O})_{\mathfrak{m}} \otimes_{\mathcal{O}[\Delta_Q]} \mathcal{O} \longrightarrow \mathrm{H}_1(\Gamma_Q, \mathcal{O})_{\mathfrak{m}} \otimes_{\mathcal{O}[\Delta_Q]} \mathbb{F} \end{aligned}$$

and we can reinterpret the second line (using the part of proposition 65 that we have already proved) as

$$\mathrm{H}_1(\Gamma_0(Q), \mathcal{O})_{\mathfrak{m}} \xrightarrow{\varpi} \mathrm{H}_1(\Gamma_0(Q), \mathcal{O})_{\mathfrak{m}} \longrightarrow \mathrm{H}_1(\Gamma_0(Q), \mathbb{F})_{\mathfrak{m}}$$

But multiplication by ϖ on $\mathrm{H}_1(\Gamma_0(Q), \mathcal{O})_{\mathfrak{m}}$ is injective (by proposition 58 and the properties listed in the beginning of this subsection). Therefore, we see from the exact sequence above that

$$\mathrm{Tor}_1^{\mathcal{O}[\Delta_Q]}(\mathrm{H}_1(\Gamma_Q, \mathcal{O})_{\mathfrak{m}}, \mathbb{F}) = 0$$

as desired. This completes the proof of proposition 64.

Remark. The proof of proposition 64 was completely formal, using only the key fact that $\mathrm{H}^i(Y_Q, \mathbb{F})_\mathfrak{m} = 0$ if $i \neq 1$ (but without using modular forms).

Recall that, given a fixed global deformation problem

$$\mathscr{S} = \big(\overline{\rho}, S, \psi, \mathscr{O}, (D_v)_{v\in S}\big),$$

we have the augmented deformation problem

$$\mathscr{S}_Q = \big(\overline{\rho}, S, \psi, \mathscr{O}, (D_v)_{v\in S} \cup (D_v^{\psi})_{v\in Q}\big)$$

and the universal ring $R_{\mathscr{S}_Q}$ is an $\mathscr{O}[\Delta_Q]$–algebra (depending on the choice of the α_v) satisfying that

$$R_{\mathscr{S}_Q}/\mathfrak{a}_Q \cong R_{\mathscr{S}},$$

where $\mathfrak{a}_Q$ is the augmentation ideal of $\mathscr{O}[\Delta_Q]$. We also have Galois representations

$$\rho_\mathfrak{m} \colon G_{\mathbb{Q},S} \to \mathrm{GL}_2\big(\mathbb{T}^S(\Gamma)_\mathfrak{m}\big)$$

and

$$\rho_{\mathfrak{m}_Q} \colon G_{\mathbb{Q},S\cup Q} \to \mathrm{GL}_2\big(\mathbb{T}_Q^{S\cup Q}(\Gamma)_{\mathfrak{m}_Q}\big).$$

If we know that $\rho_\mathfrak{m}$ and $\rho_{\mathfrak{m}_Q}$ are of types $\mathscr{S}$ and $\mathscr{S}_Q$, respectively, then the actions of $R_{\mathscr{S}_Q}$ on $\mathrm{H}_1(Y_Q, \mathscr{O})_{\mathfrak{m}_Q}$ and of $R_{\mathscr{S}}$ on $\mathrm{H}_1(Y, \mathscr{O})_\mathfrak{m}$ are compatible (with respect to the projections modulo $\mathfrak{a}_Q$). That is, we obtain a commutative diagram

$$\begin{tikzcd}
R_{\mathscr{S}_Q} \arrow[d, two heads, "\text{mod } \mathfrak{a}_Q"] \arrow[r, phantom, "\circlearrowright"] & \mathrm{H}_1(Y_Q, \mathscr{O})_{\mathfrak{m}_Q} \arrow[d, two heads, "\text{mod } \mathfrak{a}_Q"] \\
R_{\mathscr{S}} \arrow[r, phantom, "\circlearrowright"] & \mathrm{H}_1(Y, \mathscr{O})_\mathfrak{m}
\end{tikzcd}$$

of actions.

4.2 Local-global compatibility

Theorem 66. *Let f be a cuspidal Hecke eigenform with associated cuspidal automorphic representation π_f of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$. Let p be a prime number and consider a fixed isomorphism $\iota \colon \mathbb{C} \cong \overline{\mathbb{Q}}_p$. Let $\rho_{f,\iota} \colon G_\mathbb{Q} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ be the associated Galois representation.*

(1) *The representation $\rho_{f,\iota}|_{G_{\mathbb{Q}_p}}$ is de Rham with Hodge–Tate weights 0 and $k-1$, where k is the weight of f.*

(2) *For every prime number ℓ,*

$$\mathrm{WD}\big(\rho_{f,\iota}|_{G_{\mathbb{Q}_\ell}}\big)^{\text{Fr-ss}} \cong \mathrm{LL}\big(\pi_\ell \otimes |\det|_\ell^{-1/2}\big) \otimes_{\mathbb{C},\iota} \overline{\mathbb{Q}}_p,$$

where WD *means the Weil–Deligne representation,* Fr-ss *denotes the Frobenius-semisimplification and* LL *means the local Langlands correspondence (suitably normalized).*

Let us write down some more concrete consequences of theorem 66. Take a newform $f \in S_2(\Gamma_1(N), \overline{\mathbb{Q}}_p)$ (i.e., f does not come from $\Gamma_1(M)$ for any $M < N$). Let η be the nebentype of f and let C be its conductor (in particular, $C \mid N$). Write again η for the corresponding Galois character. Let ϵ_p be the p–adic cyclotomic character. Take a prime $\ell \neq p$.

- If $\ell \nmid N$, then ρ_f is unramified at ℓ and

$$\mathrm{CharPoly}(\rho_f(\mathrm{Frob}_\ell)) = X^2 - a_\ell X + \eta(\ell)\ell,$$

  where a_ℓ is the T_ℓ–eigenvalue of f.
- If $\ell \parallel N$ but $\ell \nmid C$ (i.e., at ℓ, f is new of level $\Gamma_0(\ell)$), then

$$\rho_f|_{G_{\mathbb{Q}_\ell}} \cong \begin{pmatrix} \gamma & * \\ 0 & \gamma\epsilon_p^{-1} \end{pmatrix},$$

  where γ is the unramified character such that $\gamma(\mathrm{Frob}_\ell)$ is the U_ℓ–eigenvalue of f and

$$1 \neq \rho_f(I_\ell) \subseteq \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}.$$

- If $\ell \parallel N$ and $\ell \parallel C$, then

$$\rho_f|_{G_{\mathbb{Q}_\ell}} \cong \gamma \oplus \gamma^{-1}\epsilon_p^{-1}\eta,$$

  where γ is the unramified character such that $\gamma(\mathrm{Frob}_\ell)$ is the U_ℓ–eigenvalue of f.
- If $\rho_f|_{G_{\mathbb{Q}_\ell}}$ is irreducible, then $\ell^2 \mid N$.
- If $p \nmid N$ and the T_p–eigenvalue a_p of f is a p–adic unit, then

$$\rho_f|_{G_{\mathbb{Q}_p}} \cong \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix},$$

  where χ_1 is the unramified character such that $\chi_1(\mathrm{Frob}_p)$ is the unit root of $X^2 - a_pX + \eta(p)p$ and $\chi_2|_{I_p} = \epsilon_p^{-1}$.

4.3 Patching

Fix a newform $g \in S_2(\Gamma_1(N), \overline{\mathbb{Q}}_p)$ with nebentype η, a prime number p and a finite extension $E/\mathbb{Q}_p$ with ring of integers $\mathscr{O}$, uniformizer ϖ and residue field $\mathbb{F}$. Let $\overline{\rho} = \overline{\rho}_g \colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be the associated representation modulo p. We assume that $\mathbb{F}$ is sufficiently large so that the eigenvalues of $\overline{\rho}(\sigma)$ are defined in $\mathbb{F}$ for all $\sigma \in G_{\mathbb{Q}}$.

We require the following hypotheses:

- $p > 2$ and $p \nmid N$;
- $\overline{\rho}|_{G_{\mathbb{Q}(\zeta_p)}}$ is absolutely irreducible with enormous (equivalently, adequate) image (e.g., automatic condition if $p \geq 7$);
- N is square-free, $\overline{\rho}$ is ramified at every prime $\ell \mid N$ (restrictive condition) and η has order prime to p; equivalently, $\overline{\rho}$ is modular of weight 2 and the level $N = N(\overline{\rho})$, its Artin conductor, is square-free;
- we have

$$\overline{\rho}|_{G_{\mathbb{Q}_p}} \cong \begin{pmatrix} \overline{\chi}_1 & * \\ 0 & \overline{\chi}_2 \end{pmatrix} \quad \text{with } \overline{\chi}_1|_{I_p} = 1 \text{ and } \overline{\chi}_2|_{I_p} = \overline{\epsilon}_p^{-1}$$

  (this is actually unnecessary but simplifies things).

We define a global deformation problem

$$\big(\overline{\rho}, S, \psi, \mathscr{O}, (D_v)_{v \in S}\big)$$

with $S = \{ \ell \mid N \} \cup \{ p \}$, $\psi = \eta\epsilon_p^{-1}$ and

$$D_v = \begin{cases} D_v^{\min} & \text{if } v \mid N, \\ D_v^{\mathrm{ord}} & \text{if } v = p. \end{cases}$$

(Taking the deformation problems $D_v^{\min}$ is quite restrictive for modularity lifting purposes, but still has interesting consequences.) Let

$$\Gamma = \mathrm{Ker}\big(\Gamma_0(N) \twoheadrightarrow \Gamma_0(N)/\Gamma_1(N) \cong (\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\eta} \overline{\mathbb{Q}}_p\big)$$

and assume that Γ is torsion-free (not completely necessary, but it simplifies things). Let $\mathfrak{m}$ be the maximal ideal of $\mathbb{T}^{S,\mathrm{univ}}$ corresponding to $\overline{\rho}$.

Theorem 67. *The Galois representation* $\rho_{\mathfrak{m}} \colon G_{\mathbb{Q},S} \to \mathrm{GL}_2\big(\mathbb{T}^S(\Gamma)_{\mathfrak{m}}\big)$ *lifting* $\overline{\rho}$ *is of type* $\mathscr{S}$*. Consequently, there is a morphism*

$$R_{\mathscr{S}} \twoheadrightarrow \mathbb{T}^S(\Gamma)_{\mathfrak{m}}$$

in $\mathrm{CNL}_{\mathscr{O}}$.

Remark. Our goal is to prove that the morphism $R_{\mathscr{S}} \to \mathbb{T}^S(\Gamma)_{\mathfrak{m}}$ is actually an isomorphism.

Proof. This theorem is a consequence of two results:

(1) we can express
$$\mathbb{T}^S(\Gamma)_{\mathfrak{m}} \otimes_{\mathscr{O}} \overline{\mathbb{Q}}_p \cong \prod_{\text{eigen.}} \overline{\mathbb{Q}}_p,$$
where the product is over the Hecke eigensystems in $S_2(\Gamma, \overline{\mathbb{Q}}_p)$ that are congruent to that of g modulo p, and $\mathbb{T}^S(\Gamma)_{\mathfrak{m}}$ is p–torsion-free, and

(2) the local-global compatibility for such eigensystems.

Take a Hecke eigenform $f \in S_2(\Gamma, \mathscr{O})$ congruent to g modulo ϖ.

- The representation ρ_f is unramified outside of pN, as $\Gamma_1(N) \subseteq \Gamma$.
- The nebentype χ of f factors through $(\mathbb{Z}/N\mathbb{Z})^\times / \operatorname{Ker}(\eta)$, by the definition of Γ. But $\chi \equiv \eta \mod \varpi$ because $f \equiv g \mod \varpi$. Since the order of η is prime to p, we have that $\operatorname{Ker}(\eta) = \operatorname{Ker}(\overline{\eta})$. All in all, $\chi = \eta$ and $\det(\rho_f) = \eta\epsilon_p^{-1}$.
- Take $\ell \mid N$ and write C for the conductor of η. If $\ell \nmid C$, we know from section 4.2 that
$$1 \neq \rho_f(I_\ell) \subseteq \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}.$$
In particular, $\overline{\rho}(I_\ell) \neq 1$ and ρ_f defines a minimal deformation (in the sense of example 34). If $\ell \mid C$, we know from section 4.2 that
$$\rho_f|_{I_\ell} = 1 \oplus \eta$$
and η has order prime to p, so $\eta(I_\ell) = \overline{\eta}(I_\ell)$; therefore, ρ_f defines a minimal deformation (in the sense of example 35).
- At p, one can use Fontaine–Laffaille's theory to deduce that
$$\rho_f|_{G_{\mathbb{Q}_p}} \cong \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix} \quad \text{with } \chi_1|_{I_p} = 1 \text{ and } \chi_2|_{I_p} = \epsilon_p^{-1}.$$
Therefore, $\rho_f|_{G_{\mathbb{Q}_p}}$ gives an ordinary deformation (in the sense of example 33).

Now we have a Galois representation
$$\rho_{\mathfrak{m}} \colon G_{\mathbb{Q},S} \to \mathrm{GL}_2\big(\mathbb{T}^S(\Gamma)_{\mathfrak{m}}\big)$$
lifting $\overline{\rho}$ and unramified outside S, which induces a morphism
$$R_{\overline{\rho}}^{\text{univ}} \twoheadrightarrow \mathbb{T}^S(\Gamma)_{\mathfrak{m}}.$$

This morphism is surjective because $\mathbb{T}^S(\Gamma)_\mathfrak{m}$ is generated by the operators T_ℓ and S_ℓ for $\ell \notin S$ and the fact that

$$\mathrm{CharPoly}(\rho_\mathfrak{m}(\mathrm{Frob}_\ell)) = X^2 - \mathrm{T}_\ell\, X + \ell\, \mathrm{S}_\ell$$

implies that T_ℓ and S_ℓ appear in the image. It remains to show that the map $R_{\overline{\rho}}^{\mathrm{univ}} \twoheadrightarrow \mathbb{T}^S(\Gamma)_\mathfrak{m}$ factors through $R_{\mathscr{S}}$. As $\mathbb{T}^S(\Gamma)_\mathfrak{m}$ is embedded in $\mathbb{T}^S(\Gamma)_\mathfrak{m} \otimes_{\mathscr{O}} \overline{\mathbb{Q}}_p$, it suffices to prove that the composition

$$R_{\overline{\rho}}^{\mathrm{univ}} \to \mathbb{T}^S(\Gamma)_\mathfrak{m} \to \mathbb{T}^S(\Gamma)_\mathfrak{m} \otimes_{\mathscr{O}} \overline{\mathbb{Q}}_p$$

factors through $R_{\mathscr{S}}$. But

$$\rho_\mathfrak{m} \otimes \overline{\mathbb{Q}}_p = \prod_f \rho_f \colon G_{\mathbb{Q},S} \to \mathrm{GL}_2\Big(\prod_f \overline{\mathbb{Q}}_p\Big),$$

where the product runs over the eigenforms f congruent to g modulo ϖ as above, and each ρ_f is of type $\mathscr{S}$. This completes the proof of the theorem. □

4.3.1 The minimal case

Consider a Taylor–Wiles datum $(Q, (\alpha_v)_{v\in Q})$. From this, we defined an augmented global deformation problem $\mathscr{S}_Q$, a finite p–group Δ_Q and $\mathbb{T}_Q^{S\cup Q}(\Gamma_Q)_{\mathfrak{m}_Q}$ (cf. section 4.1). Let $\mathbb{T}^{S\cup Q}(\Gamma_Q)_{\mathfrak{m}_Q}$ be the subalgebra of $\mathrm{End}_{\mathscr{O}}\big(\mathrm{H}_1(\Gamma_Q, \mathscr{O})_{\mathfrak{m}_Q}\big)$ generated by the operators T_ℓ and S_ℓ for the primes $\ell \notin S \cup Q$ and the operators $\langle\delta\rangle$ for $\delta \in \Delta_Q$.

Theorem 68. *There exists a continuous Galois representation*

$$\rho_Q \colon G_{\mathbb{Q},S\cup Q} \to \mathrm{GL}_2\big(\mathbb{T}^{S\cup Q}(\Gamma)_{\mathfrak{m}_Q}\big)$$

satisfying the following properties:

(1) *for every prime $\ell \notin S \cup Q$, ρ_Q is unramified at ℓ and*

$$\mathrm{CharPoly}(\rho_Q(\mathrm{Frob}_\ell)) = \mathrm{X}^2 - \mathrm{T}_\ell\, \mathrm{X} + \ell\, \mathrm{S}_\ell\,;$$

(2) *for every $v \in S$, the lift $\rho_Q|_{G_{\mathbb{Q}_v}}$ is in D_v, and*

(3) *for every $v \in Q$,*

$$\rho_Q|_{I_{\mathbb{Q}_v}} \cong \begin{pmatrix} 1 & 0 \\ 0 & \chi_v \end{pmatrix} \quad \textit{with } \chi_v \circ \mathrm{rec}_{\mathbb{Q}_v}(\delta) = \langle\delta\rangle,$$

where $\mathrm{rec}_{\mathbb{Q}_v} \colon \mathscr{O}_{\mathbb{Q}_v}^{\times} \to G_{\mathbb{Q}_v}^{\mathrm{ab}}$ *denotes Artin's local reciprocity map.*

Proof. It works like theorem 67, using the local-global compatibility. $\square$

Note that $\eta_Q = \det(\rho_Q)\psi^{-1}$ is a finite character of p–power order and so (as $p > 2$) admits a square root $\eta_Q^{1/2}$. The twist

$$\rho_Q' = \rho_Q \otimes \eta_Q^{-1/2}$$

is then of type $\mathscr{S}_Q$ and we obtain a morphism $R_{\mathscr{S}_Q} \to \mathbb{T}^{S \cup Q}(\Gamma)_{\mathfrak{m}_Q}$ (representing ρ_Q). That is, $\mathrm{H}_1(Y_Q, \mathscr{O})_{\mathfrak{m}_Q}$ is an $R_{\mathscr{S}_Q}$–module in a way that is compatible with the structure of $\mathscr{O}[\Delta_Q]$–module (cf. proposition 65).

Proposition 69. *There exist* $q \in \mathbb{Z}_{\geq 0}$ *and a commutative diagram*

$$\begin{tikzcd}
S_\infty \arrow[r] & R_\infty \arrow[r, phantom, "\circlearrowright"] \arrow[d, two heads] & H_\infty \arrow[d, two heads, "\bmod \mathfrak{a}"] \\
 & R = R_{\mathscr{S}} \arrow[r, phantom, "\circlearrowright"] & H = \mathrm{H}_1(Y, \mathscr{O})_{\mathfrak{m}}
\end{tikzcd}$$

of actions, where

1. $S_\infty = \mathscr{O}[\![y_1, \ldots, y_q]\!]$ *and* $\mathfrak{a} = (y_1, \ldots, y_q) \subset S_\infty$,
2. $R_\infty = \mathscr{O}[\![x_1, \ldots, x_q]\!]$ *and it admits a surjection* $R_\infty \twoheadrightarrow R$ *whose kernel contains* $\mathfrak{a}R_\infty$, *and*
3. H_∞ *is a finitely generated* R_∞*–module that becomes finite free as an* S_∞*–module and admits a surjection* $H_\infty \twoheadrightarrow H$ *with kernel* $\mathfrak{a}H_\infty$.

Theorem 70. *The natural map* $R_{\mathscr{S}} \to \mathbb{T}^S(\Gamma)_{\mathfrak{m}}$ *(cf. theorem 67) is an isomorphism of local complete intersection rings.*

Proof. Since H_∞ is free over S_∞ and the S_∞–module structure factors through R_∞ by proposition 69, we have

$$\begin{aligned} 1 + q = \dim(R_\infty) \geq \dim_{R_\infty}(H_\infty) &\geq \mathrm{depth}_{R_\infty}(H_\infty) \\ &\geq \mathrm{depth}_{S_\infty}(H_\infty) = \dim(S_\infty) = 1 + q \end{aligned}$$

and so all these inequalities must be equalities.

Since R_∞ is regular, then H_∞ has a projective resolution of finite length by Serre's theorem. More precisely, we can use the Auslander–Buchsbaum formula:

$$\mathrm{projdim}_{R_\infty}(H_\infty) = \mathrm{depth}(R_\infty) - \mathrm{depth}_{R_\infty}(H_\infty) = (1 + q) - (1 + q) = 0.$$

Therefore, H_∞ is a projective R_∞–module and must be free because R_∞ is local.

By proposition 69, $H \cong H_\infty/\mathfrak{a}H_\infty$ is a free module over $R \cong R_\infty/\mathfrak{a}R_\infty$. But the R–action on $H = \mathrm{H}_1(Y, \mathscr{O})_\mathfrak{m}$ is defined via the surjection $R_\mathscr{S} \twoheadrightarrow \mathbb{T}^S(\Gamma)_\mathfrak{m}$, which must thus have trivial kernel. In conclusion, $R_\mathscr{S} \cong \mathbb{T}^S(\Gamma)_\mathfrak{m}$. Moreover, these rings are complete intersection rings because we have a presentation

$$R_\mathscr{S} \cong R_\infty/\mathfrak{a} = \mathscr{O}[\![x_1, \dots, x_q]\!]/(y_1, \dots, y_q)$$

and $\dim(R_\mathscr{S}) = \dim(\mathbb{T}^S(\Gamma)_\mathfrak{m}) = 1$. □

Next we want to prove proposition 69. We use the results of the last part of section 3.7 for $T = \varnothing$ (in particular, see corollary 56 and the paragraphs that follow). Set $q = h^1_{\mathscr{S}^\perp}(\mathbb{Q}, \mathrm{ad}^0(\overline{\rho})(1))$. We work with $S_\infty = \mathscr{O}[\![\mathbb{Z}_p^q]\!]$ (that can be identified with $\mathscr{O}[\![y_1, \dots, y_q]\!]$). For every $N \in \mathbb{Z}_{\geq 1}$, define

$$\begin{aligned} \mathfrak{a}_N &= \mathrm{Ker}\big(S_\infty \twoheadrightarrow \mathscr{O}[(\mathbb{Z}/p^N\mathbb{Z})^q]\big), \\ S_N &= S_\infty/(\varpi^N, \mathfrak{a}_N), \\ \mathfrak{d}_N &= \big(\varpi^N, \mathrm{Ann}_R(H)^N\big) \subset R \quad \text{(open ideal).} \end{aligned}$$

Definition 71. A *patching datum of level* N is a triple (f, X, g), where

- $f\colon R_\infty \twoheadrightarrow R/\mathfrak{d}_N$ is a surjective morphism in $\mathrm{CNL}_\mathscr{O}$,
- X is an $(R_\infty \otimes_\mathscr{O} S_N)$–module that becomes finite free over S_N such that
  - $\mathrm{Im}\big(S_N \to \mathrm{End}_\mathscr{O}(X)\big) \subseteq \mathrm{Im}\big(R_\infty \to \mathrm{End}_\mathscr{O}(X)\big)$ and
  - $\mathrm{Im}\big(\mathfrak{a} \to \mathrm{End}_\mathscr{O}(X)\big) \subseteq \mathrm{Im}\big(\mathrm{Ker}(f) \to \mathrm{End}_\mathscr{O}(X)\big)$,

  and
- $g\colon X/\mathfrak{a} \to H/(\varpi^N)$ is an isomorphism of R_∞–modules, with R_∞–module structure on $H/(\varpi^N)$ given by f.

We say that two patching data (f, X, g) and (f', X', g') of level N are *isomorphic* if $f = f'$ and there is an isomorphism $X \cong X'$ of $(R_\infty \otimes_\mathscr{O} S_N)$–modules by means of which g and g' are compatible.

Remark. Given $N \in \mathbb{Z}_{\geq 1}$, there are only finitely many isomorphism classes of patching data of level N. In addition, given $M \in \mathbb{Z}_{\geq 1}$ with $M \geq N$, every patching datum $D = (f, X, g)$ of level M induces a patching datum

$$(D \mod N) = (f \mod \mathfrak{d}_N, X \otimes_{S_M} S_N, g \otimes_{S_M} S_N)$$

of level N.

Proof of proposition 69. For each $M \geq 1$, we can choose a Taylor–Wiles datum $(Q_M, (\alpha_v)_{v \in Q_M})$ of level M such that $|Q_M| = q$ and $h^1_{\mathscr{S}^\perp_{Q_M}}(\mathbb{Q}, \mathrm{ad}^0(\overline{\rho})(1)) = 0$, by proposition 55. We define a patching datum $D_M = (f_M, X_M, g_M)$ as follows:

- we construct the composition
$$f_M \colon R_\infty \twoheadrightarrow R_{\mathscr{S}_{Q_M}} \twoheadrightarrow R_{\mathscr{S}_{Q_M}}/\mathfrak{a}_{Q_M} \cong R \twoheadrightarrow R/\mathfrak{d}_M,$$
where the first arrow is given by corollary 56 and the other arrows are the canonical projections;
- let $X_M = \mathrm{H}_1(Y_{Q_M}, \mathscr{O})_{\mathfrak{m}_{Q_M}} \otimes_{S_\infty} S_M$, which is a finite free S_M–module because $\mathrm{H}_1(Y_{Q_M}, \mathscr{O})_{\mathfrak{m}_{Q_M}}$ is a free $\mathscr{O}[\Delta_{Q_M}]$–module (see proposition 64) and the projection $S_\infty \twoheadrightarrow S_M$ factors through $\mathscr{O}[\Delta_{Q_M}]$, while the R_∞–action is via
$$R_\infty \twoheadrightarrow R_{\mathscr{S}_{Q_M}} \twoheadrightarrow \mathbb{T}^{S \cup Q_M}(\Gamma_{Q_M})_{\mathfrak{m}_{Q_M}},$$
and
- the isomorphism $g_M \colon X_M/\mathfrak{a} \to H/(\varpi^M)$ is induced by the isomorphism
$$\mathrm{H}_1(Y_{Q_M}, \mathscr{O})_{\mathfrak{m}_{Q_M}}/(\mathfrak{a}_{Q_M}) \cong \mathrm{H}_1(Y, \mathscr{O})_{\mathfrak{m}} = H$$
from proposition 65.

Now, for $M \geq N \geq 1$, we can define a patching datum of level N
$$D_{M,N} = (D_M \mod N) = (f_{M,N}, X_{M,N}, g_{M,N}).$$

Since there are only finitely many isomorphism classes of patching data of a fixed level N but there are infinitely many $M \in \mathbb{Z}$ such that $M \geq N$, we can find a subsequence $(M_i, N_i)_{i \geq 1}$ with $M_i \geq N_i$ and $N_{i+1} > N_i$ and such that
$$(D_{M_{i+1}, N_{i+1}} \mod N_i) \cong D_{M_i, N_i}.$$

Finally, we define
$$H_\infty = \varprojlim_{i \geq 1} X_{M_i}.$$

The desired projections are
$$\varprojlim_{i \geq 1} f_{M_i, N_i} \colon R_\infty \twoheadrightarrow R \quad \text{and} \quad \varprojlim_{i \geq 1} g_{M_i, N_i} \colon H_\infty \twoheadrightarrow H.$$

From

$$\operatorname{Im}\big(S_{M_i} \to \operatorname{End}_{\mathscr{O}}(X_{M_i,N_i})\big) \subseteq \operatorname{Im}\big(R_\infty \to \operatorname{End}_{\mathscr{O}}(X_{M_i,N_i})\big),$$

we deduce that

$$\operatorname{Im}\big(S_\infty \to \operatorname{End}_{\mathscr{O}}(H_\infty)\big) \subseteq \operatorname{Im}\big(R_\infty \to \operatorname{End}_{\mathscr{O}}(H_\infty)\big).$$

Since S_∞ is a ring of power series, we can choose the morphism $S_\infty \to R_\infty$ lifting $S_\infty \to \operatorname{End}_{\mathscr{O}}(H_\infty)$. Similarly, from

$$\operatorname{Im}\big(\mathfrak{a} \to \operatorname{End}_{\mathscr{O}}(X_{M_i,N_i})\big) \subseteq \operatorname{Im}\big(\operatorname{Ker}(f_{M_i,N_i}) \to \operatorname{End}_{\mathscr{O}}(X_{M_i,N_i})\big),$$

we deduce that

$$\operatorname{Im}\big(\mathfrak{a} \to \operatorname{End}_{\mathscr{O}}(H_\infty)\big) \subseteq \operatorname{Im}\big(\operatorname{Ker}(R_\infty \twoheadrightarrow R) \to \operatorname{End}_{\mathscr{O}}(H_\infty)\big),$$

whence the diagram

$$\begin{tikzcd}
S_\infty \arrow[r] & R_\infty \arrow[d, two heads] \arrow[r, phantom, "\circlearrowright"] & H_\infty \arrow[d, two heads] \\
 & R \arrow[r, phantom, "\circlearrowright"] & H
\end{tikzcd}$$

commutes. ☐

As a consequence of theorem 70, we have the following result:

Theorem 72. *Let p be a prime number > 2 and let $\rho\colon G_{\mathbb{Q}} \to \operatorname{GL}_2(\overline{\mathbb{Q}}_p)$ be a continuous representation satisfying the following conditions:*

1. *ρ is unramified outside finitely many primes,*
2. *we can express*

$$\rho|_{G_{\mathbb{Q}_p}} \cong \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix} \quad \text{with } \chi_1|_{I_p} = 1 \text{ and } \chi_2|_{I_p} = \epsilon_p^{-1},$$

3. *$\overline{\rho}|_{G_{\mathbb{Q}(\zeta_p)}}$ is absolutely irreducible with adequate image,*
4. *for every prime $\ell \neq p$ at which ρ is ramified,*
    - *either*

$$\rho|_{I_\ell} \cong \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix}$$

      and the reduction map $\rho(I_\ell) \to \overline{\rho}(I_\ell)$ is an isomorphism
    - *or*

$$\rho|_{I_\ell} \cong \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$$

but $\overline{\rho}(I_\ell) \neq 1$*,*

whereas at p we have

$$\overline{\rho}|_{G_{\mathbb{Q}_p}} \cong \begin{pmatrix} \overline{\chi}_1 & * \\ 0 & \overline{\chi}_2 \end{pmatrix} \quad \textit{with } \overline{\chi}_1\overline{\chi}_2^{-1} \neq 1 \textit{ or } \overline{\epsilon}_p,$$

and

(5) $\overline{\rho} \cong \overline{\rho}_g$ *for some* $g \in S_2(\Gamma_1(N), \overline{\mathbb{Q}}_p)$ *with*

$$N = \prod_{\ell \neq p} \ell,$$

where the product runs over the primes where ρ *is ramified.*

Then $\rho \cong \rho_f$ *for some Hecke eigenform* $f \in S_2(\Gamma_1(N), \overline{\mathbb{Q}}_p)$*.*

Sketch of the proof. One checks that the assumptions of the theorem imply, after fixing a model $\rho\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathscr{O}')$, for $\mathscr{O}'$ the ring of integers of some finite extension of $\mathbb{Q}_p$, that ρ defines a morphism $R_{\mathscr{S}} \to \mathscr{O}'$ of $\mathscr{O}$–algebras, where $\mathscr{S}$ is a global deformation problem as in the rest of this section. Combining this with the isomorphism $R_{\mathscr{S}} \cong \mathbb{T}^S(\Gamma)_{\mathfrak{m}}$, where $S = \{\, \ell \mid N \,\} \cup \{\, p \,\}$, we obtain a morphism $\lambda\colon \mathbb{T}^S(\Gamma)_{\mathfrak{m}} \to \mathscr{O}'$ of $\mathscr{O}$–algebras with the property that, for every prime $\ell \notin S$,

$$\mathrm{CharPoly}(\rho(\mathrm{Frob}_\ell)) = X^2 - \lambda(\mathrm{T}_\ell)X + \ell\lambda(\mathrm{S}_\ell).$$

Such a λ is the eigensystem of some $f \in S_2(\Gamma_1(N), \overline{\mathbb{Q}}_p)$. $\square$

Remark. Condition (4) is very restrictive. To remove it, Wiles defined a "numerical criterion" (which seemed hard to generalize but experiences a revival in current research). Alternatively, Kisin presented global deformation rings as algebras over local framed deformation rings (that is what we will do next).

4.3.2 The non-minimal case

Continue to assume that $\overline{\rho}$ is modular (i.e., $\cong \overline{\rho}_g$) and that $\overline{\rho}|_{G_{\mathbb{Q}(\zeta_p)}}$ is absolutely irreducible with adequate image, but let us drop the "minimality hypotheses", so maybe the integer N appearing in the level $\Gamma_1(N)$ is not square-free and we want to allow lifts ramified at primes ℓ at which $\overline{\rho}$ is unramified.

Say that we have a global deformation problem

$$\mathscr{S} = (\overline{\rho}, S, \psi, \mathscr{O}, (D_v)_{v \in S})$$

such that we can prove that

$$\rho_{\mathfrak{m}} \colon G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{T}^S(\Gamma)_{\mathfrak{m}})$$

is of type $\mathscr{S}$ and that we expect all deformations of $\overline{\rho}$ of type $\mathscr{S}$ come from $\mathbb{T}^S(\Gamma)_{\mathfrak{m}}$. Assume that, for every place $v \in S$, we have $D_v \subseteq D_v^{\square,\psi}$ and the ring R_v representing D_v is $\mathscr{O}$–flat and pure with

$$\dim(R_v) = \begin{cases} 1+3 & \text{if } v \neq p, \\ 1+3+1 & \text{if } v = p, \end{cases}$$

where the first 1 comes from $\mathscr{O}$, the 3 comes from the dimension of the space of values of Frobenius (which are 2×2 matrices with a fixed determinant) and the last 1 is the index of $\mathbb{Q}_v$ over $\mathbb{Q}_v$ itself (cf. the assumptions of section 3.7).

We consider frames at $T = S$ and

$$R_{\mathscr{S}}^{S-\mathrm{loc}} = \bigotimes_{v \in S} R_v,$$

which is $\mathscr{O}$–flat of dimension $2 + 3|S|$. Recall also that

$$R_{\mathscr{S}}^S \cong R_{\mathscr{S}} \mathbin{\widehat{\otimes}_{\mathscr{O}}} \mathcal{T}, \quad \text{where } \mathcal{T} = \mathscr{O}[\![z_1, \ldots, z_{4|S|-1}]\!].$$

Proposition 73. *There exist $q \in \mathbb{Z}_{\geq 0}$ and a commutative diagram*

$$\begin{tikzcd} S_\infty \arrow[r] & R_\infty \arrow[d, two heads, "\bmod \mathfrak{a}"] \arrow[r, phantom, "\circlearrowright"] & H_\infty \arrow[d, two heads, "\bmod \mathfrak{a}"] \\ & R = R_{\mathscr{S}} \arrow[r, phantom, "\circlearrowright"] & H = \mathrm{H}_1(Y, \mathscr{O})_{\mathfrak{m}} \end{tikzcd}$$

of actions, where

1. $S_\infty = \mathcal{T}[\![y_1, \ldots, y_q]\!]$ *and* $\mathfrak{a} = (z_1, \ldots, z_{4|S|-1}, y_1, \ldots, y_q) \subset S_\infty$,
2. $R_\infty = R_{\mathscr{S}}^{S-\mathrm{loc}}[\![x_1, \ldots, x_g]\!]$, *for some* $g \in \mathbb{Z}_{\geq 0}$ *such that*

   $$\dim(S_\infty) = 4|S| + q = g + 2 + 3|S| = \dim(R_\infty),$$

   and it admits a surjection $R_\infty \twoheadrightarrow R$ *whose kernel contains* $\mathfrak{a}R_\infty$, *and*
3. H_∞ *is a finitely generated* R_∞*–module that becomes finite free as an* S_∞*–module and admits a surjection* $H_\infty \twoheadrightarrow H$ *with kernel* $\mathfrak{a}H_\infty$.

Sketch of the proof. The proof of this result is analogous to that of proposition 69:

- One needs to use the computations of Galois cohomology from section 3.7 (especially corollary 56) in the case that $T = S$.
- We have to use the framed deformation rings $R^S_{\mathscr{S}_{Q_N}}$ to define the maps

  $$f_N \colon R_\infty \twoheadrightarrow R^S_{\mathscr{S}_{Q_N}} \twoheadrightarrow R/\mathfrak{d}_N.$$
- The modules X_N have to be defined using

  $$\mathrm{H}_1(Y_{Q_N}, \mathscr{O})_{\mathfrak{m}_{Q_N}} \widehat{\otimes}_{R_{\mathscr{S}_{Q_N}}} R^S_{\mathscr{S}_{Q_N}} \cong \mathrm{H}_1(Y_{Q_N}, \mathscr{O})_{\mathfrak{m}_{Q_N}} \widehat{\otimes}_{\mathscr{O}} \mathcal{T},$$

  where the last module is free over $\mathcal{T}[\Delta_{Q_N}]$. □

We can try to proceed as in section 4.3.1. From the chain of inequalities

$$\dim(R_\infty) \geq \dim_{R_\infty}(H_\infty) \geq \mathrm{depth}_{R_\infty}(H_\infty) \geq \mathrm{depth}_{S_\infty}(H_\infty) = \dim(S_\infty)$$

and the fact that $\dim(R_\infty) = \dim(S_\infty)$, all the inequalities must be equalities. In addition, H_∞ is a Cohen–Macaulay R_∞–module and $\mathrm{Supp}_{R_\infty}(H_\infty)$ is a union of irreducible components of $\mathrm{Spec}(R_\infty)$. However, in this case the ring R_∞ might not be regular and we cannot apply the Auslander–Buchsbaum formula as in section 4.3.1.

Proposition 74. *If* $\mathrm{Supp}_{R_\infty}(H_\infty) = \mathrm{Spec}(R_\infty)$, *then* $\mathrm{Supp}_R(H) = \mathrm{Spec}(R)$ *and the natural map*

$$R = R_{\mathscr{S}} \twoheadrightarrow \mathbb{T}^S(\Gamma)_\mathfrak{m}$$

has nilpotent kernel.

Proof. Take $\mathfrak{p} \in \mathrm{Spec}(R)$ and let $\mathfrak{p}_\infty$ be its pull-back in R_∞. By hypothesis, we know that $(H_\infty)_{\mathfrak{p}_\infty} \neq 0$. Since H_∞ is finitely generated over R_∞, by Nakayama's lemma,

$$H_\mathfrak{p} \cong (H_\infty/\mathfrak{a}H_\infty)_\mathfrak{p} = (H_\infty)_{\mathfrak{p}_\infty}/\mathfrak{a}(H_\infty)_{\mathfrak{p}_\infty} \neq 0$$

and so $\mathfrak{p} \in \mathrm{Supp}_R(H)$. Therefore, $\mathrm{Supp}_R(H) = \mathrm{Spec}(R)$. This implies that $\mathrm{Ann}_R(H)$ is nilpotent and, as the action of R on H factors through the natural map

$$R \twoheadrightarrow \mathbb{T}^S(\Gamma)_\mathfrak{m}$$

and $\mathbb{T}^S(\Gamma)_\mathfrak{m}$ acts faithfully on H, the kernel of the last map must be nilpotent. □

Remark. The ring $R^{\mathrm{red}}_{\mathscr{S}}$ (i.e., the quotient of $R_{\mathscr{S}}$ by its nilpotent ideal) is $\cong \mathbb{T}^S(\Gamma)_\mathfrak{m}$ and that is good enough for modularity lifting. (However, it is not enough for applications to adjoint Bloch–Kato conjectures.)

For modularity lifting, we want to know that H_∞ has full support in $\mathrm{Spec}(R_\infty)$. Since $\mathrm{Supp}_{R_\infty}(H_\infty)$ is a union of irreducible components of $\mathrm{Spec}(R_\infty)$, we have to prove that it contains all these irreducible components. But the map

$$\mathrm{Spec}(R_\infty) = \mathrm{Spec}\big(R_{\mathscr{S}}^{S-\mathrm{loc}}[\![x_1, \dots, x_g]\!]\big) \to \mathrm{Spec}(R_{\mathscr{S}}^{S-\mathrm{loc}})$$

induces a bijection on the sets of irreducible components and an irreducible component X of $\mathrm{Spec}(R_{\mathscr{S}}^{S-\mathrm{loc}})$ is of the form

$$X = \prod_{v \in S} X_v,$$

where X_v is an irreducible component of $\mathrm{Spec}(R_v)$ for every $v \in S$. Thus, for each $v \in S$, we want to

(1) understand the irreducible components of $\mathrm{Spec}(R_v)$ and
(2) produce congruences from g (as $\overline{\rho} \cong \overline{\rho}_g$), which lies on one component, to modular forms lying on other components.

There are two clearly different cases to consider.

- If $v \nmid p$, one can use level raising/lowering using Ihara's lemma (this method has not been generalized to higher ranks) or Taylor's Ihara avoidance trick.
- If $v \mid p$, the situation is more complicated. It is related to the Breuil–Mézard conjecture and to the weight part of Serre's conjecture.

The minimal modularity lifting follows from an $R \cong \mathbb{T}$ theorem. Similarly, the non-minimal modularity lifting follows from an $R^{\mathrm{red}} \cong \mathbb{T}$ theorem *assuming* that H_∞ has full support over R_∞. Next, we want to show how to prove this full support result in some cases.

4.4 A result over totally real fields

We will sketch the proof of the following modularity lifting theorem:

Theorem 75. *Let F be a totally real field and take a prime number $p \geq 5$ that is unramified in F. Let $\rho\colon G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ be a continuous irreducible representation satisfying that*

(1) *ρ is unramified outside finitely many primes,*
(2) *for every place $v \mid p$ of F, the restriction $\rho|_{G_{F_v}}$ is crystalline and all its labelled Hodge–Tate weights are 0 or 1,*
(3) *the representation $\overline{\rho}|_{G_{F(\zeta_p)}}$ is absolutely irreducible with adequate image and*
(4) *$\overline{\rho} \cong \overline{\rho}_g$ for a Hilbert modular cusp form of parallel weight 2 and level prime to p.*

Then $\rho \cong \rho_f$ for some Hilbert modular cusp form f of parallel weight 2.

Using cyclic base change (work of Saito and Shintani), we can prove the following result:

Theorem 76. *Let L/F be a finite solvable totally real Galois extension. Consider a representation $\rho\colon G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ and a Hilbert modular form g as in theorem 75.*

1. *If $\rho_g|_{G_L}$ is irreducible, then there exists a Hilbert modular cusp form G over L which is the base change of g to L. In particular,*
$$\rho_g|_{G_L} \cong \rho_G.$$
2. *If $\rho|_{G_L} \cong \rho_H$ for a Hilbert modular cusp form H over L, then $\rho \cong \rho_f$ for a Hilbert modular cusp form f over F.*

Lemma 77. *Let K be a number field and let S be a finite set of places of K. For each $v \in S$, let K'_v/K_v be a finite Galois extension. There exists a finite solvable Galois extension L/K with the property that, for every place w of L above a place $v \in S$, $L_w \cong K'_v$ as K_v–algebras.*

Idea of the proof. It is an application of the Grunwald–Wang theorem. □

Let $S_p = \{\, v \text{ place of } F : v \mid p \,\}$ and $S_\infty = \{\, v \text{ place of } F : v \mid \infty \,\}$. Let Σ denote the set of finite places of F at which ρ or g are ramified. In particular, $\Sigma \cap S_p = \emptyset$. Let $M/F(\zeta_p)$ be the extension cut out by $\overline{\rho}|_{G_{F(\zeta_p)}}$, which is a Galois extension. We can find a finite set V of finite places of F such that $V \cap (S_p \cup \Sigma) = \emptyset$ and every non-trivial conjugacy class in $\mathrm{Gal}(M/F)$ is of the form Frob_v for some $v \in V$. Now we apply lemma 77 with $K = F$ and $S = S_\infty \cup S_p \cup \Sigma \cup V$ with the following local extensions:

1. for every $v \in S_\infty$, $K'_v = F_v \cong \mathbb{R}$;
2. for every $v \in S_p$, $K'_v = F_v$;
3. for every $v \in \Sigma$, we take K'_v/F_v to be an extension such that, if $\rho|_{G_{K'_v}}$ is ramified, then the ramification is unipotent, and similarly for g, such that $\overline{\rho}|_{G_{K'_v}} = 1$ and such that the cardinality of the residue field of K'_v is $\equiv 1 \bmod p$;
4. for every $v \in V$, $K'_v = F_v$.

Thus, we obtain a finite solvable Galois extension L/F with the following properties:

1. L is totally real;
2. p is unramified in L;
3. for every prime w of L at which $\rho|_{G_L}$ is ramified, the ramification is unipotent; similarly, writing G for the base change of g to L, for every prime w of L at which G is ramified, G has Iwahori level at w;

(4) $M \cap L = F$ and, in particular, $\overline{\rho}|_{G_{L(\zeta_p)}}$ is absolutely irreducible with adequate image.

Therefore, after replacing F with L, we have simplified our original situation because now ramification of ρ or g is of a simple kind. By further replacing F with $F \cdot F_0$ for an appropriate quadratic extension $F_0/\mathbb{Q}$ (i.e., $F_0/\mathbb{Q}$ is disjoint from $M/\mathbb{Q}$ and p is unramified in F_0), we can assume that $[F : \mathbb{Q}]$ is even.

Now we can consider the unique (up to isomorphism) quaternion algebra D over F which is ramified exactly at S_∞. For every place v of F, write $D_v = D \otimes_F F_v$. Fix a maximal order $\mathscr{O}_D$ of D and an isomorphism

$$\mathscr{O}_D \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} \cong \mathrm{M}_2(\mathscr{O}_F \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}) \cong \prod_{v \nmid \infty} \mathrm{M}_2(\mathscr{O}_{F_v}).$$

We have an isomorphism

$$(\mathscr{O}_D \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}})^\times \cong \mathrm{GL}_2(\mathscr{O}_D \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}) \cong \prod_{v \nmid \infty} \mathrm{GL}_2(\mathscr{O}_{F_v})$$

which can be extended to

$$(D \otimes_F \mathbb{A}_F^\infty)^\times \cong \mathrm{GL}_2(\mathbb{A}_F^\infty).$$

Fix an open compact subgroup U of $(\mathscr{O}_D \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}})^\times$ and identify it with an open compact subgroup of

$$\prod_{v \nmid \infty} \mathrm{GL}_2(\mathscr{O}_{F_v}).$$

(Later we will need a precise choice of such U.)

By our assumptions on ρ and g, the characters $\det(\rho)\epsilon_p$ and $\det(\rho_g)\epsilon_p$ are finite (and unramified) and $\det(\rho)\det(\rho_g)^{-1}$ is finite of p–power order. Since we assume that $p > 2$, the character $\det(\rho)\det(\rho_g)^{-1}$ admits a square root and, up to twisting, we can assume that

$$\det(\rho) = \det(\rho_g) = \eta\epsilon_p^{-1}$$

for some unramified character η of finite order. Let $\mathscr{O}$ be the ring of integers of some finite extension of $\mathbb{Q}_p$ such that ρ takes values in $\mathrm{GL}_2(\mathscr{O})$. For every $\mathscr{O}$–algebra A, define

$$S_{2,\eta}(U, A) = \Big\{ f \colon D^\times \backslash (D \otimes \mathbb{A}_F^\infty)^\times \to A \text{ continuous } : \\ f(gu) = f(g) \text{ for all } u \in U \text{ and}$$

$$f(gz) = \eta(z) f(g) \text{ for all } z \in (\mathbb{A}_F^\infty)^\times \Big\},$$

where (by abuse of notation) we write η for the composition of η with the Artin reciprocity map $\mathrm{rec}_F \colon F^\times \backslash \mathbb{A}_F^\times \to \mathscr{O}^\times$. This space has an action of the Hecke operators

$$\mathrm{T}_v = \left[\mathrm{GL}_2(\mathscr{O}_{F_v}) \begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix} \mathrm{GL}_2(\mathscr{O}_{F_v}) \right]$$

and

$$\mathrm{S}_v = \left[\mathrm{GL}_2(\mathscr{O}_{F_v}) \begin{pmatrix} \varpi_v & 0 \\ 0 & \varpi_v \end{pmatrix} \mathrm{GL}_2(\mathscr{O}_{F_v}) \right]$$

for all places v of F such that $U_v = \mathrm{GL}_2(\mathscr{O}_{F_v})$. (Observe that S_v simply acts by $\eta(\varpi_v)$.) That is to say, defining $S = \{ v \mid p \} \cup \{ v : U_v \neq \mathrm{GL}_2(\mathscr{O}_{F_v}) \}$, we have an action of

$$\mathbb{T}^{S,\mathrm{univ}} = \mathscr{O}[\mathrm{T}_v, \mathrm{S}_v : v \notin S]$$

on $S_{2,\eta}(U, A)$.

Theorem 78 (Jacquet–Langlands). *Recall that we have a fixed isomorphism $\overline{\mathbb{Q}}_p \cong \mathbb{C}$. The Hecke eigensystems appearing in the space of Hilbert modular cusp forms of parallel weight 2 and level U and nebentype η are in bijection with the Hecke eigensystems appearing in $S_{2,\eta}(U, \mathbb{C})$ that do not factor through the reduced norm of D.*

Remark. The Hecke eigensystems of $S_{2,\eta}(U, \mathbb{C})$ that factor through the reduced norm of D are Eisenstein, which means that their associated Galois representations are reducible.

Consequently, we can transfer statements of Hilbert modular forms into statements in terms of $S_{2,\eta}(U, \overline{\mathbb{Q}}_p)$. In particular, it suffices to prove that $\rho \cong \rho_f$ for some $f \in S_{2,\eta}(U, \mathscr{O})$. Also by theorem 78, we have that $\rho \cong \overline{\rho}_g$ for some $g \in S_{2,\eta}(U, \mathscr{O})$, where the level

$$U \subseteq (\mathscr{O}_D \mathbin{\widehat{\otimes}}_{\mathbb{Z}} \widehat{\mathbb{Z}})^\times \cong \prod_{v \nmid \infty} \mathrm{GL}_2(\mathscr{O}_{F_v})$$

satisfies that

- for every place $v \notin \Sigma$, $U_v = \mathrm{GL}_2(\mathscr{O}_{F_v})$, and
- for every place $v \in \Sigma$,

$$U_v = \mathrm{Iw}_v = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathscr{O}_{F_v}) : c \equiv 0 \mod \varpi_v \right\}.$$

(Since Σ might contain places at which g is unramified, this U is not the "optimal" level in which g appears; however, it is the "optimal" level in which *both* g and the desired f appear.)

Writing

$$(D \otimes_F \mathbb{A}_F^\infty)^\times = \bigsqcup_{i \in I} D^\times t_i U(\mathbb{A}_F^\infty)^\times,$$

we can decompose

$$S_{2,\eta}(U, A) \cong \bigoplus_{i \in I} A(\eta^{-1})^{(U(\mathbb{A}_F^\infty)^\times \cap t_i^{-1} D t_i)/F^\times}$$

and this isomorphism is given by $f \mapsto (f(t_i))_{i \in I}$.

Lemma 79. *Each group $\big(U(\mathbb{A}_F^\infty)^\times \cap t_i^{-1} D t_i\big)/F^\times$, for $i \in I$, is finite and (since $p \geq 5$ and is unramified in F) has order prime to p.*

Corollary 80. *The functor on $\mathcal{O}$–algebras given by $A \mapsto S_{2,\eta}(U, A)$ is exact. In particular, $S_{2,\eta}(U, \mathcal{O})$ is a free $\mathcal{O}$–module and*

$$S_{2,\eta}(U, \mathcal{O})/(\varpi) \cong S_{2,\eta}(U, \mathbb{F}).$$

Given a Taylor–Wiles datum $(Q, (\alpha_v)_{v \in Q})$ for $\overline{\rho}$, we can proceed as in the beginning of section 4 (where we defined levels $\Gamma_0(Q)$ and Γ_Q) and define levels $U_0(Q)$ and U_Q as follows:

- for $v \notin Q$, set $U_0(Q)_v = U_{Q,v} = U_v$ and
- for $v \in Q$, set $U_0(Q)_v = \mathrm{Iw}_v$ and

$$U_{Q,v} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Iw}_v : ad^{-1} \in \mathrm{Ker}(\mathcal{O}_{F_v}^\times \twoheadrightarrow \Delta_v) \right\},$$

where Δ_v is the maximal p–power order quotient of $(\mathcal{O}_{F_v}/\varpi_v)^\times$.

In particular,

$$U_0(Q)/U_Q \cong \Delta_Q = \prod_{v \in Q} \Delta_v.$$

Analogously to section 4.1, we can define maximal ideals

$$\mathfrak{m} \in \mathbb{T}^{S,\mathrm{univ}} \quad \text{and} \quad \mathfrak{m}_Q \in \mathbb{T}_Q^{S \cup Q,\mathrm{univ}}$$

and can prove that $S_{2,\eta}(U_Q, \mathcal{O})_{\mathfrak{m}_Q}$ is a free $\mathcal{O}[\Delta_Q]$–algebra with Δ_Q–coinvariants $\cong S_{2,\eta}(U, \mathcal{O})_{\mathfrak{m}}$ as $\mathbb{T}^{S \cup Q,\mathrm{univ}}$–modules (cf. proposition 65).

4.4.1 Taylor's Ihara avoidance trick

Recall that, for every $v \in \Sigma$, we have $\mathrm{N}(v) \equiv 1 \bmod p$. (This is one of the assumptions that we imposed when we potentially used lemma 77 for a base change.) Fix a non-trivial character

$$\chi_v \colon \mathscr{O}_{F_v}^\times \twoheadrightarrow (\mathscr{O}_{F_v}/\varpi_v)^\times \to \mathscr{O}^\times$$

of p–power order. From this, we obtain

$$\begin{aligned}\chi = \prod_{v \in \Sigma} \chi_v \colon U = \prod_{v \nmid \infty} U_v &\longrightarrow \mathscr{O}^\times \\ \left(\begin{pmatrix} a_v & b_v \\ c_v & d_v \end{pmatrix}\right)_v &\longmapsto \prod_{v \in \Sigma} \chi_v(a_v d_v^{-1})\end{aligned}$$

and we can define

$$\begin{aligned} S_{2,\eta}^{\chi}(U,A) = \Big\{ f \colon D^\times \backslash (D \otimes_F \mathbb{A}_F^\infty)^\times \to A \text{ continuous } : \\ f(guz) = \eta(z)\chi(u)^{-1} f(g) \text{ for all } g \in (D \otimes_F \mathbb{A}_F^\infty)^\times, \\ u \in U \text{ and } z \in (\mathbb{A}_F^\infty)^\times \Big\}. \end{aligned}$$

Observe that

$$S_{2,\eta}^{\chi}(U,\mathscr{O})/(\varpi) \cong S_{2,\eta}^{\chi}(U,\mathbb{F}) = S_{2,\eta}(U,\mathbb{F}) \cong S_{2,\eta}(U,\mathscr{O})/(\varpi).$$

We again argue as in the proof of proposition 65 and see that $S_{2,\eta}^{\chi}(U_Q,\mathscr{O})_{\mathfrak{m}_Q}$ is a free $\mathscr{O}[\Delta_Q]$–module with Δ_Q–coinvariants $\cong S_{2,\eta}^{\chi}(U,\mathscr{O})_{\mathfrak{m}}$. Furthermore, as in section 4.1, we have that

$$S_{2,\eta}^{\chi}(U_Q,\mathscr{O})_{\mathfrak{m}_Q}/(\varpi) \cong S_{2,\eta}(U_Q,\mathscr{O})_{\mathfrak{m}_Q} \quad \text{as } \mathscr{O}[\Delta_Q]\text{–modules.}$$

Recall that, for every $v \in \Sigma$, we assume that $\overline{\rho}|_{G_{F_v}} = 1$. (This is another of the assumptions that we imposed in the base change step.)

Theorem 81 (Taylor). *Let $v \in \Sigma$.*

(1) *There is a local deformation problem D_v^1 corresponding to the lifts ρ of $\overline{\rho}|_{G_{F_v}}$ with* $\det(\rho) = \eta\epsilon_p^{-1}$ *and such that*

$$\mathrm{CharPoly}(\rho(\sigma)) = (X-1)^2 \quad \textit{for all } \sigma \in I_{F_v}.$$

The ring R_v^1 representing D_v^1 satisfies the following properties:

- *all irreducible components of* $\mathrm{Spec}(R_v^1)$ *have dimension* 3 *and generic point of characteristic* 0, *and*
- *every irreducible component of the special fibre* $\mathrm{Spec}(R_v^1/(\varpi))$ *is contained in a unique irreducible component of* $\mathrm{Spec}(R_v^1)$.

(2) *There is a local deformation problem $D_v^{\chi_v}$ corresponding to the lifts ρ of $\overline{\rho}|_{G_{F_v}}$ with* $\det(\rho) = \eta\epsilon_p^{-1}$ *and such that*

$$\mathrm{CharPoly}(\rho(\sigma)) = (X - \chi_v(\sigma))(X - \chi_v^{-1}(\sigma)) \quad \textit{for all } \sigma \in I_{F_v}.$$

The ring $R_v^{\chi_v}$ representing $D_v^{\chi_v}$ satisfies that $\mathrm{Spec}(R_v^{\chi_v})$ *is irreducible of dimension* 3 *and its generic point has characteristic* 0.

Remark. We are interested in studying R_v^1 but it is easier to work with $R_v^{\chi_v}$. We will use that $R_v^1/(\varpi) \cong R_v^{\chi_v}/(\varpi)$ because $\chi_v \equiv 1 \bmod \varpi$.

We define a pair of global deformation problems as. In what follows, the symbol ? means either 1 or χ. Consider the global deformation problem

$$\mathscr{S}^? = \left(\overline{\rho}, S = \{\, v \mid p \,\} \cup \Sigma, \eta\epsilon_p^{-1}, \mathscr{O}, (D_v)_{v|p} \cup (D_v^?)_{v\in\Sigma}\right),$$

where for every $v \mid p$ the local deformation problem D_v corresponds to crystalline lifts with all labelled Hodge–Tate weights equal to 0 or 1. It turns out that such D_v are represented by

$$R_v \cong \mathscr{O}[\![z_1, \ldots, z_{3+[F_v:\mathbb{Q}_p]}]\!].$$

One can show that the Galois representations valued in

$$\mathrm{Im}\left(\mathbb{T}^{S,\mathrm{univ}} \to \mathrm{End}_{\mathscr{O}}(S_{2,\eta}(U, \mathscr{O})_{\mathfrak{m}})\right)$$

are of type $\mathscr{S}^1$ and, similarly, the Galois representations valued in

$$\mathrm{Im}\left(\mathbb{T}^{S,\mathrm{univ}} \to \mathrm{End}_{\mathscr{O}}(S_{2,\eta}^{\chi}(U, \mathscr{O})_{\mathfrak{m}})\right)$$

are of type $\mathscr{S}^{\chi}$. Also, our fixed ρ is of type $\mathscr{S}^1$.

As before, we can augment with a Taylor–Wiles datum to obtain global deformation problems $\mathscr{S}_Q^1$ and $\mathscr{S}_Q^{\chi}$. We patch both (towers of) deformation problems simultaneously incorporating an isomorphism modulo ϖ between the two patching data. In this way, we get a pair of diagrams

$$\begin{tikzcd} S_\infty \arrow[r] & R^1_\infty \arrow[r, phantom, "\circlearrowleft"] \arrow[d, two heads] & H^1_\infty \arrow[d, two heads] \\ & R_{\mathscr{S}^1} \arrow[r, phantom, "\circlearrowleft"] & S_{2,\eta}(U,\mathcal{O})_\mathfrak{m} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} S_\infty \arrow[r] & R^\chi_\infty \arrow[r, phantom, "\circlearrowleft"] \arrow[d, two heads] & H^\chi_\infty \arrow[d, two heads] \\ & R_{\mathscr{S}^\chi} \arrow[r, phantom, "\circlearrowleft"] & S^\chi_{2,\eta}(U,\mathcal{O})_\mathfrak{m} \end{tikzcd}$$

that are identified modulo ϖ. We know that $H^?_\infty$ is supported on a non-empty union of irreducible components of $\mathrm{Spec}(R^?_\infty)$. Our goal is to prove that H^1_∞ has full support in $\mathrm{Spec}(R^1_\infty)$. But

$$R^?_\infty = \left(\left(\bigotimes_{v|p} R_v\right) \widehat{\otimes}_\mathcal{O} \left(\bigotimes_{v\in\Sigma} R^?_v\right)\right)[\![x_1,\dots,x_g]\!]$$

and, as the R_v for $v \mid p$ are formally smooth, the irreducible components of $R^?_\infty$ arise from the $R^?_v$ for $v \in \Sigma$. More precisely, the canonical morphism

$$\mathrm{Spec}(R^?_\infty) \to \prod_{v\in\Sigma} \mathrm{Spec}(R^?_v)$$

induces a bijection on irreducible components.

The second part of theorem 81 implies that $\mathrm{Spec}(R^\chi_\infty)$ is irreducible and so H^χ_∞ has full support in it, as it must be a union of irreducible components. Thus, $H^1_\infty/(\varpi) \cong H^\chi_\infty/(\varpi)$ has full support in $\mathrm{Spec}(R^1_\infty/(\varpi)) \cong \mathrm{Spec}(R^\chi_\infty/(\varpi))$. But $\mathrm{Supp}_{R^1_\infty}(H^1_\infty)$ must be a union of irreducible components too and, by the first part of theorem 81, each irreducible component of $\mathrm{Spec}(R^1_\infty/(\varpi))$ is contained in a unique irreducible component of $\mathrm{Spec}(R^1_\infty)$. All in all, H^1_∞ has full support in $\mathrm{Spec}(R^1_\infty)$, as desired.

Once we know this, we can apply the analogue of proposition 74 (whose proof was formal at this point) to deduce that the action of $R_{\mathscr{S}^1}$ on $S_{2,\eta}(U,\mathcal{O})_\mathfrak{m}$ via the Hecke algebra has nilpotent kernel and so ρ arises from an eigenform $f \in S_{2,\eta}(U,\mathcal{O})_\mathfrak{m}$.

4.5 More general number fields

Previously our field F was either $\mathbb{Q}$ or a totally real number field. This assumption was used in two places:

- On the Galois side, we used it in the "minimal case" to obtain that

  $$h^1_{\mathscr{S}}(\mathrm{ad}^0(\overline{\rho})) = h^1_{\mathscr{S}^\perp}(\mathrm{ad}^0(\overline{\rho})(1)).$$

  Thus, if we "kill" the dual Selmer group with a set Q of $q = h^1_{\mathscr{S}^\perp}(\mathrm{ad}^0(\overline{\rho})(1))$ Taylor–Wiles primes, then $R_{\mathscr{S}_Q}$ is a quotient of $\mathscr{O}[\![x_1, \dots, x_q]\!]$ and we can deduce that $\dim(R_\infty) = \dim(S_\infty)$. In the "non-minimal case", we used it analogously.
- On the automorphic side, after localizing at a non-Eisenstein maximal ideal $\mathfrak{m}$, we used that the cohomology

  $$\mathrm{H}^\bullet(Y, \mathbb{F})_{\mathfrak{m}}$$

  is concentrated in a single degree d to deduce that, at the Taylor–Wiles level, $\mathrm{H}_d(Y_Q, \mathscr{O})_{\mathfrak{m}_Q}$ is a free $\mathscr{O}[\Delta_Q]$–module with Δ_Q–coinvariants $\cong \mathrm{H}_d(Y, \mathscr{O})_{\mathfrak{m}}$.

Combining these results, we obtained a commutative diagram

$$\begin{tikzcd}
S_\infty \arrow[r] & R_\infty \arrow[d, two heads, "\mathrm{mod}\ \mathfrak{a}"'] & \circlearrowright & H_\infty \arrow[d, two heads] \\
& R_{\mathscr{S}} & \circlearrowright & \mathrm{H}_d(Y, \mathscr{O})_{\mathfrak{m}}
\end{tikzcd}$$

of actions that we used for the patching step.

Now say that F is any number field and write $[F : \mathbb{Q}] = r + 2s$, where r (resp. s) is the number of real (resp. complex) places of F. Let $\overline{\rho}\colon G_F \to \mathrm{GL}_2(\mathbb{F})$ be a continuous representation such that

- the restriction $\overline{\rho}|_{G_{F(\zeta_p)}}$ is absolutely irreducible and
- for every real place of F and a choice of complex conjugation c_v at v,

  $$\det(\overline{\rho}(c_v)) = -1.$$

Suppose that we are in a "minimal regular" situation:

- for every place $v \mid p$, we consider regular crystalline deformations of fixed weight and
- for every ramified place v,

  $$\dim_{\mathbb{F}}(L_v) - h^0(F_v, \mathrm{ad}^0(\overline{\rho})) = \begin{cases} [F_v : \mathbb{Q}_p] & \text{if } v \mid p, \\ 0 & \text{if } v \nmid p. \end{cases}$$

Then (see theorem 50 and the computations in section 3.7)

$$\begin{aligned}
h^1_{\mathscr{S}}(\operatorname{ad}^0(\overline{\rho})) &= h^1_{\mathscr{S}^\perp}(\operatorname{ad}^0(\overline{\rho})(1)) + \sum_{v \in S} \big(\dim_{\mathbb{F}}(L_v) - h^0(F_v, \operatorname{ad}^0(\overline{\rho}))\big) \\
&\qquad - \sum_{v \mid \infty} h^0(F_v, \operatorname{ad}^0(\overline{\rho})) \\
&= h^1_{\mathscr{S}^\perp}(\operatorname{ad}^0(\overline{\rho})(1)) + [F : \mathbb{Q}] - r - 3s = h^1_{\mathscr{S}^\perp}(\operatorname{ad}^0(\overline{\rho})(1)) - s.
\end{aligned}$$

Thus, if $s \neq 0$, the rings S_∞ and R_∞ will not have the same dimensions.

On the other hand, let

$$X = \left(\prod_{v \mid \infty} \operatorname{PGL}_2(F_v) \right) / U_\infty,$$

where U_∞ is a maximal compact open subset. Then

$$X \cong \big(\operatorname{PGL}_2(\mathbb{R}) / \operatorname{PO}(2)\big)^r \times \big(\operatorname{PGL}_2(\mathbb{C}) / \operatorname{PU}(2)\big)^s \cong \mathbb{H}_2^r \times \mathbb{H}_3^s,$$

where $\mathbb{H}_k$ denotes the hyperbolic k–space. Choose an open compact subgroup U of

$$\prod_{v \in \infty} \operatorname{PGL}_2(\mathscr{O}_{F_v})$$

that is "sufficiently small". We obtain a smooth manifold

$$Y(U) = \operatorname{PGL}_2(F) \backslash X \times \operatorname{PGL}_2(\mathbb{A}_F^\infty) / U.$$

Let $S = \{\, v \mid p \,\} \cup \{\, v : U_v \neq \operatorname{PGL}_2(\mathscr{O}_{F_v}) \,\}$ and consider for every place $v \notin S$ the Hecke operator

$$\mathrm{T}_v = \left[\operatorname{PGL}_2(\mathscr{O}_{F_v}) \begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix} \operatorname{PGL}_2(\mathscr{O}_{F_v}) \right].$$

For every $\mathscr{O}$–algebra A,

$$\mathrm{H}^\bullet(Y(U), A)$$

has an action of $\mathbb{T}^{S,\mathrm{univ}} = \mathscr{O}[\mathrm{T}_v : v \notin S]$. Fix an isomorphism $\eta \colon \overline{\mathbb{Q}}_p \cong \mathbb{C}$.

Theorem 82 (Harder, Franke). *There are $\mathbb{T}^{S,\mathrm{univ}}$–stable decompositions*

$$\mathrm{H}^\bullet(Y(U), \mathbb{C}) \cong \mathrm{H}^\bullet_{\mathrm{cusp}}(Y(U), \mathbb{C}) \oplus \mathrm{H}^\bullet_{\mathrm{Eis}}(Y(U), \mathbb{C})$$

with

(1) $\mathrm{H}^i_{\mathrm{cusp}}(Y(U),\mathbb{C}) \cong \bigoplus_{\pi} \big((\pi^{\infty})^U\big)^{m_i(\pi_\infty)}$ *as* $\mathbb{T}^{S,\mathrm{univ}}$*–modules, the sum ranging over the cuspidal automorphic representations of* $\mathrm{PGL}_2(\mathbb{A}_F)$*, and*

(2) *the* $\mathbb{T}^{S,\mathrm{univ}}$*–action of* $\mathrm{H}^i_{\mathrm{Eis}}(Y(U),\mathbb{C})$ *"is Eisenstein".*

Theorem 83 (Borel–Wallach). *Set* $q_0 = r + s$*. Let* $\lambda\colon \mathbb{T}^{S,\mathrm{univ}} \to \mathbb{C}$ *be an eigensystem corresponding to a cuspidal automorphic representation* π *of* $\mathrm{PGL}_2(\mathbb{A}_F)$ *such that* π_∞ *is tempered. If* $\mathrm{H}^\bullet_{\mathrm{cusp}}(Y(U),\mathbb{C})[\lambda] \neq 0$*, then* $\mathrm{H}^i_{\mathrm{cusp}}(Y(U),\mathbb{C})[\lambda] \neq 0$ *exactly for* $i \in [q_0, q_0 + s]$*.*

Remark. The key philosophy (also valid for more general ranks or groups) is that, in "nice situations", the difference between the dimensions of the dual Selmer and of the Selmer groups is δ if and only if the cohomology appears in $\delta + 1$ consecutive degrees.

For PGL_2 / F, we have $\delta = s$ (the number of complex places of F).

Conjecture 84 (Ash, Calegari–Geraghty). *Let* $\mathfrak{m}$ *be a maximal ideal of* $\mathbb{T}^{S,\mathrm{univ}}$ *such that* $\mathrm{H}^\bullet(Y(U),\mathbb{F})_{\mathfrak{m}} \neq 0$*. There exists a continuous semisimple representation*

$$\overline{\rho}_{\mathfrak{m}}\colon G_{F,S} \to \mathrm{GL}_2(\mathbb{F})$$

such that, for every place $v \notin S$*,*

$$\mathrm{CharPoly}(\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_v)) = X^2 - \mathrm{T}_v\, X + \mathrm{N}(v) \bmod \mathfrak{m}.$$

Remark. When $\delta > 0$, there can exist classes in $\mathrm{H}^\bullet(Y(U),\mathbb{F})$ that *do not* lift to characteristic 0.

Assuming conjecture 84, we say that $\mathfrak{m}$ is *non-Eisenstein* if $\overline{\rho}_{\mathfrak{m}}$ is absolutely irreducible.

Conjecture 85 (Calegari–Geraghty). *Let* $\mathfrak{m}$ *be a maximal ideal of* $\mathbb{T}^{S,\mathrm{univ}}$*. If* $\mathfrak{m}$ *is non-Eisenstein, then*

$$\mathrm{H}^i(Y(U),\mathbb{F})_{\mathfrak{m}} = 0 \quad \text{if } i \notin [q_0, q_0 + \delta].$$

Our goal next is to construct a diagram of actions

$$\begin{tikzcd} S_\infty \arrow[r] & R_\infty \circlearrowright \mathrm{H}_\bullet(C_\infty) \arrow[d, two heads, "\bmod \mathfrak{a}"] \\ & R_{\mathscr{S}} \circlearrowright \mathrm{H}_\bullet(C) \cong \mathrm{H}_\bullet(Y(U),\mathscr{O})_{\mathfrak{m}} \end{tikzcd}$$

in which

- S_∞ is a power series ring over $\mathscr{O}$ with augmentation ideal $\mathfrak{a}$,
- $\dim(R_\infty) - \dim(S_\infty) = -\delta$,
- C_∞ is a complex of finite free S_∞–modules concentrated in degrees $[q_0, q_0+\delta]$ and $C \cong C_\infty \otimes_{S_\infty} S_\infty/\mathfrak{a}$ computes $\mathrm{H}_\bullet(Y(U), \mathscr{O})_\mathfrak{m}$ and
- $\mathrm{H}_\bullet(C_\infty)$ is a finite R_∞–module.

Theorem 86. *Assuming that we have a diagram of actions as above,*

1. $\mathrm{Supp}_{R_\infty}(\mathrm{H}_{q_0}(C_\infty))$ *is a non-empty union of irreducible components of* $\mathrm{Spec}(R_\infty)$,
2. *if every irreducible component of* $\mathrm{Spec}(R_\infty)$ *is in* $\mathrm{Supp}_{R_\infty}(\mathrm{H}_{q_0}(C_\infty))$, *then*
$$\mathrm{Ker}\Big(R_\mathscr{S} \to \mathrm{End}_\mathscr{O}\big(\mathrm{H}_{q_0}(Y(U), \mathscr{O})_\mathfrak{m}\big)\Big)$$
is nilpotent, and
3. *if* $R_\infty \cong \mathscr{O}[\![x_1, \ldots, x_g]\!]$ *(with* $1 + g = \dim(S_\infty) - \delta$*), then* $\mathrm{H}_{q_0}(Y(U), \mathscr{O})_\mathfrak{m}$ *is a free* $R_\mathscr{S}$*–module.*

Proof. We claim that $\mathrm{depth}_{S_\infty}(\mathrm{H}_{q_0}(C_\infty)) = \dim(S_\infty) - \delta$. We will prove this claim later; now we use it to prove the statements of the theorem (in a way that is similar to the proofs of theorem 70 and proposition 74).

1. The S_∞–action on $\mathrm{H}_\bullet(C_\infty)$ factors through R_∞ and $\mathrm{H}_\bullet(C_\infty)$ is a finitely generated R_∞–module. Thus, for every $i \in \mathbb{Z}$,
$$\begin{aligned}\mathrm{depth}_{S_\infty}(\mathrm{H}_i(C_\infty)) \leq \mathrm{depth}_{R_\infty}(\mathrm{H}_i(C_\infty)) &\leq \dim_{R_\infty}(\mathrm{H}_i(C_\infty)) \\ &\leq \dim(R_\infty) = \dim(S_\infty) - \delta.\end{aligned}$$
In particular, for $i = q_0$, all the inequalities must be equalities (by the claim) and the result is a restating of $\dim_{R_\infty}(\mathrm{H}_i(C_\infty)) = \dim(R_\infty)$.
2. Take $\mathfrak{p} \in \mathrm{Spec}(R_\mathscr{S})$ and let $\mathfrak{p}_\infty$ denote its pull-back to R_∞. By assumption, $\mathrm{H}_{q_0}(C_\infty) \neq 0$, whence
$$\begin{aligned}\mathrm{H}_{q_0}(Y(U), \mathscr{O})_\mathfrak{p} \cong \mathrm{H}_{q_0}(C_\infty \otimes_{S_\infty} \mathscr{O})_\mathfrak{p} &= \big((C_\infty)_{q_0}/(\mathfrak{a}, \mathrm{Im}(d_{q_0-1}))\big)_\mathfrak{p} \\ &= \big(\mathrm{H}_{q_0}(C_\infty)/\mathfrak{a}\big)_\mathfrak{p} \cong \mathrm{H}_{q_0}(C_\infty)_{\mathfrak{p}_\infty}/\mathfrak{a} \neq 0\end{aligned}$$
by Nakayama's lemma. Here, we used that q_0 is the smallest degree in which the complex C_∞ is non-trivial (in general it is not true that taking quotients commutes with homology).
3. Since R is regular and $\dim_{R_\infty}(\mathrm{H}_{q_0}(C_\infty)) = \mathrm{depth}_{R_\infty}(\mathrm{H}_{q_0}(C_\infty))$ (as we saw in the proof of (1)), we can use the Auslander–Buchsbaum formula to deduce

that $\mathrm{H}_{q_0}(C_\infty)$ is a projective R_∞–module and so must be free because R_∞ is local. Therefore,

$$\mathrm{H}_{q_0}(Y(U), \mathscr{O})_{\mathfrak{m}} \cong \mathrm{H}_{q_0}(C_\infty)/\mathfrak{a}$$

is a free $(R_\infty/\mathfrak{a})$–module. But the $(R_\infty/\mathfrak{a})$–action factors through $R_{\mathscr{S}}$ and we conclude that $R_\infty/\mathfrak{a} \cong R_{\mathscr{S}}$. □

Lemma 87. *Let S be a local regular noetherian ring of dimension n. Let $P = P_\bullet$ be a homological complex of finite free S–modules concentrated in degrees $[0, \delta]$. In this situation, $\dim_S(\mathrm{H}_\bullet(P)) \geq n - \delta$ and moreover, if equality holds, then*

1. *P is a projective resolution of $\mathrm{H}_0(P)$ and*
2. *$\mathrm{H}_0(P)$ has depth $n - \delta$ (as an S–module).*

Proof. Write $d_n \colon P_n \to P_{n-1}$ for the differentials of the complex $P_\bullet$. Let $m \in \mathbb{Z}_{\geq 0}$ be the largest integer such that $\mathrm{H}_m(P) \neq 0$. Then

$$0 \longrightarrow P_\delta \longrightarrow P_{\delta-1} \longrightarrow \cdots \longrightarrow P_m$$

is a projective resolution of $M = P_m/\operatorname{Im}(d_{m+1})$. Therefore,

$$\operatorname{projdim}_S(M) \leq \delta - m.$$

On the other hand, $\mathrm{H}_m(P) = \operatorname{Ker}(d_m)/\operatorname{Im}(d_{m+1}) \subseteq M$, which implies that $\dim_S(\mathrm{H}_m(P)) \geq \operatorname{depth}_S(M)$. Using the Auslander–Buchsbaum formula, we obtain that

$$\dim_S(\mathrm{H}_m(P)) \geq \operatorname{depth}_S(M) = n - \operatorname{projdim}_S(M) \geq n - \delta + m.$$

If $\dim_S(\mathrm{H}_\bullet(P)) \leq n - \delta$, then we must have $m = 0$, which means that P is a projective resolution of $M = \mathrm{H}_0(P)$ and all the inequalities above must be equalities. In particular, $\operatorname{depth}_S(\mathrm{H}_0(P)) = n - \delta$. □

It remains to discuss, at least conjecturally, how to create the patched diagram that we used to prove theorem 86.

On the Galois side, just as for $\mathrm{GL}_2(\mathbb{Q})$, we can use that $\overline{\rho}|_{G_{F(\zeta_p)}}$ is absolutely irreducible with enormous image to prove that, for every $N \in \mathbb{Z}_{\geq 1}$, we can find a Taylor–Wiles datum Q_N of level N such that

$$h^1_{\mathscr{S}^\perp_{Q_N}}(\mathrm{ad}^0(\overline{\rho})(1)) = 0$$

in such a way that $|Q_N| = q$ is independent of N. Then $R^S_{\mathscr{S}_{Q_N}}$ is a quotient of $R_\infty = R^{S-\mathrm{loc}}_{\mathscr{S}}[\![x_1, \dots, x_g]\!]$ with $g = h^1_{\mathscr{S}^\perp_{Q_N}}(\mathrm{ad}^0(\overline{\rho})) = q + |S| - 1 - \delta$. Therefore,

$$\dim(R_\infty) = \dim(S_\infty) - \delta.$$

On the automorphic side, we consider $G = \mathrm{PGL}_2$ and the quotient X of $G(F \otimes_{\mathbb{Q}} \mathbb{R})$ by some maximal compact. Let U be a sufficiently small subgroup of $G(\mathbb{A}_F^\infty)$. We obtain a smooth manifold $Y(U) = G(F)\backslash X \times G(\mathbb{A}_F^\infty)/U$. Every Taylor–Wiles datum Q still gives rise to levels $U_Q \subseteq U_0(Q) \subseteq U$ such that $U_0(Q)$ is the Iwahori level at every $v \in Q$ and $U_0(Q)/U_Q \cong \Delta_Q$ (formed from the maximal p–power quotients of the residue fields).

Again, we can define a maximal ideal $\mathfrak{m}_Q$ of $\mathbb{T}_Q^{S\cup Q,\mathrm{univ}}$. The problem is that

$$\begin{aligned} \mathrm{H}_\bullet(Y(U_Q), \mathscr{O})_{\mathfrak{m}_Q} \otimes_{\mathscr{O}[\Delta_Q]} \mathscr{O} &\cong \mathrm{H}_\bullet(Y(U_0(Q)), \mathscr{O}[\Delta_Q])_{\mathfrak{m}_Q} \otimes_{\mathscr{O}[\Delta_Q]} \mathscr{O} \\ &\not\cong \mathrm{H}_\bullet(Y(U_0(Q)), \mathscr{O})_{\mathfrak{m}_Q} \end{aligned}$$

because homology and tensor products do not commute (unless, say, homology is concentrated in one single degree). A workaround is to use a complex C_Q of free $\mathscr{O}[\Delta_Q]$–modules that computes $\mathrm{H}_\bullet(Y(U_Q), \mathscr{O})_{\mathfrak{m}_Q}$, in which case $C_Q \otimes_{\mathscr{O}[\Delta_Q]} \mathscr{O}$ computes $\mathrm{H}_\bullet(Y(U_0(Q)), \mathscr{O})_{\mathfrak{m}_Q}$.

- We will have to define a Hecke action on C_Q and an action on $R_{\mathscr{S}_Q}$ via a map to a Hecke algebra with operators outside $S \cup Q$. If we use singular chains to define C_Q (i.e., the usual complex that computes singular homology), then we automatically have a Hecke action.
- However, for patching, we need C_Q to be a bounded complex of finite free $\mathscr{O}[\Delta_Q]$–modules (in order to have only finitely many isomorphism classes of patching data of a fixed level) and that will not be preserved by $\mathbb{T}^{S,\mathrm{univ}}$.

The most natural way to resolve this "contradictory" requirements on the complexes is to work in the derived categories $\mathrm{D}(\mathscr{O})$ and $\mathrm{D}(\mathscr{O}[\Delta_Q])$ of $\mathscr{O}$–modules and $\mathscr{O}[\Delta_Q]$–modules, respectively.

Roughly, for a ring A, the category $\mathrm{D}(A)$ is constructed as follows. Let $\mathrm{Ch}(A)$ be the category of chain complexes of A–modules and let $\mathrm{K}(A)$ be the category whose objects are the same as in $\mathrm{Ch}(A)$ but whose homomorphisms are morphisms of complexes up to chain homotopy. The category $\mathrm{D}(A)$ is obtained from $\mathrm{K}(A)$ by formally inverting quasi-isomorphisms (i.e., chain morphisms that induce an isomorphism on homology). In particular, every morphism $f\colon X \to Y$ in

$\mathrm{D}(A)$ is represented by a roof

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

formed of a quasi-isomorphism $Z \to X$ and a general morphism of complexes $Z \to X$. There are full subcategories $\mathrm{D}^-(A)$ (or $\mathrm{K}^-(A)$) and $\mathrm{D}^+(A)$ (or $\mathrm{K}^+(A)$) of bounded above and bounded below, respectively, complexes. Let $\mathrm{K}^{-,\mathrm{proj}}(A)$ be the full subcategory of $\mathrm{K}^-(A)$ consisting of complexes of projective A–modules. It turns out that the obvious functor $\mathrm{K}^{-,\mathrm{proj}}(A) \to \mathrm{D}^-(A)$ is an equivalence of categories.

Let $C \in \mathrm{Ob}(\mathrm{D}^-(A))$. (We often identify chain and cochain complexes by $C_i = C^{-i}$.) Choose a complex P of projective A–modules isomorphic to C in $\mathrm{D}(A)$. Given an A–module M, we define

$$C \otimes_A^{\mathbb{L}} M = P \otimes_A M \quad \text{(i.e., } (C \otimes_A^{\mathbb{L}} M)_i = P_i \otimes_A M\text{)}$$

and $\mathrm{R}\,\mathrm{Hom}_A(C, M)$ by

$$\mathrm{R}\,\mathrm{Hom}_A(C, M)_i = \mathrm{Hom}_A(P_{-i}, M)$$

with $d(f) = (-1)^{\deg(f)+1} f \circ d$. These objects are independent of the choice of P up to unique isomorphism in $\mathrm{D}(A)$. If $M = B$ is an A–algebra, we obtain a functor

$$\cdot \otimes_A^{\mathbb{L}} B \colon \mathrm{D}^-(A) \to \mathrm{D}^-(B).$$

There is a spectral sequence

$$(E_2)_{i,j} = \mathrm{Tor}_j^A(\mathrm{H}_i(C), M) \implies \mathrm{H}_{i+j}(C \otimes_A^{\mathbb{L}} M)$$

(and a similar spectral sequence computing the same homology in terms of a resolution of M).

The category $\mathrm{D}(A)$ is idempotent complete: if $e \in \mathrm{End}_{\mathrm{D}(A)}(C)$ satisfies that $e^2 = e$, there exists a decomposition $C = eC \oplus (1-e)C$ in $\mathrm{D}(A)$.

We say that C is *perfect* if it is isomorphic in $\mathrm{D}(A)$ to a bounded complex of finite projective A–modules. If A is local and noetherian, the complex C is called *minimal* if it is a bounded complex of finite projective (equivalently, free) A–modules and the differentials are 0 modulo $\mathfrak{m}_A$. When A is local and noetherian, every perfect complex is isomorphic in $\mathrm{D}(A)$ to a minimal one. Moreover, if C is a perfect complex such that $\mathrm{H}_\bullet(C \otimes^{\mathbb{L}}_A A/\mathfrak{m}_A)$ is concentrated in degrees $[a, b]$, then C is isomorphic in $\mathrm{D}(A)$ to a complex concentrated in degrees $[a, b]$. Observe that there is a natural map

$$\mathrm{End}_{\mathrm{D}(A)}(C) \to \mathrm{End}_A(\mathrm{H}_\bullet(C))$$

and, if C is perfect and concentrated in degrees $[0, d]$ and $f \in \mathrm{End}_{\mathrm{D}(A)}(C)$ acts as 0 on $\mathrm{H}_\bullet(C)$, then $f^{d+1} = 0$ in $\mathrm{End}_{\mathrm{D}(A)}(C)$. More generally, for a perfect complex C, the kernel of

$$\mathrm{End}_{\mathrm{D}(A)}(C) \to \mathrm{End}_A(\mathrm{H}_\bullet(C))$$

is nilpotent.

Going back to our automorphic setting, it turns out that there exists a perfect complex $C(U) \in \mathrm{Ob}(\mathrm{D}(\mathscr{O}))$ such that

$$\mathrm{H}_\bullet(C(U)) = \mathrm{H}_\bullet(Y(U), \mathscr{O})$$

and then $\mathrm{H}^\bullet(Y(U), \mathscr{O})$ is computed by

$$\mathrm{R}\,\mathrm{Hom}_{\mathscr{O}}(C(U), \mathscr{O}).$$

There exists a morphism

$$\mathbb{T}^{S,\mathrm{univ}} \to \mathrm{End}_{\mathrm{D}(\mathscr{O})}(C(U))$$

of $\mathscr{O}$–algebras and we define $\mathbb{T}^S(U)$ to be its image. In particular, $\mathbb{T}^S(U)$ is an $\mathscr{O}$–algebra of finite rank and so must be semilocal (equal to the product of its local subrings). For the maximal ideal $\mathfrak{m}$ of $\mathbb{T}^{S,\mathrm{univ}}$ that we constructed before, the localization $C(U)_\mathfrak{m}$ makes sense in $\mathrm{D}(\mathscr{O})$ and $\mathrm{H}_\bullet(C(U)_\mathfrak{m}) = \mathrm{H}_\bullet(Y(U), \mathscr{O})_\mathfrak{m}$. Therefore, the kernel of

$$\mathbb{T}^S(U)_\mathfrak{m} \to \mathrm{End}_{\mathscr{O}}\big(\mathrm{H}_\bullet(Y(U), \mathscr{O})_\mathfrak{m}\big)$$

is (at least) nilpotent.

Conjecture 88 (Calegari–Geraghty). *There exists a continuous Galois representation*

$$\rho_\mathfrak{m} \colon G_{F,S} \to \mathrm{GL}_2(\mathbb{T}^S(U)_\mathfrak{m})$$

such that, for every $v \notin S$,

$$\mathrm{CharPoly}(\rho_{\mathfrak{m}}(\mathrm{Frob}_v)) = X^2 - \mathrm{T}_v\, X + \mathrm{N}(v).$$

Moreover,

(1) *for every place $v \mid p$ such that $U_v = G(\mathscr{O}_{F_v})$, if p is unramified in F, then $\rho_{\mathfrak{m}}|_{G_{F_v}}$ is Fontaine–Laffaille with all labelled Hodge–Tate weights equal to* 0 *or* 1*, and*

(2) *for every $v \in S$ such that $v \mid \ell \neq p$ and U_v contains the pro-ℓ Iwahori group,*

$$\mathrm{CharPoly}(\rho_{\mathfrak{m}}(\sigma)) = (X - \langle \mathrm{rec}_{F_v}^{-1}(\sigma)\rangle)(X - \langle \mathrm{rec}_{F_v}^{-1}(\sigma^{-1})\rangle)$$

for every $\sigma \in I_{F_v}$ and we can also describe $\mathrm{CharPoly}(\rho_{\mathfrak{m}}(\mathrm{Frob}_v))$ *similarly. (Here,* $\mathrm{rec}_{F_v} \colon \mathscr{O}_{F_v}^{\times} \to G_{F_v}^{\mathrm{ab}}$ *is Artin's local reciprocity map and $\langle \cdot \rangle$ denotes the diamond operator.)*

Conjecture 88 would give a morphism

$$R_{\mathscr{S}} \to \mathbb{T}^S(U)_{\mathfrak{m}}$$

for a suitable type $\mathscr{S}$, whence we obtain an action

$$R_{\mathscr{S}} \circlearrowright \mathrm{H}_{\bullet}(C(U)_{\mathfrak{m}}) = \mathrm{H}_{\bullet}(Y(U), \mathscr{O})_{\mathfrak{m}}.$$

Adding a Taylor–Wiles datum Q, we can similarly construct a perfect complex $C(U_Q)_{\mathfrak{m}_Q} \in \mathrm{Ob}\big(\mathrm{D}(\mathscr{O}[\Delta_Q])\big)$ such that

$$C(U_Q)_{\mathfrak{m}_Q} \otimes^{\mathbb{L}}_{\mathscr{O}[\Delta_Q]} \mathscr{O} \cong C(U)_{\mathfrak{m}}.$$

In particular,

$$C(U_Q)_{\mathfrak{m}_Q} \otimes^{\mathbb{L}}_{\mathscr{O}[\Delta_Q]} \mathbb{F} \cong C(U)_{\mathfrak{m}} \otimes^{\mathbb{L}}_{\mathscr{O}} \mathbb{F}$$

and the latter computes $\mathrm{H}_{\bullet}(Y(U), \mathbb{F})_{\mathfrak{m}}$. Then conjecture 85 implies that $C(U_Q)_{\mathfrak{m}_Q}$ is concentrated in degrees $[q_0, q_0 + \delta]$. Therefore, assuming conjectures 85 and 88, we can patch the (minimal) complexes and get the desired diagram of actions.

Conjecture 85 can be proved if F is a quadratic imaginary field because then $\dim(Y(U)) = 3$ and we only need to understand H^0 (and H^3). But it is very hard to prove it in general. There is a workaround due to Khare and Thorne if one only wants to prove that $R_{\mathscr{S}}^{\mathrm{red}} \cong \mathbb{T}^S(U)_{\mathfrak{m}}^{\mathrm{red}}$ (or $R_{\mathscr{S}}[p^{-1}] \cong \mathbb{T}^S(U)_{\mathfrak{m}}[p^{-1}]$). We at least know that

$$\mathrm{H}_i(Y(U))\mathbb{F} = 0 \quad \text{for } i \notin [0, d],$$

where $d = \dim(Y(U))$. We can still patch to obtain a diagram of actions

$$\begin{tikzcd}
S_\infty \arrow[r] & R_\infty \circlearrowright \mathrm{H}_\bullet(C_\infty) \arrow[d, two heads, "\text{mod } \mathfrak{a}"] \\
& R_{\mathscr{S}} \circlearrowright \mathrm{H}_\bullet(C) \cong \mathrm{H}_\bullet(Y(U), \mathscr{O})_{\mathfrak{m}}
\end{tikzcd}$$

but we only know that C_∞ is concentrated in degrees $[0, d]$. We need concentration in $[q_0, q_0 + \delta]$ instead. Suppose that we know that $\mathrm{H}_{q_0}(Y(U), \mathscr{O})_{\mathfrak{m}}[p^{-1}] \neq 0$ and localize the previous diagram at the augmentation ideal $\mathfrak{a}$ of S_∞ to obtain

$$\begin{tikzcd}
S_{\infty,\mathfrak{a}} \arrow[r] & R_{\infty,\mathfrak{a}} \circlearrowright \mathrm{H}_\bullet(C_\infty)_{\mathfrak{a}} \cong \mathrm{H}_\bullet(C_{\infty,\mathfrak{a}}) \arrow[d, two heads] \\
& R_{\mathscr{S}}[p^{-1}] \circlearrowright \mathrm{H}_\bullet(Y(U), \mathscr{O})_{\mathfrak{m}}[p^{-1}]
\end{tikzcd}$$

(where we used that localizing at $\mathfrak{a}$ and then modding out by $\mathfrak{a}$ is equivalent to just inverting p). Let $E = \mathscr{O}[p^{-1}]$. Then

$$C_{\infty,\mathfrak{a}} \otimes^{\mathbb{L}}_{S_{\infty,\mathfrak{a}}} E \cong C \otimes^{\mathbb{L}}_{\mathscr{O}} E$$

and theorems 82 and 83 imply that $\mathrm{H}_\bullet(C \otimes^{\mathbb{L}}_{\mathscr{O}} E) \cong \mathrm{H}_\bullet(Y(U), \mathscr{O})_{\mathfrak{m}}[p^{-1}]$ is concentrated in degrees $[q_0, q_0 + \delta]$. Now we can apply the arguments from proposition 74 (cf. theorem 86 too), based on commutative algebra results, to prove that the morphism

$$R_{\mathscr{S}}[p^{-1}] \to \mathbb{T}^S(U)_{\mathfrak{m}}[p^{-1}]$$

has nilpotent kernel if $\mathrm{H}_{q_0}(C_{\infty,\mathfrak{a}})$ has full support in $\mathrm{Spec}(R_{\infty,\mathfrak{a}})$.

Remark. We need Galois representations for this argument to work; then one uses that $\mathfrak{m}$ is non-Eisenstein to show that

$$\mathrm{H}^\bullet(Y(U), \mathscr{O})_{\mathfrak{m}}[p^{-1}]$$

is all cuspidal.

Conjecture 88 can be proved if F is a CM field with many technical conditions up to replacing $\mathbb{T}^S(U)_{\mathfrak{m}}$ with $\mathbb{T}^S(U)_{\mathfrak{m}}/I$ for a nilpotent ideal I with nilpotence degree depending only on F and the rank $n = 2$ (appearing in PGL_n). Then one can build I into the patching argument and still get a theorem of the form $R^{\mathrm{red}}_{\mathscr{S}} \cong \mathbb{T}^S(U)^{\mathrm{red}}$. This crucially relies on viewing $\mathrm{Res}_{F/F^+} \mathrm{GL}_n$ (where F^+ is the maximal totally real field of F) as a Levi on a unitary $2n$–dimensional unitary group over F^+. Via the Borel–Serre compactification, one can find the cohomology of the locally symmetric space associated with GL_n in the cohomology of the unitary Shimura variety. That is why we restricted to CM fields.