Residues and duality

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Throughout this document, we consider a fixed function field K in one variable over a ground field k. We use the same notation as Goldschmidt's book [1] and the previous talks.

The main objectives of this talk are to prove the residue theorem, which roughly states that the sum of the local residues of a differential form is zero, and to exhibit a duality between differential forms and Weil differentials in the case that K / k is geometric.

Residues

In the last talk, residues were introduced in general. Here, we restrict our attention to the local residues of differential forms. In order to define them, we need a previous observation.

Lemma 1. *For every $P \in \mathbb{P}_K$, $\hat{\mathcal{O}}_P$ is a near $\hat{K}_P$–submodule of $\hat{K}_P$.*

Proof. Fix a local parameter t at P. Since $\dim_k(\hat{\mathcal{O}}_P \,/\, t\hat{\mathcal{O}}_P) = \dim_k(F_P) < \infty$, for $n \in \mathbb{N}$ we get that $\dim_k(\hat{\mathcal{O}}_P \,/\, t^n\hat{\mathcal{O}}_P) = n\dim_k(\hat{\mathcal{O}}_P \,/\, t\hat{\mathcal{O}}_P) < \infty$. This implies that $\hat{\mathcal{O}}_P \sim t^n\hat{\mathcal{O}}_P$ for all $n \geq 0$. Moreover, multiplication by t^n induces an isomorphism $t^{-n}\hat{\mathcal{O}}_P \,/\, \hat{\mathcal{O}}_P \cong \hat{\mathcal{O}}_P \,/\, t^n\hat{\mathcal{O}}_P$, so $\hat{\mathcal{O}}_P \sim t^n\hat{\mathcal{O}}_P$ for all $n \in \mathbb{Z}$. Finally, for every $x \in \hat{K}_P$, there exists some $n \in \mathbb{Z}$ such that $x\hat{\mathcal{O}}_P = t^n\hat{\mathcal{O}}_P$; in particular, $x\hat{\mathcal{O}}_P \preccurlyeq \hat{\mathcal{O}}_P$. □

Definition 2. The *local residue map* at $P \in \mathbb{P}_K$ is the k–linear map

$$\begin{aligned} \operatorname{Res}_P\colon \Omega_K &\longrightarrow k \\ \omega &\longmapsto \operatorname{Res}_{\hat{\mathcal{O}}_P}^{\hat{K}_P}(\omega) \end{aligned}$$

(which is well-defined by the previous lemma).

Now we can already start working towards the proof of the residue theorem.

Lemma 3. *If $x, y \in \hat{\mathcal{O}}_P$, $\mathrm{Res}_P(y\,dx) = 0$. Consequently, for every $\alpha \in A_K$ and $x \in K$, $\mathrm{Res}_P(\alpha_P\,dx) = 0$ for all but finitely many primes $P \in \mathbb{P}_K$.*

Proof. If $x, y \in \hat{\mathcal{O}}_P$, $\hat{\mathcal{O}}_P$ is invariant under both x and y and so the trace which defines $\mathrm{Res}_P(y\,dx)$ is zero (as proved in the previous talk; see lemma 1.4.9 in [1]).

The second assertion follows from the first because both α and x have only finitely many poles. □

Theorem 4 (Tate). *Let $S \subseteq \mathbb{P}_K$ and define*

$$\mathcal{O}_S = \bigcap_{P \in S} \mathcal{O}_P\,.$$

$\mathcal{O}_S$ is a near K–submodule of K and

$$\sum_{P \in S} \mathrm{Res}_P(\omega) = \mathrm{Res}^K_{\mathcal{O}_S}(\omega)$$

for every $\omega \in \Omega_K$.

Proof. Let $A = A_K$ be the adèle ring of K and consider the subring

$$A_S = \{\, \alpha \in A : \alpha_P = 0 \ \text{ for all } \ P \notin S \,\}\,.$$

For any $D \in \mathrm{Div}(K)$, write $A_S(D) = A(D) \cap A_S$. Consider also the natural projection $\pi\colon A \to A_S$ and $K_S = \pi(K)$. Clearly, π induces isomorphisms $K \cong K_S$ and $\mathcal{O}_S \cong K_S \cap A_S(0)$, so we can view these spaces as subspaces of A_S. Let us check that they are near K–submodules of A_S.

Indeed, K_S is even K–invariant. As for $A_S(0)$, take $x \in K$ and we see that, for every $\alpha \in A_S(0)$ and every $P \in S$,

$$v_P(x\alpha) = v_P(x) + v_P(\alpha) \geq v_P(x) \geq -v_P([x]_\infty)\,,$$

whence $xA_S(0) \subseteq A_S([x]_\infty)$ and so $xA_S(0) + A_S(0) \subseteq A_S([x]_\infty)$. From this, using that $A_S([x]_\infty) = A([x]_\infty) \cap A_S$ and $A_S(0) = A(0) \cap A_S$, we obtain that

$$\dim_k \frac{xA_S(0) + A_S(0)}{A_S(0)} \leq \dim_k \frac{A_S([x]_\infty)}{A_S(0)} \leq \dim_k \frac{A([x]_\infty)}{A(0)} = \deg[x]_\infty < \infty\,.$$

Therefore, K_S and $A_S(0)$ are near K–submodules of A_S, and so are $K_S \cap A_S(0)$ and $K_S + A_S(0)$ too. In addition,

$$\mathrm{Res}^{A_S}_{K_S + A_S(0)} + \mathrm{Res}^{A_S}_{K_S \cap A_S(0)} = \mathrm{Res}^{A_S}_{K_S} + \mathrm{Res}^{A_S}_{A_S(0)}\,.$$

Since K_S is K–invariant, $\mathrm{Res}^{A_S}_{K_S} = 0$. Similarly, from

$$\dim_k \frac{A_S}{A_S(0) + K_S} \leq \dim_k \frac{A}{A(0) + K} = g - \delta(0) = g < \infty\,,$$

we deduce that $A_S(0) + K_S \sim A_S$ and then $\operatorname{Res}^{A_S}_{A_S(0)+K_S} = \operatorname{Res}^{A_S}_{A_S} = 0$ too. Also, as the spaces where the residues are computed can be enlarged,

$$\operatorname{Res}^{K}_{\mathcal{O}_S} = \operatorname{Res}^{K_S}_{K_S \cap A_S(0)} = \operatorname{Res}^{A_S}_{K_S \cap A_S(0)} .$$

All in all,

$$\operatorname{Res}^{K}_{\mathcal{O}_S} = \operatorname{Res}^{A_S}_{A_S(0)} .$$

Take $\omega = y\,dx \in \Omega_K$ for some $x, y \in K$. Let $\{ P_1, \ldots, P_n \} \subseteq S$ be the finite set of primes of S where at least one of x and y has a pole and let T be its complement in S. We observe that

$$A_S(0) = A_T(0) \oplus \left(\bigoplus_{i=1}^{n} \hat{\mathcal{O}}_{P_i} \right)$$

and, as $\operatorname{Res}^{A_S}_0 = 0$, the formula for the residue map on the sum of near submodules yields

$$\operatorname{Res}^{A_S}_{A_S(0)} = \operatorname{Res}^{A_S}_{A_T(0)} + \sum_{i=1}^{n} \operatorname{Res}^{A_S}_{\hat{\mathcal{O}}_{P_i}} = \operatorname{Res}^{A_S}_{A_T(0)} + \sum_{i=1}^{n} \operatorname{Res}_{P_i} .$$

Furthermore, $A_T(0)$ is invariant under both x and y because neither x nor y has poles in T, whence $\operatorname{Res}^{A_S}_{A_T(0)}(y\,dx) = 0$. For the same reason, lemma 3 implies that $\operatorname{Res}_P(y\,dx) = 0$ for all $P \in T$. In conclusion,

$$\operatorname{Res}^{K}_{\mathcal{O}_S}(\omega) = \operatorname{Res}^{A_S}_{A_S(0)}(\omega) = \sum_{i=1}^{n} \operatorname{Res}_{P_i}(\omega) = \sum_{P \in S} \operatorname{Res}_P(\omega)$$

as claimed. □

As a particular case of this theorem, we obtain the result which is usually referred to as the residue theorem.

Corollary 5. *For every $\omega \in \Omega_K$,*

$$\sum_{P \in \mathbb{P}_K} \operatorname{Res}_P(\omega) = 0 .$$

Proof. It is immediate from the theorem because

$$\bigcap_{P \in \mathbb{P}_K} \mathcal{O}_P = k \sim 0$$

and so $\operatorname{Res}^{K}_{\mathcal{O}_{\mathbb{P}_K}} = 0$. □

Duality

From now on, assume further that $K\,/\,k$ is a geometric function field of genus g. In this case, the theory of residues allows us to construct an explicit duality isomorphism between the space Ω_K of differential forms on K and the space W_K of Weil differentials on K.

Theorem 6. *The map*

$$\begin{aligned} *\colon \Omega_K &\longrightarrow W_K \\ \omega &\longmapsto \omega^* \end{aligned}$$

defined by

$$\begin{aligned} \omega^*\colon A_K &\longrightarrow k \\ \alpha &\longmapsto \sum_{P\in\mathbb{P}_K} \mathrm{Res}_P(\alpha_P\,\omega) \end{aligned}$$

is a K–linear isomorphism.

Proof. First, we must prove that the duality map is well-defined, i.e., that ω^* is a Weil differential for every $\omega \in \Omega_K$. Let $\omega = y\,dx \neq 0$ for some $x, y \in K$ and define $D = [x]_\infty + [y]_\infty + 1$. We want to show that ω^* vanishes on $A_K(-D)$. Indeed, take $\alpha \in A_K(-D)$. Let $P \in \mathbb{P}_K$ and choose a local parameter t at P. We can write $x = ut^n$ with $n = v_P(x) \in \mathbb{Z}$ and then

$$\omega = yt^n\,du + nyut^{n-1}\,dt\,.$$

Thus, by our choice of D, $v_P(\alpha_P yt^n), v_P(n\alpha_P yut^{n-1}) \geq 0$ and so $\mathrm{Res}_P(\alpha_P\omega) = 0$ by lemma 3. On the other hand, theorem 4 shows that ω^* vanishes on K too. Therefore, ω^* vanishes on $A_K(-D) + K$, which means that it is a Weil differential.

From the definition, one sees easily that the map $*$ is K–linear. Since both Ω_K and W_K are K–vector spaces of dimension 1, it suffices to prove that $*$ is not trivial. To do so, take a separable prime $P \in \mathbb{P}_K^{\mathrm{sep}}$ with local parameter t. For every $x \in \mathcal{O}_P$, let $\alpha(x)$ be the adèle $\alpha \in A_K$ with $\alpha_P = t^{-1}x$ and $\alpha_Q = 0$ for $Q \neq P$. Then,

$$(dt)^*(\alpha(x)) = \mathrm{Res}_P(t^{-1}x\,dt) = \mathrm{tr}_{\mathcal{O}_P/t\mathcal{O}_P}(x) = \mathrm{tr}_{F_P}(x)$$

(as proved in the previous talk; see theorem 1.4.12 in [1]). But $F_P\,/\,k$ is a separable extension, and this is equivalent to the trace form tr_{F_P} being non-zero. Therefore, we can find some $x \in \mathcal{O}_P$ such that $\mathrm{tr}_{F_P}(x) \neq 0$. □

The techniques used to prove that the duality map is an isomorphism can be used to obtain some additional results.

Corollary 7. *Let $P \in \mathbb{P}_K$ and let t be a local parameter at P. $v_P((dt)^*) = 0$ if and only if P is a separable prime. Moreover, in this case, t is a separating variable of $K\,/\,k$.*

Proof. Recall that, for a Weil differential $w \in W_K$ and $e \in \mathbb{Z}$, $v_P(w) \geq e$ if and only if w vanishes on $t^{-e}\hat{\mathcal{O}}_P$. As in the proof of theorem 6, define $\alpha(x)$ (for every $x \in \hat{K}_P$) to be the adèle $\alpha \in A_K$ defined by $\alpha_P = t^{-1}x$ and $\alpha_Q = 0$ for $Q \neq P$, so that

$$(dt)^*(\alpha(x)) = \mathrm{tr}_{F_P}(x)\,.$$

If P is separable, tr_{F_P} is not zero, which shows that $(dt)^*$ does not vanish on $t^{-1}\hat{\mathcal{O}}_P$. In particular, we obtain that $dt \neq 0$ or, equivalently, that t is a separating variable (as $K\,/\,k$ is geometric). On the other hand, $(dt)^*$ vanishes on $\hat{\mathcal{O}}_P$ by lemma 3. Therefore, $v_P((dt)^*) = 0$.

Conversely, if P is not separable, the trace form tr_{F_P} is zero, which implies that $(dt)^*$ vanishes on $t^{-1}\hat{\mathcal{O}}_P$ or, equivalently, that $v_P((dt)^*) \geq 1$. □

Corollary 8. *Let $P \in \mathbb{P}_K$ and let $x \in K$ be a separating variable.*

(i) *There exists $y \in \hat{K}_P$ such that* $\mathrm{Res}_P(y\,dx) \neq 0$.

(ii) *The separable closure F^{sep} of k in F_P is the maximal finite extension of k contained in $\hat{K}_P$.*

Proof. For (i), we have that $dx \neq 0$ and so $(dx)^* \neq 0$. Setting $e = v_P((dx)^*) \in \mathbb{Z}$, we get that $(dx)^*$ does not vanish on $P^{-(e+1)}\hat{\mathcal{O}}_P$, from which the claim follows.

Now we turn to (ii). Consider k' to be the maximal finite extension of k contained in $\hat{K}_P$. We must prove that $k' = F^{\mathrm{sep}}$. Recall that, if $x \in \hat{K}_P$ is algebraic over k, $v_P(x) = 0$. Hence, k' lies in $\hat{\mathcal{O}}_P$ and so defines a subfield of F_P.

On the one hand, we know that F^{sep} has a unique lift to a subfield of $\hat{\mathcal{O}}_P$ (see lemma 1.2.12 in [1]), so that $F^{\mathrm{sep}} \subseteq k'$ by the maximality of k'.

On the other hand, we can choose a differential form $y\,dx$ with $\mathrm{Res}_P(y\,dx) \neq 0$ as in (i). But, since $k' \subseteq \hat{\mathcal{O}}_P$, $\hat{\mathcal{O}}_P$ is k'–invariant and we saw in the previous talk that, in this situation, the residue in K/k can be computed in terms of the residue in K/k': $\mathrm{Res}_P(y\,dx) = \mathrm{tr}_{k'/k}(\mathrm{Res}'_P(y\,dx))$ (see lemma 1.4.16 in [1]). From this, we see that $\mathrm{tr}_{k'/k}$ is not zero. Therefore, $k'\,/\,k$ is separable and so $k' \subseteq F^{\mathrm{sep}}$. □

Using the duality between differential forms and Weil differentials, we can express the properties and results involving Weil differentials in terms of differential forms. To conclude, we write down some of these results, which were actually proved in another form in the preceding talks of the seminar.

Definition 9. For every $P \in \mathbb{P}_K$, we define the *valuation* v_P on Ω_K as follows:

$$v_P(\omega) = v_P(\omega^*) \ \text{ for all } \ \omega \in \Omega_K\,.$$

The *divisor* of $\omega \in \Omega_K$ is

$$[\omega] = \sum_{P \in \mathbb{P}_K} v_P(\omega) P \,.$$

For any divisor $D \in \mathrm{Div}(K)$, we define $\Omega_K(D)$ to be the preimage of $W_K(D)$ under the duality isomorphism. In particular, the differential forms in $\Omega_K(0)$ are called *regular* (or *holomorphic* if $k = \mathbb{C}$).

Corollary 10. *Let $P \in \mathbb{P}_K$. For all $\omega, \omega' \in \Omega_K$ and $x \in K$,*

$$v_P(x\omega) = v_P(x) + v_P(\omega) \;\; \textit{and} \;\; v_P(\omega + \omega') \geq \min\{\, v_P(\omega), v_P(\omega') \,\} \,.$$

In particular, $[\omega]$ is a canonical divisor and $\deg[\omega] = 2g - 2$ for all $\omega \in \Omega_K$.

Corollary 11. $\dim_k(\Omega_K(0)) = g$.

Corollary 12 (Riemann–Roch). *For every $D \in \mathrm{Div}(K)$,*

$$\dim_k(L_K(D)) = \deg(D) + 1 - g + \dim_k(\Omega_K(D)) \,.$$

Corollary 13. *Let $D \in \mathrm{Div}(K)$. If $D \geq 0$ and $\deg(D) < g$, then $\dim_k(\Omega_K(D)) > 0$.*

Proof. Since $D \geq 0$, $k = L_K(0) \subseteq L_K(D)$. Therefore,

$$\dim_k(\Omega_K(D)) = g - 1 - \deg(D) + \dim_k(L_K(D)) \geq \dim_k(L_K(D)) > 0 \,. \qquad \square$$

References

[1] Goldschmidt, D. M. *Algebraic functions and projective curves*. Springer-Verlag New York, 2003. Chap. 2.5, pp. 58–63.

[2] Tate, J. "Residues of differentials on curves". In: *Ann. scient. Éc. Norm. Sup.* 1.1 (1968), pp. 149–159.