# Rational points of bounded size on the base of an abelian-by-finite family

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**Abstract**

These are the informal notes for a one-and-a-half-hour talk given in the students seminar[1] on the new proof of Mordell's conjecture by Brian Lawrence and Akshay Venkatesh. The objective of the talk is to explain the technical core of the proof. More precisely, I go over the proof of the finiteness of the set of rational points of bounded size on the base of an abelian-by-finite family with some additional properties. The notes follow very closely section 6 of Lawrence–Venkatesh's article [1] and contain no original results.

## 1 The main result

We consider the same setting as in the previous talks. In particular, throughout $K$ denotes a number field and $S$ denotes a finite set of places of $K$ containing all the archimedean ones.

Last time in David's talk we saw how to deduce the proof of Mordell's conjecture from the next result.

**Main theorem.** Let $Y$ be a (smooth projective) curve over $K$ of genus $g \geq 2$. Let $X \to Y' \to Y$ be an abelian-by-finite family, consisting of a finite étale morphism $\pi\colon Y' \to Y$ and a polarized abelian scheme $X \to Y'$ of relative dimension $d$. Suppose that this family has *full monodromy* and that it admits a *good model* over $\mathcal{O} = \mathcal{O}_K[S^{-1}]$. Let $v \notin S$ be a *friendly place* of $K$ (over a rational prime $p$). The set

$$Y(K)^* = \left\{ y \in Y(K) : \mathrm{size}_v\big(\pi^{-1}(y)\big) < \frac{1}{d+1} \right\}$$

is finite.

[1] I am grateful to Henri Darmon, Mike Lipnowski and Giovanni Rosso for organizing the seminar.

*Remark.* The notion of a *good model* was defined in previous talks, but we still have not defined what *full monodromy* or a place being *friendly* mean. These concepts will appear later as we need them in the proof (although we will not give the formal definition of the latter for the sake of conciseness).

All in all, the proof of Mordell's conjecture will be complete once

- we prove the main theorem above (the topic of this talk) and
- we construct the Kodaira–Parshin family with the properties described in the previous talk (the topic of the next couple of talks).

# 2 Outline of the strategy

The rough strategy that we follow (and that appeared already in earlier talks) can be summarized as follows. We are going to consider a period mapping $\Phi_v$ from a $v$–adic disk $\Omega_v$ in $Y(K_v)$ to some sort of flag variety and prove that

$$\dim_{K_v}\left(\overline{\mathrm{Z}(\varphi_v)\cdot\Phi_v(y)}\right) < \dim_{K_v}\left(\overline{\Phi_v(\Omega_v)}\right)$$

(where the bar denotes the Zariski closure inside the flag variety and $\mathrm{Z}(\varphi_v)$ denotes the centralizer of the crystalline Frobenius $\varphi_v$). This inequality implies that the set of points of $\Omega_v$ with local Galois representation isomorphic to $\rho_y|_{G_{K_v}}$, which is precisely $\Phi_v^{-1}\left(\mathrm{Z}(\varphi_v)\cdot\Phi_v(y)\right)$, is contained in a proper Zariski-closed subvariety of $\Omega_v$. Since $\Omega_v$ is a curve (dimension 1), any such subvariety is of dimension 0. Therefore, $\Phi_v^{-1}\left(\mathrm{Z}(\varphi_v)\cdot\Phi_v(y)\right)$ must be finite. That is, the map $y\mapsto\left[\rho_y|_{G_{K_v}}\right]$ has finite fibres.

To prove the inequality above, we are going to argue in the same way as in Jim's talk on the especial case of the modified Legendre family.

(1) **"$\Phi_v(\Omega_v)$ is large".** By the lemma at the end of Ju-Feng's talk, we can compare $\Phi_v$ with the corresponding complex period mapping $\Phi_{\mathbb{C}}$. Namely,

$$\dim_{K_v}\left(\overline{\Phi_v(\Omega_v)}\right)\geq\dim_{\mathbb{C}}\left(\Gamma\cdot h_0^{\iota}\right),$$

where $h_0=\Phi_{\mathbb{C}}(y_0)$ and $\Gamma$ is the Zariski-closure of the image of the monodromy map. Here we are going to use the hypothesis of full monodromy.

(2) **"$\mathrm{Z}(\varphi_v)$ is small".** We can use the semilinearity of $\varphi_v$ together with the fact that most extensions of residue fields $K(y')\,/\,K(y)$ are large to obtain bounds on $\dim_{\mathbb{Q}_p}\left(\mathrm{Z}(\varphi_v)\right)$. The last fact comes from the hypothesis on $\mathrm{size}_v\left(\pi^{-1}(y)\right)$ for $y\in Y(K)^*$.

**Problem.** We cannot conclude by Faltings's lemma as in Marta's talk because $\rho_y$ might not be semisimple. That is, the map $y \mapsto \left[\rho_y|_{G_{K_v}}\right]$ could have infinite image even if it has finite fibres. To prove that this is not the case, we are going to use the hypothesis that $v$ is *friendly* and some results of $p$–adic Hodge theory.

Checking that most of the $\rho_y$ are simple is the most complicated part of the proof (and we will have to content ourselves with a sketch).

# 3 Setting and notation

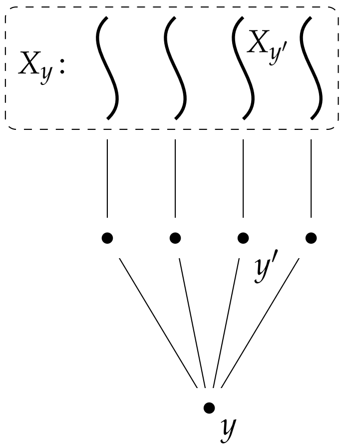
Fix $y_0 \in Y(K)^*$ and $\Omega_v = \{\, y \in Y(K_v) : y \equiv y_0 \mod v \,\}$. It suffices to prove that $Y(K)^* \cap \Omega_v$ is finite, as we can cover $Y(K)^*$ with finitely many such $v$–adic disks.

Consider a rational point $y \in Y(K) \cap \Omega_v$. Taking fibres over $y$, we can express $\pi^{-1}(y) = \mathrm{Spec}(E_y)$ for a finite étale $K$–algebra $E_y$ and then the fibre $X_y$ of $X$ over $y$ is an $E_y$–scheme. More precisely,

$$E_y = \prod_{y' | y} K(y');$$

that is, $E_y$ is a product of finite separable extensions of $K$, each corresponding to the residue field of a point $y' \in Y'(\overline{K})$ such that $\pi(y') = y$. In turn, $X_y \,/\, E_y$ is the disjoint union of the $d$–dimensional polarized abelian varieties $X_{y'} \,/\, K(y')$ for $y' \mid y$. The geometric setting is summarized in the next diagram:

$$\begin{tikzcd}
X_{y'} \arrow[r] \arrow[d, "\rho_{y'} \leftsquigarrow"'] \arrow[dr, phantom, "\ulcorner"] & X_y \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d, "\substack{d\text{–dim. pol.}\\ \text{ab. sch.}}"] \\
y' \colon \mathrm{Spec}(K(y')) \arrow[r] & \mathrm{Spec}(E_y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Y' \arrow[d, "\pi"', "\text{f. ét.}"] \\
 & y \colon \mathrm{Spec}(K) \arrow[r] & Y
\end{tikzcd}$$




*Remark.* I somewhat "lied" in the rough strategy of the proof in section 2. We are not going to use the map

$$y \longmapsto \rho_y = \mathrm{H}^1_{\text{ét}}\big((X_y)_{\overline{K}}, \mathbb{Q}_p\big)$$

but rather

$$y' \longmapsto \rho_{y'} = \mathrm{H}^1_{\text{ét}}\big((X_{y'})_{\overline{K}}, \mathbb{Q}_p\big)$$

(of course this is not precise: in any case we consider only the isomorphism classes of the restrictions of all these representations to $\mathrm{G}_{K_v} = \mathrm{Gal}\big(\overline{K_v} \,/\, K_v\big)$, so that we can use the usual $p$–adic Hodge theory equivalence with isomorphism classes of filtered $\varphi$–modules). As a matter of fact, for $y \in Y(K)$ we are going to decompose everything over the points $y' \mid y$.

By the definition of an abelian-by-finite family, we obtain the following extra structure:

- Since $E_y \,/\, K$ is finite étale, it is unramified and $\Omega^1_{E_y \,/\, K} = 0$. Therefore,
$$\mathrm{H}^i_{\mathrm{dR}}(X_y \,/\, K) = \mathrm{H}^i_{\mathrm{dR}}(X_y \,/\, E_y) = \prod_{y' \mid y} \mathrm{H}^i_{\mathrm{dR}}\big(X_{y'} \,/\, K(y')\big)$$
(in these equalities, there is a slight abuse of notation: we identify these sheaves with their global sections). We always regard $\mathrm{H}^i_{\mathrm{dR}}(X_y \,/\, K)$ as a free $E_y$–module of finite rank.
- The polarization of $X \,/\, Y'$ provides a symplectic $E_y$–bilinear pairing
$$\omega\colon\ \mathrm{H}^1_{\mathrm{dR}}(X_y \,/\, E_y) \times \mathrm{H}^1_{\mathrm{dR}}(X_y \,/\, E_y) \to E_y$$
compatible with the decompositions as products over $y' \mid y$.

We consider the analogous local situation at $v$ and fix some additional notation.

- Set
$$E_{y,v} = E_y \otimes_K K_v = \prod_{(y',w) \mid (y,v)} K(y')_w,$$
where the product is over the pairs $(y', w)$ consisting of a point $y' \in Y'\big(\overline{K}\big)$ with $\pi(y') = y$ and a valuation $w$ of $K(y')$ above the valuation $v$ of $K = K(y)$. Note that we can also identify $E_{y,v} = \mathrm{H}^0_{\mathrm{dR}}\big((X_y)_{K_v} \,/\, K_v\big)$.
- Set
$$V_{y,v} = \mathrm{H}^1_{\mathrm{dR}}\big((X_y)_{K_v} \,/\, K_v\big) = \mathrm{H}^1_{\mathrm{dR}}\big((X_y)_{K_v} \,/\, E_{y,v}\big)$$
(regarded as an $E_{y,v}$–module).
- For every $(y', w) \mid (y, v)$, set
$$V_{y',w} = \mathrm{H}^1_{\mathrm{dR}}\big((X_{y'})_{K(y')_w} \,/\, K(y')_w\big).$$
- All in all, we obtain a decomposition
$$V_{y,v} = \prod_{(y',w) \mid (y,v)} V_{y',w} \quad \text{compatible with} \quad E_{y,v} = \prod_{(y',w) \mid (y,v)} K(y')_w.$$

# 4 Period mappings

Let $y \in Y(K) \cap \Omega_v$ as before. As in previous talks, we want to use $p$–adic Hodge theory to identify the crystalline representation $\rho_y|_{\mathrm{G}_{K_v}}$ with a filtered $\varphi$–module via the comparison theorems between different cohomology theories. However, we are going to use the extra structure described above to obtain "finer-tuned" period mappings.

Regard $V_{y,v}$ as a free $E_{y,v}$–module of rank $2d$ endowed with a symplectic form $\omega$ and a crystalline Frobenius $\varphi_v$. Observe that $\mathrm{F}^1 V_{y,v}$, the first filtered piece of the Hodge filtration, is a $d$–dimensional lagrangian submodule (i.e., the restriction of $\omega$ to $\mathrm{F}^1 V_{y,v}$ is identically 0). The Gauss–Manin connection provides compatible (canonical) identifications

$$E_{y,v} \cong E_{y_0,v} \quad \text{and} \quad (V_{y,v}, \omega, \varphi_v) \cong (V_{y_0,v}, \omega, \varphi_v).$$

Therefore, the image of $\mathrm{F}^1 V_{y,v}$ in $V_{y_0,v}$ is a lagrangian $E_{y_0,v}$–submodule and we obtain a refined period mapping

$$\begin{array}{rcl} & & \mathcal{G}_v = \mathrm{Res}_{K_v}^{E_{y_0,v}} \mathrm{Gr}(V_{y_0,v}, d) \\ & & \cup \\ \Phi_v \colon \Omega_v & \longrightarrow & \mathcal{H}_v = \mathrm{Res}_{K_v}^{E_{y_0,v}} \mathrm{LGr}(V_{y_0,v}, \omega) \\ y & \longmapsto & \mathrm{F}^1 V_{y,v} \end{array}$$

(here, Gr and LGr denote the grassmannian and the lagrangian grassmannian varieties over $E_{y_0,v}$, while $\mathrm{Res}_{K_v}^{E_{y_0,v}}$ denotes the Weil restriction to $K_v$–varieties).

Similarly, the Gauss–Manin connection identifies

$$\{\, (y', w) \mid (y, v) \,\} \cong \{\, (y'_0, w_0) \mid (y_0, v) \,\}$$

because

$$\prod_{(y',w)\mid(y,v)} K(y')_w = E_{y,v} \cong E_{y_0,v} = \prod_{(y'_0,w_0)\mid(y_0,v)} K(y'_0)_{w_0}.$$

We decompose

$$\mathcal{H}_v = \prod_{(y'_0,w_0)\mid(y_0,v)} \mathcal{H}_{y'_0,w_0}, \quad \text{where} \quad \mathcal{H}_{y'_0,w_0} = \mathrm{Res}_{K_v}^{K(y'_0)_{w_0}} \mathrm{LGr}(V_{y'_0,w_0}, \omega).$$

Furthermore, if $(y', w) \mid (y, v)$ corresponds to $(y'_0, w_0) \mid (y_0, v)$, the Gauss–Manin connection identifies

$$(V_{y',w}, \omega, \varphi_\omega) \cong (V_{y'_0,w_0}, \omega, \varphi_{w_0})$$

compatibly and the image of $\mathrm{F}^1 V_{y',w}$ in $V_{y'_0,w_0}$ is a lagrangian $K(y'_0)_{w_0}$–subspace. All in all, we can define a refined period mapping

$$\begin{array}{rcl} \Phi_{y'_0,w_0} \colon \Omega_v & \xrightarrow{\Phi_v} \mathcal{H}_v \twoheadrightarrow & \mathcal{H}_{y'_0,w_0} \\ y & \longmapsto & \mathrm{F}^1 V_{y',w} \end{array}$$

for every component $\mathcal{H}_{y_0',w_0}$ of $\mathcal{H}_v$ (where $(y',w)$ corresponds to $(y_0',w_0)$).

# 5 Full monodromy

The assumption that $X \to Y' \to Y$ has full monodromy means that

$$\Gamma = \overline{\mathrm{Im}\Big(\pi_1\big(Y(\mathbb{C}),y_0\big) \longrightarrow \mathrm{GL}\big(\mathrm{H}^1_{\mathrm{B}}(X_{y_0}(\mathbb{C}),\mathbb{Q})\big)\Big)}$$

(the Zariski-closure of the image of the complex monodromy) contains

$$\prod_{y_0'|y_0} \mathrm{Sp}\big(\mathrm{H}^1_{\mathrm{B}}(X_{y_0'}(\mathbb{C}),\mathbb{Q}),\omega\big),$$

where $\omega$ is the symplectic form coming from the polarization of $X \to Y'$. By the comparison theorem between Betti cohomology and de Rham cohomology over $\mathbb{C}$, this hypothesis says that $\Gamma$ acts transitively on the flag variety $\mathcal{H}_{\mathbb{C}} = \mathrm{LGr}\big(V_{\mathbb{C}},\omega\big)$, where

$$V_{\mathbb{C}} = \mathrm{H}^1_{\mathrm{dR}}\big(X_{y_0}(\mathbb{C})\big) = \prod_{y_0'|y_0} \mathrm{H}^1_{\mathrm{dR}}\big(X_{y_0'}(\mathbb{C})\big).$$

We conclude that $\Phi_v(\Omega_v)$ is dense in $\mathcal{H}_v$ because

$$\dim_{K_v}\big(\overline{\Phi_v(\Omega_v)}\big) \geq \dim_{\mathbb{C}}\big(\Gamma \cdot h_0^\iota\big).$$

This was the first point of the outline of the strategy (cf. section 2).

# 6 The proof of the main theorem

The rest of the proof of the main theorem follows from two "kind-of-complicated" lemmata. We now state them and deduce the conclusion from them.

**Lemma A (generic simplicity).** For all but finitely many points $y \in \Omega_v \cap Y(K)^*$, there exists a pair $(y',w) \mid (y,v)$ satisfying that
(A1) $[K(y')_w : K_v] \geq 8$ and
(A2) $\rho_{y'}$ is a simple representation of $\mathrm{G}_{K(y')}$.

**Lemma B (variation of Galois representations).** Fix an extension $K_v' \,/\, K_v$ with $8 \leq [K_v' : K_v] < \infty$ and a representation $\rho'$ of $\mathrm{G}_{K_v'}$. There are only finitely many points $y \in \Omega_v \cap Y(K)$ for which there exists a pair $(y',w) \mid (y,v)$ with properties (A1) and (A2) and such that

$$\big(K(y')_w, \rho_{y'}|_{\mathrm{G}_{K(y')_w}}\big) \cong \big(K_v',\rho'\big).$$

As $y$ and $y'$ vary among the points considered in lemma A, there are only finitely many possibilities for the isomorphism class of $K(y')$ (which is isomorphic to $K(y'_0)$ for some $y'_0 \mid y_0$) and so also for the isomorphism class of $\big(K(y'), \rho_{y'}\big)$ by Faltings's lemma. Thus, lemma B concludes the proof of the theorem.

In the next two sections, we give the proofs of lemma B and lemma A.

# 7 Variation of representations (proof of lemma B)

We have correspondences

$$(y', w) \longleftrightarrow (y'_0, w_0)$$

and

$$\rho_{y'}|_{G_{K(y')_w}} \longleftrightarrow (V_{y',w}, \varphi_w, \mathrm{F}^1\, V_{y',w}) \longleftrightarrow \Phi_{y'_0,w_0}(y).$$

Consider the point $h$ of $\mathcal{H}_{y'_0,w_0}$ corresponding to $\rho'$. We have to prove that the set

$$\{\, y \in \Omega_v \cap Y(K) : \Phi_{y'_0,w_0}(y) \in \mathrm{Z}(\varphi_{w_0}) \cdot h \,\}$$

is finite, where

$$\mathrm{Z}(\varphi_{w_0}) = \{\, f \in \mathrm{GL}\big(V_{y'_0,w_0}\big) : f \circ \varphi_{w_0} = \varphi_{w_0} \circ f \,\}.$$

Since $\varphi_{w_0}$ is Frobenius-semilinear, $\mathrm{Z}(\varphi_{w_0})$ is a $\mathbb{Q}_p$–vector space but not a $K_v$–vector space, so we modify it a bit.

Write $K'_{w_0} = K(y'_0)_{w_0}$, $r = [K'_{w_0} : K_v] \geq 8$ and $n = [K_v : \mathbb{Q}_p]$ to simplify the notation. We use that $\mathrm{Z}(\varphi_{w_0}) \subset \mathrm{Z}(\varphi^n_{w_0})$ (the latter is a $K_v$–vector space) and prove instead that the set

$$\{\, y \in \Omega_v \cap Y(K) : \Phi_{y'_0,w_0}(y) \in \mathrm{Z}(\varphi^n_{w_0}) \cdot h \,\}$$

is finite. We do so with the already familiar argument of comparing dimensions:

$$\begin{array}{c} \dim_{K_v}\big(\mathrm{Z}(\varphi^n_{w_0})\big) = \dim_{K'_{w_0}}\big(\mathrm{Z}(\varphi^{nr}_{w_0})\big) \leq \dim_{K'_{w_0}}\big(\mathrm{GL}(V_{y'_0,w_0})\big) = (2d)^2 \\ \wedge \\ \dim_{K_v}\big(\mathcal{H}_{y'_0,w_0}\big) = r \cdot \dim_{K'_{w_0}}\big(\mathrm{LGr}(V_{y'_0,w_0}, \omega)\big) \geq 8 \cdot \dfrac{d(d+1)}{2} = 4d(d+1) \end{array}$$

and so we deduce that the set above lies in a proper Zariski-closed subvariety of a curve, which means that it is finite. This was the second point of the outline of the strategy (cf. section 2).

## 8 Generic simplicity (proof of lemma A)

What remains is to find a workaround for the problem described in the outline of the strategy (cf. section 2). Namely, we want to prove that, up to discarding a finite number of points, the considered representations are simple and so we can apply Faltings's lemma. This is the trickiest part and we will have to content ourselves with a sketch of the proof, as we have not seen the $p$–adic Hodge theory background involved. We divide the proof of lemma A into two sublemmata.

**Sublemma 1.** If $y \in \Omega_v \cap Y(K)^*$ is a *bad* point (i.e., does not satisfy the conclusion of lemma A), then at least there exists a pair $(y', w) \mid (y, v)$ such that
(A1) $[K(y')_w : K_v] \geq 8$ and
(A2′) there is a non-trivial proper subspace $W_{y',w}$ of $V_{y',w}$ which is $\varphi_w$–stable and with the property that

$$\dim\big(\mathrm{F}^1 W_{y',w}\big) \geq \frac{1}{2} \dim\big(V_{y',w}\big).$$

(Here and in the proof, all occurrences of dim mean $\dim_{K(y')_w}$.)

*Idea of the proof.* For each $y' \mid y$, let $W_{y'}$ be a subrepresentation of $\rho_{y'}$ that is minimal of positive dimension. For each $w \mid v$, $W_{y'}$ induces by $p$–adic Hodge theory a $\varphi_w$–stable subspace $W_{y',w}$ of $V_{y',w}$.

If $y$ is bad but $[K(y')_w : K_v] \geq 8$, then $\rho_{y'}$ must fail to be simple. This together with the choice of $W_{y'}$ minimal and the fact that $\rho_{y'}$ preserves the bilinear form $\omega$ imply that

$$\dim\big(W_{y'}\big) \leq \frac{1}{2} \dim\big(\rho_{y'}\big),$$

whence $\dim\big(W_{y',w}\big) \leq d$ (recall that $X_{y'}$ is an abelian variety of dimension $d$).

Now suppose, for the sake of contradiction, that (as $y'$ varies over the points of $Y'$ lying over a fixed bad $y$) the implication

$$[K(y')_w : K_v] \geq 8 \implies \dim\big(\mathrm{F}^1 W_{y',w}\big) < \frac{1}{2} \dim\big(V_{y',w}\big)$$

holds. Observe that, whenever $[K(y')_w : K_v] \geq 8$,

$$\frac{\dim\big(\mathrm{F}^1 W_{y',w}\big)}{\dim\big(W_{y',w}\big)} < \frac{1}{2} \quad \text{and} \quad \dim\big(W_{y',w}\big) \leq d,$$

whence we deduce that

$$\frac{\dim\big(\mathrm{F}^1 W_{y',w}\big)}{\dim\big(W_{y',w}\big)} \leq \frac{1}{2} - \frac{1}{2d}$$

because all dimensions are integer numbers.

The place $v$ being friendly is a technical condition that constrains the possible Hodge–Tate weights of any (global) Galois representation that is crystalline at all primes above $p$ and pure of some weight. In particular, one can prove that

$$\sum_{w|v}[K(y')_w : K_v] \cdot \frac{\dim\big(\mathrm{F}^1 W_{y',w}\big)}{\dim\big(W_{y',w}\big)} = \frac{1}{2}[K(y') : K].$$

Summing over all $y' \mid y$, we obtain that

$$\sum_{(y',w)\big|(y,v)} [K(y')_w : K_v] \cdot \frac{1}{2} = \sum_{(y',w)\big|(y,v)} [K(y')_w : K_v] \cdot \frac{\dim\big(\mathrm{F}^1 W_{y',w}\big)}{\dim\big(W_{y',w}\big)} \leq$$

$$\leq \sum_{[K(y')_w:K_v]\geq 8} [K(y')_w : K_v] \cdot \Big(\frac{1}{2} - \frac{1}{2d}\Big) + \sum_{[K(y')_w:K_v]<8} [K(y')_w : K_v] \cdot 1$$

and rearranging

$$\sum_{[K(y')_w:K_v]<8} [K(y')_w : K_v] \cdot \Big(\frac{1}{2} + \frac{1}{2d}\Big) \geq \sum_{(y',w)\big|(y,v)} [K(y')_w : K_v] \cdot \frac{1}{2d}.$$

Interpreting $[K(y')_w : K_v]$ as the size of the $\mathrm{Frob}_v$–orbit of $y'$, the last inequality says that

$$\mathrm{size}_v\big(\pi^{-1}(y)\big) \geq \frac{1}{d+1},$$

which contradicts the choice of $y \in Y(K)^*$. □

Fix $(y_0', w_0) \mid (y_0, v)$ such that $[K(y_0')_{w_0} : K_v] \geq 8$. By sublemma 1, it suffices to prove that the set

$$\big\{\, y \in \Omega_v \cap Y(K) : \Phi_{y_0',w_0}(y) \in \mathcal{H}^{\mathrm{bad}}_{y_0',w_0} \,\big\}$$

is finite, where the subvariety $\mathcal{H}^{\mathrm{bad}}_{y_0',w_0}$ of $\mathcal{H}_{y_0',w_0}$ parametrizes lagrangian subspaces $F \subset V_{y_0',w_0}$ for which there exists $0 \neq W \subsetneq V_{y_0',w_0}$ that is $\varphi_{w_0}$–stable and such that $\dim\big(F \cap W\big) \geq \frac{1}{2}\dim\big(W\big)$. As usual, we prove that $\mathcal{H}^{\mathrm{bad}}_{y_0',w_0}$ lies in a proper closed subvariety of $\mathcal{H}_{y_0',w_0}$.

**Sublemma 2.** Let $L_w \,/\, K_v$ be a finite unramified extension with $\big[L_w : K_v\big] = r \geq 8$. Let $(V, \omega)$ be a symplectic $L_w$–vector space with $\dim_{L_w}\big(V\big) = 2d$ and let $\varphi\colon V \to V$ be a $\mathrm{Frob}_w$–semilinear bijective map. For lagrangian subspaces $F_1, \ldots, F_n \subset V$ and $0 \neq W \subsetneq V$ $\varphi$–stable, let $\mathcal{P}(F_1, \ldots, F_n; W)$ be the property

$$\dim\big(F_j \cap W\big) \geq \frac{1}{2}\dim\big(W\big) \quad \text{ for every } 1 \leq j \leq n.$$

Then

$$\big\{\, F \subset V \text{ lagrangian} : \text{there exists } 0 \neq W \subsetneq V \ \varphi\text{–stable such that } \mathcal{P}(F; W) \,\big\}$$

defines a proper Zariski-closed subset of $\operatorname{Res}_{K_v}^{L_w}\big(\mathrm{LGr}(V,\omega)\big)$.

*Idea of the proof.* Since we want to work on points of the lagrangian grassmannian variety, we have to base change everything to the algebraic closure $\overline{K_v}$. We can decompose

$$V \otimes_{K_v} \overline{K_v} = \bigoplus_{\sigma_i\colon L_w \hookrightarrow \overline{K_v}} V_i,$$

where every $(V_i,\omega)$ is a symplectic $\overline{K_v}$–vector space with $\dim\big(V_i\big) = 2d$. We can even number the $r$ embeddings $\sigma_i$ in such a way that $\varphi|_{V_i}\colon V_i \xrightarrow{\cong} V_{i+1}$. Similarly, we decompose

$$W \otimes_{K_v} \overline{K_v} = \bigoplus_{i=1}^{r} W_i \quad \text{and} \quad F \otimes_{K_v} \overline{K_v} = \bigoplus_{i=1}^{r} F_i$$

with $W_i, F_i \subset V_i$ and $\varphi|_{W_i}\colon W_i \xrightarrow{\cong} W_{i+1}$. Moreover, $\mathcal{P}(F;W)$ implies that $\mathcal{P}(F_i;W_i)$ for every $1 \le i \le r$.

Identifying each $V_i$ with $V_1$ via $\varphi^{i-1}$, it suffices to prove that

$$E = \left\{\, (F_1,\dots,F_r) \in \mathrm{LGr}(V_1,\omega)^r : \mathcal{P}(F_1,\dots,F_r;W_1) \text{ for some } 0 \ne W_1 \subsetneq V_1 \,\right\}$$

defines a proper Zariski-closed subset of $\big(\operatorname{Res}_{K_v}^{L_w}\mathrm{LGr}(V,\omega)\big) \otimes_{K_v} \overline{K_v} = \mathrm{LGr}(V_1,\omega)^r$. Note that we identify these varieties over $\overline{K_v}$ with their closed points.

That $E$ is closed follows by upper-semicontinuity of $\dim\big(F_i \cap W_1\big)$. To see that $E$ is proper, we construct $(F_1,\dots,F_r) \notin E$ with an ad-hoc argument[2] (linear algebra). As a matter of fact, the argument works so long as $r \ge 5$.

Let $e_1, e_1', \dots, e_d, e_d'$ be a standard symplectic basis, with $\omega(e_i,e_i') = 1$. Consider the lagrangian subspaces

$$\begin{aligned}
F_1 &= \langle e_1, e_2, \dots, e_d\rangle\\
F_2 &= \langle e_1', e_2', \dots, e_d'\rangle\\
F_3 &= \langle e_1+e_1', e_2+e_2', \dots, e_d+e_d'\rangle\\
F_4 &= \langle e_1+2e_1', e_2+4e_2', \dots, e_d+2de_d'\rangle.
\end{aligned}$$

Observe that $F_i \cap F_j = 0$ if $i \ne j$ and $V = F_1 \oplus F_2 = F_3 \oplus F_4$. Let $\pi_1\colon V \twoheadrightarrow F_1$ and $\pi_2\colon V \twoheadrightarrow F_2$ be the two projections from $F_1 \oplus F_2$. We obtain isomorphisms

$$\begin{array}{ccc}
\Phi_3\colon F_1 \overset{\pi_1}{\cong} F_3 \overset{\pi_2}{\cong} F_2 & & \Phi_4\colon F_1 \overset{\pi_1}{\cong} F_4 \overset{\pi_2}{\cong} F_2\\
e_i \longmapsto e_i' & \text{and} & e_i \longmapsto 2ie_i'
\end{array}$$

(where $\pi_1$ and $\pi_2$ denote the corresponding restrictions to $F_3$ or $F_4$).

[2]In the seminar talk, I stopped here without explaining the argument any further.

We claim that there are only finitely many non-trivial subspaces $W$ such that $\mathcal{P}(F_1, F_2, F_3, F_4; W)$. Indeed, since

$$\dim\big(W \cap F_i\big) \geq \frac{1}{2}\dim\big(W\big),$$

comparing dimensions we see that

$$W = \big(W \cap F_1\big) \oplus \big(W \cap F_2\big) \quad \text{and} \quad W = \big(W \cap F_3\big) \oplus \big(W \cap F_4\big)$$

and so in fact

$$\dim\big(W \cap F_i\big) = \frac{1}{2}\dim\big(W\big).$$

Again by comparison of dimensions, we see that $\pi_1$ and $\pi_2$ induce isomorphisms

$$\big(W \cap F_1\big) \overset{\pi_1}{\cong} \big(W \cap F_3\big) \overset{\pi_2}{\cong} \big(W \cap F_2\big)$$

and

$$\big(W \cap F_1\big) \overset{\pi_1}{\cong} \big(W \cap F_4\big) \overset{\pi_2}{\cong} \big(W \cap F_2\big).$$

Thus, $\Phi_3^{-1} \circ \Phi_4$ induces an automorphism of $W \cap F_1$. But $\Phi_3^{-1} \circ \Phi_4$ is "diagonal" with pairwise distinct eigenvalues (it is given by $e_i \mapsto 2ie_i$), so there are only finitely many possibilities for $W \cap F_1$. Same for $W \cap F_2 = \Phi_3(W \cap F_1)$ and so for $W = \big(W \cap F_1\big) \oplus \big(W \cap F_2\big)$ too.

Let $W_1, \ldots, W_N$ be the possibilities for $W$ satisfying that $\mathcal{P}(F_1, F_2, F_3, F_4; W)$. The condition

$$\mathcal{P}(F_5; W_i): \quad \dim\big(F_5 \cap W_i\big) \geq \frac{1}{2}\dim\big(W_i\big)$$

cuts out a proper Zariski-closed subset of $\mathrm{LGr}(V_1, \omega)$. Therefore, we can choose $F_5 \in \mathrm{LGr}(V_1, \omega)$ such that $\mathcal{P}(F_5; W_i)$ does not hold for any $1 \leq i \leq N$. Finally, choosing any other lagrangians $F_6, \ldots, F_r$, we conclude that $(F_1, \ldots, F_r) \notin E$ by construction. □

# References


[1] Lawrence, B. and Venkatesh, A. *Diophantine problems and p–adic period mappings*. Version 2 (preprint). 2019. arXiv: `1807.02721v2 [math.NT]`.