Classical and overconvergent modular symbols and p–adic L–functions attached to eigenforms

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Abstract

These are the (extended) notes for a talk given in the students seminar[1] on the Mazur–Tate–Teitelbaum conjecture. I present briefly the theory of modular symbols and show how they can be used to construct p–adic L–functions. The notes are a summary of various sources and contain no original results.

Notation. Throughout this document, fix a positive integer N and a prime number p such that $p > 2$ and $p \nmid N$. Fix algebraic closures $\overline{\mathbb{Q}}$ and $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}$ and $\mathbb{Q}_p$, respectively, together with embeddings $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ and $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Set $\Gamma = \Gamma_0(N)$ and $\Gamma_0 = \Gamma_0(Np)$.[2] Take an integer $k \geq 0$.

0 Motivation

We begin with the simplest case. Consider the space $S_2(\Gamma)$ of cusp forms of level 2 for Γ, which is a complex vector space of finite dimension. Let $\mathbb{T}_{\mathbb{C}}$ be the subalgebra of $\operatorname{End}_{\mathbb{C}}(S_2(\Gamma))$ generated by the Hecke operators T_ℓ for the primes $\ell \nmid N$ and U_ℓ for the primes $\ell \mid N$.

The $\mathbb{T}_{\mathbb{C}}$–algebra $S_2(\Gamma)$ is an important object of study in number theory and we naturally want to gain a better understanding of it. One way to do so is by computing an explicit basis in terms of which all interesting operations have a concrete and simple description. The following result hints at a first step in that direction.

[1] I thank Henri Darmon and Adrian Iovita for organizing the seminar.

[2] We use the groups $\Gamma_0(-)$ for simplicity, but the theory works in the same way for $\Gamma_1(-)$.

Theorem 1. *The map*

$$\langle -,- \rangle \colon S_2(\Gamma) \times \mathbb{T}_{\mathbb{C}} \longrightarrow \mathbb{C}$$

given by $\langle f, T \rangle = a_1(\mathrm{T} f)$ *(where* $a_1(-)$ *denotes the coefficient of* q *in the* q*–expansion of a cusp form) defines a perfect pairing of complex vector spaces and so induces an isomorphism* $S_2(\Gamma) \cong \mathrm{Hom}_{\mathbb{C}}(\mathbb{T}_{\mathbb{C}}, \mathbb{C})$.

Proof. The pairing is bilinear because $\mathbb{T}_{\mathbb{C}} \subset \mathrm{End}_{\mathbb{C}}(S_2(\Gamma))$ and a_1 is also a linear map.

Let $f \in S_2(\Gamma)$ such that $\langle f, \mathrm{T} \rangle = 0$ for all $\mathrm{T} \in \mathbb{T}_{\mathbb{C}}$. In particular,

$$a_n(f) = a_1(\mathrm{T}_n f) = \langle f, \mathrm{T}_n \rangle = 0$$

for all $n \in \mathbb{N}$, which means that $f = 0$.

Similarly, let $\mathrm{T} \in \mathbb{T}_{\mathbb{C}}$ with the property that $\langle f, \mathrm{T} \rangle = 0$ for all $f \in S_2(\Gamma)$. Take $g \in S_2(\Gamma)$. We want to prove that $\mathrm{T}\, g = 0$ (as g is arbitrary, this implies that T is the 0 operator in $\mathrm{End}_{\mathbb{C}}(S_2(\Gamma))$). Indeed, using the commutativity of $\mathbb{T}_{\mathbb{C}}$ we get that

$$a_n(\mathrm{T}\, g) = a_1(\mathrm{T}_n\, \mathrm{T}\, g) = a_1(\mathrm{T}\ \mathrm{T}_n\, g) = \langle \mathrm{T}_n\, g, \mathrm{T} \rangle = 0$$

for all $n \in \mathbb{N}$.

The previous two paragraphs show that $\langle -,- \rangle$ is a perfect pairing. But the $\mathbb{C}$–vector space $S_2(\Gamma)$ is finite-dimensional. Therefore, we obtain an induced isomorphism $S_2(\Gamma) \cong \mathrm{Hom}_{\mathbb{C}}(\mathbb{T}_{\mathbb{C}}, \mathbb{C})$ of complex vector spaces. □

This theorem tells us that, to understand the space $S_2(\Gamma)$, it suffices to study $\mathbb{T}_{\mathbb{C}}$ instead. And we can do so by making Hecke operators act in the same way on other spaces, which we hope will be easier to work with. More precisely, we are going to introduce a new (larger) space with an analogous action of Hecke operators T_ℓ for the primes $\ell \nmid N$ and U_ℓ for the primes $\ell \mid N$ that contains an isomorphic copy of $S_2(\Gamma)$ (as $\mathbb{T}_{\mathbb{C}}$–modules).

For example, consider the compactified modular curve $X(\Gamma) = \Gamma \setminus \mathbb{H}^*$, which is a compact Riemann surface. Recall that $S_2(\Gamma)$ corresponds naturally to the space $\Omega^1(X(\Gamma))$ of (global) holomorphic differential forms on $X(\Gamma)$. If $g(\Gamma)$ denotes the genus of $X(\Gamma)$, then $\Omega^1(X(\Gamma))$ is a $\mathbb{C}$–vector space of dimension $g(\Gamma)$ and the first homology group $\mathrm{H}_1(X(\Gamma), \mathbb{Z})$ is a free $\mathbb{Z}$–module of rank $2g(\Gamma)$. We can relate $\Omega^1(X(\Gamma))$ to $\mathrm{H}_1(X(\Gamma), \mathbb{Z})$ by means of the following result from the theory of compact Riemann surfaces.

Theorem 2. *The integration pairing*

$$I \colon \mathrm{H}_1(X(\Gamma), \mathbb{R}) \times S_2(\Gamma) \longrightarrow \mathbb{C}$$

defined by

$$I(\gamma, f) = 2\pi i \int_\gamma f(z)\, dz$$

is non-degenerate and induces an isomorphism $\mathrm{H}_1(X(\Gamma), \mathbb{R}) \cong \mathrm{Hom}_{\mathbb{C}}(S_2(\Gamma), \mathbb{C})$ *as real vector spaces.*

The space $\mathrm{H}_1(X(\Gamma), \mathbb{R})$ is given roughly by $\mathbb{R}$–linear combinations of closed paths on $X(\Gamma)$ modulo some homology relations and we can view $\mathrm{H}_1(X(\Gamma), \mathbb{Z})$ embedded in $\mathrm{H}_1(X(\Gamma), \mathbb{R})$ in a natural way. The composition

$$\begin{array}{ccccc} \mathrm{H}_1(X(\Gamma), \mathbb{Z}) & \hookrightarrow & \mathrm{H}_1(X(\Gamma), \mathbb{R}) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathbb{C}}(S_2(\Gamma), \mathbb{C}) \\ \wr\Vert & & & & \wr\Vert \\ \mathbb{Z}^{2g(\Gamma)} & & \hookrightarrow & & \mathbb{C}^{g(\Gamma)} \end{array}$$

is used to obtain a full rank lattice and define the jacobian of $X(\Gamma)$.

By duality, we obtain an action of Hecke operators on $\mathrm{H}_1(X(\Gamma), \mathbb{R})$. However, there are many possible paths on $X(\Gamma)$, so we could try to restrict to a narrower class of paths making use of the construction of $X(\Gamma)$ as a quotient of $\mathbb{H}^*$. For instance, we could focus on paths coming from geodesics between cusps in $\mathbb{H}^*$ and determine their relations. This was the original idea of Manin. A detailed account of this approach can be found in Manin's original article [3]. Stein's book [9] also explains Manin's theory and some generalizations from a computational point of view. Here, we do something *dual*, corresponding to cohomology with compact supports instead of homology.

Let $\Delta_0 = \mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$. This group is generated by the elements of the form $\{s\} - \{r\}$ for $r, s \in \mathbb{P}^1(\mathbb{Q})$. (The divisor $\{s\} - \{r\}$ should be thought of as a path from r to s in $\mathbb{H}^*$.) We endow Δ_0 with an action of $\mathrm{GL}_2(\mathbb{Q})$ as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (\{s\} - \{r\}) = \left\{\frac{as+b}{cs+d}\right\} - \left\{\frac{ar+b}{cr+d}\right\}.$$

We define the space of (classical) modular symbols for Γ to be $\mathrm{Hom}_\Gamma(\Delta_0, \mathbb{C})$, where we regard Γ acting trivially on $\mathbb{C}$.

Given $f \in S_2(\Gamma)$, we can construct a modular symbol $\psi_f \in \mathrm{Hom}_\Gamma(\Delta_0, \mathbb{C})$ defined by

$$\psi_f(\{s\} - \{r\}) = 2\pi i \int_r^s f(z)\, dz.$$

The morphism ψ_f is indeed invariant under the action of Γ:

$$2\pi i \int_{\gamma r}^{\gamma s} f(z)\, dz = 2\pi i \int_r^s f(\gamma z)\, d(\gamma z) = 2\pi i \int_r^s f(z)\, dz \quad \text{for all } \gamma \in \Gamma.$$

Now we can define an action of Hecke operators on $\mathrm{Hom}_\Gamma(\Delta_0, \mathbb{C})$ coinciding with the action on $S_2(\Gamma)$. For $\psi \in \mathrm{Hom}_\Gamma(\Delta_0, \mathbb{C})$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Q})$, we define

$\psi|\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Hom}_\Gamma(\Delta_0, \mathbb{C})$ by

$$\left(\psi|\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right)(D) = \psi\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)D\right) \quad \text{for all } D \in \Delta_0.$$

Let ℓ denote a prime number. The action of Hecke operators on $\psi \in \mathrm{Hom}_\Gamma(\Delta_0, \mathbb{C})$ is given by

$$\psi|\,\mathrm{T}_\ell = \sum_{b=1}^{\ell-1} \psi|\left(\begin{smallmatrix} 1 & b \\ 0 & \ell \end{smallmatrix}\right) + \psi|\left(\begin{smallmatrix} \ell & 0 \\ 0 & 1 \end{smallmatrix}\right)$$

if $\ell \nmid N$ and

$$\psi|\,\mathrm{U}_\ell = \sum_{b=1}^{\ell-1} \psi|\left(\begin{smallmatrix} 1 & b \\ 0 & \ell \end{smallmatrix}\right)$$

if $\ell \mid N$.

We will get more involved constructions of modular symbols if we replace $\mathbb{C}$ with other Γ–modules. For instance, to work with $S_{k+2}(\Gamma)$ for $k > 0$, we need to use the space $V_k(\mathbb{C}) = \mathbb{C}[X, Y]_k$ of homogeneous polynomials of degree k in two variables X and Y endowed with some action of Γ. We are going to see this and other examples in the following sections.

1 The Eichler–Shimura isomorphism

Let R be a ring (we will only be interested in the cases $R = \mathbb{Q}$, $\mathbb{Q}_p$ or $\mathbb{C}$) and set

$$S_0(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{Z}) : p \nmid a, p \mid c \text{ and } ad - bc \neq 0 \right\}.$$

Consider any R–module V with a right action of $S_0(p)$. Hecke operators act on V as follows: for $v \in V$ and a prime number ℓ,

$$\begin{cases} v|\,\mathrm{T}_\ell = v|\left(\begin{smallmatrix} \ell & 0 \\ 0 & 1 \end{smallmatrix}\right) + \sum_{b=0}^{\ell-1} v|\left(\begin{smallmatrix} 1 & b \\ 0 & \ell \end{smallmatrix}\right) & \text{if } \ell \nmid Np, \\ v|\,\mathrm{U}_\ell = \sum_{b=0}^{\ell-1} v|\left(\begin{smallmatrix} 1 & b \\ 0 & \ell \end{smallmatrix}\right) & \text{if } \ell \mid Np. \end{cases}$$

To any R–module V with a right action of $S_0(p)$, one can attach a locally free sheaf $\widetilde{V}$ on $Y(\Gamma_0) = \Gamma_0 \backslash \mathbb{H}$. By functoriality, we obtain an induced right action of Hecke operators on $\mathrm{H}^1_c(Y(\Gamma_0), \widetilde{V})$, the first cohomology group with compact supports of $Y(\Gamma)$ with coefficients in $\widetilde{V}$. Define $\mathrm{H}^1_c(\Gamma_0, V) = \mathrm{H}^1_c(Y(\Gamma_0), \widetilde{V})$.

We are going to define spaces of modular symbols in a *more combinatorial* way. These spaces of modular symbols are much easier to work with, but in any case they correspond to the cohomology groups with compact supports introduced above.

Let $V_k(R) = R[X,Y]_k$ (the subscript k indicates the subspace of homogeneous polynomials of degree k). We can endow $V_k(R)$ with a right action of $S_0(p)$ as follows: for $P \in V_k(R)$ and $\gamma \in S_0(p)$, we define $P|\gamma \in V_k(\mathbb{R})$ by

$$(P|\gamma)(X,Y) = P\big((X,Y)\cdot\gamma^*\big) = P(dX - cY, -bX + aY),$$

where

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \gamma^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Theorem 3 (Eichler–Shimura). *There is a canonical isomorphism of Hecke modules*

$$\mathrm{H}^1_{\mathrm{c}}(\Gamma_0, V_k(\mathbb{C})) \cong M_{k+2}(\Gamma_0) \oplus \overline{S}_{k+2}(\Gamma_0),$$

where $\overline{S}_{k+2}(\Gamma_0) = \{\, f(z) : f(-\overline{z}) \in S_{k+2}(\Gamma_0) \,\}$.

Proof. See chapter 8 of Shimura's book [7], especially theorem 8.4.

In section 8.1 of ibid., Shimura provides an ad hoc description of $\mathrm{H}^1_{\mathrm{par}}(\Gamma_0, V)$, the first parabolic cohomology group of the group Γ_0 with coefficients in V, using the same kind of cochains as in standard group cohomology but with some extra conditions related to the parabolic elements. There is a canonical map $\mathrm{H}^1_{\mathrm{c}}(\Gamma_0, V) \to \mathrm{H}^1(\Gamma_0, V)$ whose image is precisely $\mathrm{H}^1_{\mathrm{par}}(\Gamma_0, V)$. Theorem 8.4 of ibid. shows that $\mathrm{H}^1_{\mathrm{par}}(\Gamma_0, V) \cong S_{k+2}(\Gamma_0) \oplus \overline{S}_{k+2}(\Gamma_0)$. A modification of that argument yields the desired result (the kernel of $\mathrm{H}^1_{\mathrm{c}}(\Gamma_0, V) \to \mathrm{H}^1(\Gamma_0, V)$ accounts for the extra Eisenstein series). □

2 Modular symbols

Recall from section 0 that Δ_0 is the group of divisors of degree 0 on $\mathbb{P}^1(\mathbb{Q})$ endowed with the action of $\mathrm{GL}_2(\mathbb{Q})$ given by linear fractional transformations.

Given an R–module V with a right action of $S_0(p)$ as in section 1, we consider the set $\mathrm{Hom}(\Delta_0, V)$ of group homomorphism from Δ_0 to V with the right action of $S_0(p)$ defined as follows: for $\varphi \in \mathrm{Hom}(\Delta_0, V)$ and $\gamma \in S_0(p)$, $\varphi|\gamma \in \mathrm{Hom}(\Delta_0, V)$ is given by

$$(\varphi|\gamma)(D) = \varphi(\gamma D)|\gamma \quad \text{for all } D \in \Delta_0.$$

Definition 4. The *space of modular symbols* on Γ_0 with values in V is

$$\mathrm{Symb}_{\Gamma_0}(V) = \mathrm{Hom}_{\Gamma_0}(\Delta_0, V) = \{\, \varphi \in \mathrm{Hom}(\Delta_0, V) : \varphi|\gamma = \varphi \text{ for all } \gamma \in \Gamma_0 \,\}.$$

The action of $S_0(p)$ naturally induces a right action of Hecke operators on

$\mathrm{Symb}_{\Gamma_0}(V)$: for $\varphi \in \mathrm{Symb}_{\Gamma_0}(V)$ and a prime number ℓ,

$$\begin{cases} \varphi | \mathrm{T}_\ell = \varphi | \left(\begin{smallmatrix} \ell & 0 \\ 0 & 1 \end{smallmatrix}\right) + \sum_{b=0}^{\ell-1} \varphi | \left(\begin{smallmatrix} 1 & b \\ 0 & \ell \end{smallmatrix}\right) & \text{if } \ell \nmid Np, \\ \varphi | \mathrm{U}_\ell = \sum_{b=0}^{\ell-1} \varphi | \left(\begin{smallmatrix} 1 & b \\ 0 & \ell \end{smallmatrix}\right) & \text{if } \ell \mid Np. \end{cases}$$

Theorem 5. *If the orders of all torsion elements of Γ_0 are invertible in R (e.g., if R is a field of characteristic 0), then there is a canonical isomorphism of Hecke modules*

$$\mathrm{Symb}_{\Gamma_0}(V) \cong \mathrm{H}^1_c(\Gamma_0, V).$$

Proof. See proposition 4.2 of Ash–Stevens's article [1]. □

There is another important operator on $\mathrm{Symb}_{\Gamma_0}(V)$: for $\varphi \in \mathrm{Symb}_{\Gamma_0}(V)$, we define

$$\varphi | \iota = \varphi | \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right).$$

The operator ι is an involution on $\mathrm{Symb}_{\Gamma_0}(V)$. Thus, if $2 \in \mathrm{R}^\times$, ι is *diagonalizable* and we obtain a decomposition

$$\mathrm{Symb}_{\Gamma_0}(V) = \mathrm{Symb}_{\Gamma_0}(V)^+ \oplus \mathrm{Symb}_{\Gamma_0}(V)^-,$$

where $\mathrm{Symb}_{\Gamma_0}(V)^\pm$ is the ±1–eigenspace for ι. That is, every $\varphi \in \mathrm{Symb}_{\Gamma_0}(V)$ can be written uniquely as $\varphi = \varphi^+ + \varphi^-$ with $\varphi^\pm | \iota = \pm\varphi^\pm$.

In particular, for $V = V_k(\mathbb{C})$, the Eichler–Shimura theorem gives us two modular symbols for each $f \in S_{k+2}(\Gamma_0)$ which can be interpreted in this way. Namely, one can check that $\psi_f \in \mathrm{Hom}(\Delta_0, V_k(\mathbb{C}))$ defined by

$$\psi_f(\{s\} - \{r\}) = 2\pi i \int_r^s f(z)(zX + Y)^k \, dz$$

is a modular symbol and so induces $\psi_f^\pm \in \mathrm{Symb}_{\Gamma_0}(V_k(\mathbb{C}))^\pm$. Moreover, a change of variables in the integral shows that the assignment $f \mapsto \psi_f$ is Hecke-equivariant. The two modular symbols ψ_f^+ and ψ_f^- are given by the images of $f \in S_{k+2}(\Gamma_0)$ and $\overline{f} \in \overline{S}_{k+2}(\Gamma_0)$ under the Eichler–Shimura isomorphism (cf. theorem 3).

3 Classical modular symbols and p–adic L–functions

Since we want to get p–adic L–functions, we work with $V_k = V_k(\mathbb{Q}_p) = \mathbb{Q}_p[X, Y]_k$ and focus on elements of $S_{k+2}(\Gamma)$ whose q–expansions have coefficients in $\mathbb{Q}_p$. Recall from section 1 that we have a right action of $S_0(p)$ (in fact, even of $\mathrm{GL}_2(\mathbb{Q}_p)$) on V_k given by

$$(P|\gamma)(X, Y) = P((X, Y) \cdot \gamma^*).$$

Thus, we can apply the theory of section 2 to V_k.

The following result tells us that, for the class of modular forms we are most interested in, we *essentially* obtain modular symbols in $\mathrm{Symb}_\Gamma(V_k)$.

Theorem 6 (Shimura). *Let $f \in S_{k+2}(\Gamma_0)$ be a normalized eigenform and consider the associated modular symbols $\psi_f^\pm \in \mathrm{Symb}_{\Gamma_0}(V_k(\mathbb{C}))^\pm$. There exist periods $\Omega_f^\pm \in \mathbb{C}^\times$ such that the modular symbols*

$$\varphi_f^\pm = \frac{\psi_f^\pm}{\Omega_f^\pm}$$

are defined over $R = \mathbb{Z}[a_n(f) : n \geq 1]$ (i.e., $\varphi_f^\pm \in \mathrm{Symb}_{\Gamma_0}(V_k(R))^\pm$ via the natural inclusions $\mathrm{Symb}_{\Gamma_0}(V_k(R))^\pm \subset \mathrm{Symb}_{\Gamma_0}(V_k(\overline{\mathbb{Q}}))^\pm \subset \mathrm{Symb}_{\Gamma_0}(V_k(\mathbb{C}))^\pm$). In particular, the Hecke eigenspace corresponding to f in $\mathrm{Symb}_{\Gamma_0}(V_k(\mathrm{Frac}(R)))^\pm$ via the Eichler–Shimura isomorphism is generated by $\varphi_f^\pm$.

Proof. See Shimura's article [8]. □

Now suppose that $f \in S_{k+2}(\Gamma)$ is a p–ordinary normalized eigenform, so that $a_p(f) \in \mathbb{Z}_p^\times$. Consider $f_1, f_2 \in S_{k+2}(\Gamma_0)$ given by $f_1(z) = f(z)$ and $f_2(z) = f(pz)$. By construction, the forms f_1 and f_2 are eigenvectors of the Hecke operators T_ℓ, $\ell \nmid Np$, and U_ℓ, $\ell \mid N$, but not necessarily of U_p. Using the definitions of T_p and U_p, one checks that

$$\begin{pmatrix} f_1 \big| \mathrm{U}_p & f_2 \big| \mathrm{U}_p \end{pmatrix} = \begin{pmatrix} f_1 & f_2 \end{pmatrix} \begin{pmatrix} a_p(f) & 1 \\ -p^{k+1} & 0 \end{pmatrix}.$$

Therefore, the characteristic polynomial of U_p acting on the subspace of $S_{k+2}(\Gamma_0)$ generated by f_1 and f_2 is of the form

$$X^2 - a_p(f)X + p^{k+1} = (X - \alpha)(X - \beta)$$

with $v_p(\alpha) = 0$ and $v_p(\beta) = k + 1$ and the corresponding eigenvectors are

$$f_\alpha = f_1 - \beta f_2 \quad \text{and} \quad f_\beta = f_1 - \alpha f_2.$$

In fact, the p–stabilizations f_α and f_β are eigenforms over Γ_0 (i.e., simultaneous eigenvectors of the Hecke operators T_ℓ for the primes ℓ such that $\ell \nmid Np$ and U_ℓ for the primes ℓ such that $\ell \mid Np$).

Next take the modular symbol $\varphi_{f_\alpha} \in \mathrm{Symb}_{\Gamma_0}(V_k)$ corresponding to f_α according to theorem 6. Since $\varphi_{f_\alpha} \big| \mathrm{U}_p = \alpha \cdot \varphi_{f_\alpha}$, for every $a \in \mathbb{Z}$ such that $p \nmid a$ and every $n \in \mathbb{N}$,

$$\alpha \cdot \varphi_{f_\alpha}\Big(\Big\{\frac{a}{p^n}\Big\} - \{\infty\}\Big) = \sum_{b=0}^{p-1} \varphi_{f_\alpha}\Big(\Big\{\frac{a + bp^n}{p^{n+1}}\Big\} - \{\infty\}\Big).$$

Except for the factor α on the left-hand side, this equation looks like the compatibility condition required to define a distribution on balls. Thus, we obtain a measure on $\mathbb{Z}_p$ defined on (characteristic functions of) balls by

$$\mu_{f_\alpha}(\mathbb{1}_{a+p^n\mathbb{Z}_p}) = \frac{1}{\alpha^n}\varphi_{f_\alpha}\Big(\Big\{\frac{a}{p^n}\Big\} - \{\infty\}\Big)\Big|_{\substack{X=0\\Y=1}}.$$

The measure μ_{f_α} yields the p–adic L–function[3] of f via a p–adic Mellin transform as follows. Recall that there is a canonical decomposition $\mathbb{Z}_p^\times \cong \mathbb{F}_p^\times \times U_1$, where U_1 is the multiplicative subgroup $1 + p\mathbb{Z}_p$. Namely, every $x \in \mathbb{Z}_p^\times$ can be expressed as $x = \omega(x \bmod p) \cdot \langle x \rangle$, where $\omega\colon \mathbb{F}_p^\times \hookrightarrow \mathbb{Z}_p^\times$ is the Teichmüller character and $\langle - \rangle\colon \mathbb{Z}_p^\times \to U_1 \subset \mathbb{Z}_p^\times$. The p–adic L–function of f is

$$L_p(f,s) = \int_{\mathbb{Z}_p^\times} \langle x \rangle^{s-1}\, d\mu_{f_\alpha}(x),$$

where $\langle x \rangle^{s-1} = \exp_p\big((s-1)\log_p\langle x \rangle\big)$.

With this construction one interpolates special values of the classical L–function attached to f. The following result is an example of this kind of p–adic interpolation.

Proposition 7. *For a Dirichlet character χ of conductor p^n and $0 \le j \le k$,*

$$\mu_{f_\alpha}(z^j \cdot \chi) = \frac{1}{\alpha^n} \cdot \frac{(p^n)^{j+1}}{(-2\pi i)^j} \cdot \frac{j!}{\tau(\chi^{-1})} \cdot \frac{L(f,\chi^{-1},j+1)}{\Omega_f^{\mathrm{sgn}(\chi)}},$$

where

$$\tau(\chi^{-1}) = \sum_{a \bmod p^n} \chi(a)^{-1} \cdot e^{2\pi i a/p^n}.$$

Proof. See sections I.7 and I.8 of Mazur–Tate–Teitelbaum's article [4]. □

4 The weight space

In the same way as one can embed spaces of classical modular forms in (much larger) spaces of overconvergent modular forms that can be studied using p–adic analysis, we are going to embed the spaces of classical modular symbols in (much larger) spaces of overconvergent modular symbols. To define the latter, it is convenient to extend the notion of weight.

[3] In the last lecture we saw p–adic L–functions in a more general setting. There, the initial modular form was not necessarily p–ordinary. Instead, one could use a theorem of Vishik and Amice–Velú (cf. section 11 of Mazur–Tate–Teitelbaum's article [4]) to produce a p–adic distribution taking the role of μ_{f_α} so long as $v_p(\alpha) < k-1$.

Continuing with the notation from the previous section, we write U_n for the multiplicative subgroup $1 + p^n\mathbb{Z}_p$ of $\mathbb{Z}_p^\times$. One can check that $U_1 = \varprojlim_{n\geq 1} U_1 / U_n$. Recall also the Teichmüller splitting $\mathbb{Z}_p^\times \cong \mathbb{F}_p^\times \times U_1$.

Define the Iwasawa algebra Λ to be the completed group ring

$$\Lambda = \mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!] = \varprojlim_{n\geq 1} \mathbb{Z}_p\left[(\mathbb{Z} / p^n\mathbb{Z})^\times\right].$$

By the observations in the previous paragraph, we can express

$$\Lambda \cong \mathbb{Z}_p[\mathbb{F}_p^\times][\![U_1]\!] \cong \bigoplus_{\zeta\in\mathbb{F}_p^\times} \mathbb{Z}_p[\![U_1]\!] \cdot [\omega(\zeta)], \quad \text{where } \mathbb{Z}_p[\![U_1]\!] = \varprojlim_{n\geq 1} \mathbb{Z}_p[U_1 / U_n].$$

On the other hand, since $1 + p$ is a topological generator of U_1, there is an isomorphism $\mathbb{Z}_p[\![U_1]\!] \cong \mathbb{Z}_p[\![T]\!]$ given by $[1 + p] \leftrightarrow 1 + T$. We obtain in this way a very concrete description of Λ. As a matter of fact, Λ is also isomorphic to the space $\mathrm{Meas}(\mathbb{Z}_p)$ of measures on $\mathbb{Z}_p$ with values in $\mathbb{Z}_p$ via the Mahler transform

$$\mu \mapsto \int_{\mathbb{Z}_p} (1 + T)^z \, d\mu(z).$$

Definition 8. The *weight space* $\mathcal{W}$ is the rigid analytic space over $\mathbb{Q}_p$ associated with the formal $\mathrm{Spf}(\mathbb{Z}_p)$–scheme $\mathrm{Spf}(\Lambda)$, so that, for every complete p–adic algebra R,

$$\mathcal{W}(R) = \mathrm{Hom}_{\mathbb{Z}_p}(\Lambda, R) = \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, R^\times).$$

An element $\kappa \in \mathcal{W}(\mathbb{C}_p)$ is called a *weight*.

Remark. There is an embedding

$$\mathbb{Z} \hookrightarrow \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times)$$

identifying $k \in \mathbb{Z}$ with the character $\left(a \mapsto a^k\right)$ (i.e., taking k–th powers). Thus, every integer is a weight in the sense of definition 8.

5 Locally analytic distributions

Next, we recall some notions of p–adic analysis that will allow us to construct overconvergent modular symbols.

Definition 9. We say that a function $f\colon \mathbb{Z}_p \to \mathbb{C}_p$ is *locally analytic* if, for each $z_0 \in \mathbb{Z}_p$, there is a power series expansion

$$f(z) = \sum_{n\geq 0} a_n(z_0)(z - z_0)^n \in \mathbb{C}_p[\![z - z_0]\!] \quad \text{for } z \text{ in some ball around } z_0.$$

Let $\mathcal{A}$ denote the *set of locally analytic functions with coefficients in* $\mathbb{Q}_p$ (i.e., the set of functions f as above with the property that $a_n(z_0) \in \mathbb{Q}_p$ for all $z_0 \in \mathbb{Z}_p$ and all $n \geq 0$).

We want to define a topology on $\mathcal{A}$. This topology will appear naturally from the alternative description of $\mathcal{A}$ given in the following paragraphs.

For each $r \in |\mathbb{C}_p^\times|_p$, set

$$\mathrm{B}[\mathbb{Z}_p, r] = \{ z \in \mathbb{C}_p : \text{there exists } a \in \mathbb{Z}_p \text{ such that } |z - a|_p \leq r \}$$

(that is, $\mathrm{B}[\mathbb{Z}_p, r]$ is the union of all closed balls of radius r centred at some point of $\mathbb{Z}_p$). Intuitively, we want to take $r \to 0^+$ to obtain coverings of $\mathbb{Z}_p$ by smaller and smaller balls in $\mathbb{C}_p$.

Fix $r \in |\mathbb{C}_p^\times|_p$. Since $\mathbb{Z}_p$ is compact, we can decompose

$$\mathrm{B}[\mathbb{Z}_p, r] = \bigsqcup_{i=1}^{m} \mathrm{B}[z_i, r]$$

for some $z_1, \ldots, z_m \in \mathbb{Z}_p$, where $\mathrm{B}[z_i, r]$ denotes the closed ball of radius r centred at z_i in $\mathbb{C}_p$. Define $\mathbb{A}[r]$ to be the set of functions $f \colon \mathbb{Z}_p \to \mathbb{C}_p$ with the property that, for every $i \in \{ 1, \ldots, m \}$, there is a power series expansion

$$f(z) = \sum_{n \geq 0} a_n(z_i)(z - z_i)^n \in \mathbb{Q}_p[\![z - z_i]\!] \quad \text{for } z \in \mathrm{B}[z_i, r].$$

On $\mathbb{A}[r]$ there is a norm $\|-\|_r$ given by

$$\|f\|_r = \sup_{z \in \mathrm{B}[\mathbb{Z}_p, r]} |f(z)|_p.$$

To simplify the notation, write simply $\mathbb{A}$ for the set $\mathbb{A}[1]$ of rigid analytic functions on the closed unit ball and with coefficients in $\mathbb{Q}_p$.

There are natural restriction maps

$$\mathbb{A}[r_1] \hookrightarrow \mathbb{A}[r_2] \hookrightarrow \mathcal{A} \quad \text{whenever } r_1 > r_2$$

and one can easily see that

$$\varinjlim_{r \in |\mathbb{C}_p^\times|_p} \mathbb{A}[r] \cong \mathcal{A}$$

(this simply says that locally analytic functions are analytic on balls of a small enough radius). Therefore, we can endow $\mathcal{A}$ with the limit topology induced by the norms $\|-\|_r$ for $r \in |\mathbb{C}_p^\times|_p$.

Definition 10. The *space of locally analytic distributions* on $\mathbb{Z}_p$ is

$$\mathcal{D} = \mathrm{Hom}_{\mathrm{cont}}(\mathcal{A}, \mathbb{Q}_p).$$

The *space of rigid analytic distributions* on $\mathbb{Z}_p$ is

$$\mathbb{D} = \mathrm{Hom}_{\mathrm{cont}}(\mathbb{A}, \mathbb{Q}_p).$$

Since polynomials are dense in the spaces of (locally) analytic functions, such distributions can be described quite simply:

Proposition 11. *A distribution $\mu \in \mathbb{D}$ (resp. $\mu \in \mathcal{D}$) is uniquely determined by its moments $\{ \mu(z^j) \}_{j\geq 0}$.*

Remark. The natural inclusion $\mathbb{A} \hookrightarrow \mathcal{A}$ induces by duality an inclusion $\mathcal{D} \hookrightarrow \mathbb{D}$.

6 Overconvergent modular symbols

Consider a fixed weight $\kappa\colon \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times$. We define a left action of $S_0(p)$ on $\mathbb{A}$ (resp. $\mathcal{A}$) by

$$(\gamma \cdot_\kappa f)(z) = \kappa(a+cz)\cdot f\Big(\frac{b+dz}{a+cz}\Big) \qquad \text{if } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and write $\mathbb{A}_\kappa$ (resp. $\mathcal{A}_\kappa$) instead of just $\mathbb{A}$ (resp. $\mathcal{A}$) to make the action of weight κ explicit. By duality, we obtain a right action on $\mathbb{D}$ (resp. $\mathcal{D}$):

$$(\mu|_\kappa \gamma)(f) = \int_{\mathbb{Z}_p} (\gamma \cdot_\kappa f)\, d\mu.$$

Similarly, we write $\mathbb{D}_\kappa$ (resp. $\mathcal{D}_\kappa$) instead of just $\mathbb{D}$ (resp. $\mathcal{D}$) to make the action of weight κ explicit.

Definition 12. The *space of overconvergent modular symbols of weight κ* is $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_\kappa)$. The *space of rigid modular symbols of weight κ* is $\mathrm{Symb}_{\Gamma_0}(\mathbb{D}_\kappa)$.

7 The control theorem

Now take the classical weight $\kappa = k \in \mathbb{Z}$. We would like to find some relation between classical and overconvergent modular symbols of weight k. We use rigid modular symbols as an intermediate step.

There is a map $\rho_k\colon \mathbb{D}_k \to V_k$ sending $\mu \in \mathbb{D}_k$ to the polynomial

$$\int_{\mathbb{Z}_p} (Y - zX)^k \, d\mu(z) = \sum_{j=0}^{k} \binom{k}{j} (-1)^j \mu(z^j) X^j Y^{k-j} \in V_k.$$

One can check that ρ_k is $S_0(p)$–invariant (i.e., compatible with the actions of weight k).

Definition 13. The *specialization map of weight k* is the map

$$\rho_k^*\colon \operatorname{Symb}_{\Gamma_0}(\mathbb{D}_k) \to \operatorname{Symb}_{\Gamma_0}(V_k)$$

induced by duality by $\rho_k\colon \mathbb{D}_k \to V_k$.

Ideally, we would like to say that the specialization map identifies rigid modular symbols with classical modular symbols, but there seems to be little hope that this could be true: $\mathbb{D}_k$ is a much larger space than V_k. However, we can obtain such a result so long as we restrict to *classical non-critical slopes*. A precise statement is given by the control theorem at the end of this section.

Definition 14. Let M be a $\mathbb{Z}_p$–module with an action of U_p. The *slope* of a (generalized) U_p–eigenvector $m \in M$ with eigenvalue λ is the valuation $v_p(\lambda)$. For every $h \in \mathbb{R}$, let $M^{(<h)}$ denote the subspace of M where U_p acts with slope $< h$.

Proposition 15. *The slope of an eigenform $g \in S_{k+2}(\Gamma_0)$ is $\leq k+1$.*

Remark. This is a classical result that follows from a computation with characteristic polynomials of U_p as in section 3. A theorem of Coleman shows that every overconvergent eigenform of weight $k+2$ and slope $< k+1$ is in fact classical, whence $k+1$ is called the *critical slope*.

Theorem 16 (Stevens). *The specialization map ρ_k^* induces an isomorphism*

$$\operatorname{Symb}_{\Gamma_0}(\mathbb{D}_k)^{(<k+1)} \cong \operatorname{Symb}_{\Gamma_0}(V_k)^{(<k+1)}.$$

In addition, for every $h \in \mathbb{R}$, the natural inclusion $\mathcal{D}_k \hookrightarrow \mathbb{D}_k$ induces an isomorphism

$$\operatorname{Symb}_{\Gamma_0}(\mathcal{D}_k)^{(<h)} \cong \operatorname{Symb}_{\Gamma_0}(\mathbb{D}_k)^{(<h)}.$$

Proof. See theorem 5.12 of Pollack–Stevens's article [6]. □

8 p–adic L–functions revisited

Overconvergent modular symbols yield an alternative way to construct p–adic L–functions associated with eigenforms.

Theorem 17. *Let $g \in S_{k+2}(\Gamma_0)$ be an eigenform of slope $< k+1$. Let $\varphi_g \in \operatorname{Symb}_{\Gamma_0}(V_k)$ be the corresponding modular symbol (cf. theorem 6) and let $\Phi_g \in \operatorname{Symb}_{\Gamma_0}(\mathcal{D}_k)$ be its unique overconvergent lift (cf. theorem 16). Then, $\Phi_g(\{0\} - \{\infty\})|_{\mathbb{Z}_p^\times}$ is the measure μ_g giving rise to the p–adic L–function of g (cf. section 3).*

Proof. See proposition 6.3 of Pollack–Stevens's article [6]. □

Let us see the idea behind the last result in a particularly simple case: suppose that $k = 0$ and g is the p–stabilization f_α of some p–ordinary normalized eigenform $f \in S_{k+2}(\Gamma)$ corresponding to the eigenvalue α with $v_p(\alpha) = 0$, as in section 3. In this situation, we have to check that the measures $\Phi_{f_\alpha}(\{0\} - \{\infty\})$ and μ_{f_α} agree on (characteristic functions of) balls of the form $a + p^n\mathbb{Z}_p$, where $a \in \mathbb{Z}$ with $p \nmid a$ and $n \in \mathbb{N}$. Indeed,

$$\begin{aligned}(\Phi_{f_\alpha}(\{0\} - \{\infty\}))(\mathbb{1}_{a+p^n\mathbb{Z}_p}) &= \frac{1}{\alpha^n}((\Phi_{f_\alpha}|\,U_p^n)(\{0\} - \{\infty\}))(\mathbb{1}_{a+p^n\mathbb{Z}_p}) \\ &= \frac{1}{\alpha^n}\sum_{b=0}^{p^n-1}\left(\Phi_{f_\alpha}\left(\left\{\frac{b}{p^n}\right\} - \{\infty\}\right)\right)\left(\mathbb{1}_{a+p^n\mathbb{Z}_p}\Big|_0\left(\begin{smallmatrix}1 & b\\ 0 & p^n\end{smallmatrix}\right)\right)\end{aligned}$$

and by definition of the action (of weight 0)

$$\left(\mathbb{1}_{a+p^n\mathbb{Z}_p}\Big|_0\left(\begin{smallmatrix}1 & b\\ 0 & p^n\end{smallmatrix}\right)\right)(z) = \mathbb{1}_{a+p^n\mathbb{Z}_p}(b + p^n z) = \begin{cases}0 & \text{if } a \not\equiv b \mod p^n, \\ 1 & \text{if } a \equiv b \mod p^n.\end{cases}$$

Therefore,

$$\begin{aligned}(\Phi_{f_\alpha}(\{0\} - \{\infty\}))(\mathbb{1}_{a+p^n\mathbb{Z}_p}) &= \frac{1}{\alpha^n}\left(\Phi_{f_\alpha}\left(\left\{\frac{a}{p^n}\right\} - \{\infty\}\right)\right)(\mathbb{1}_{\mathbb{Z}_p}) \\ &= \frac{1}{\alpha^n}\rho_0^*\Phi_{f_\alpha}\left(\left\{\frac{a}{p^n}\right\} - \{\infty\}\right) \\ &= \frac{1}{\alpha^n}\varphi_{f_\alpha}\left(\left\{\frac{a}{p^n}\right\} - \{\infty\}\right) = \mu_{f_\alpha}(\mathbb{1}_{a+p^n\mathbb{Z}_p})\end{aligned}$$

as desired.

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