Localizing classes

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Abstract

These are the notes for a talk given in the students seminar[1] on derived categories. I present the basic properties of localizing classes and the description of the localized category using roofs. I follow the explanation of Miličić's notes [2] almost verbatim, simply rearranging the diagrams and omitting details. No originality is claimed.

Localization is too general to have interesting properties, so we restrict to certain classes of morphisms.

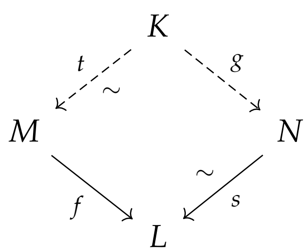
Fix a category $\mathcal{A}$.

Definition 1. A class of morphisms S in $\mathcal{A}$ is a *localizing class* if it satisfies the following properties:

(LC1) For every $M \in \mathrm{Ob}(\mathcal{A})$, $\mathrm{id}_M \in S$.

(LC2) If $(s\colon M \to N), (t\colon N \to P) \in S$, then $t \circ s \in S$.

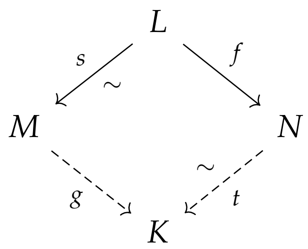
(LC3a) For every $(f\colon M \to L) \in \mathrm{Mor}(\mathcal{A})$ and every $(s\colon N \to L) \in S$, there exist $(g\colon K \to N) \in \mathrm{Mor}(\mathcal{A})$ and $(t\colon K \to M) \in S$ making the diagram



commutative.

[1] I thank Adrian Iovita for organizing the seminar.

(LC3b) For every $(f\colon L \to N) \in \mathrm{Mor}(\mathcal{A})$ and every $(s\colon L \to M) \in S$, there exist $(g\colon M \to K) \in \mathrm{Mor}(\mathcal{A})$ and $(t\colon N \to K) \in S$ making the diagram



commutative.

(LC4) Let $f, g \in \mathrm{Mor}(\mathcal{A})$. The existence of $s \in S$ such that $s \circ f = s \circ g$ is equivalent to the existence of $t \in S$ such that $f \circ t = g \circ t$.

Remark. As in the first talk, an arrow with a tilde $\sim$ represents a morphism in S.

From now on, let S be a fixed localizing class for $\mathcal{A}$. In the last talk we saw a description of $\mathcal{A}[S^{-1}]$ in which morphisms are paths formed by arrows of $\mathrm{Mor}(\mathcal{A})$ (the arrows which go to the left are inverses of morphisms in S). Composition and (LC2) allow us to combine several arrows going in the same direction, whereas (LC3) allows us to *switch* directions of adjacent arrows. Thus, all paths can be reduced by a finite number of such operations to paths of the form

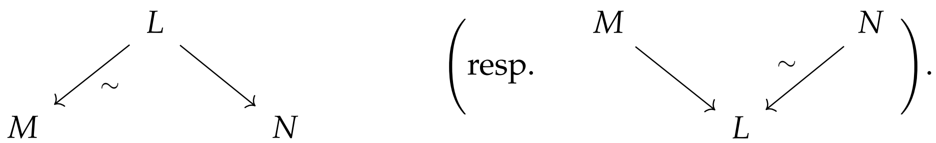
$$M \xleftarrow[\sim]{s} L \xrightarrow{f} N$$

or, alternatively, to paths of the form

$$M \xleftarrow[f]{} L \xrightarrow[\sim]{s} N.$$

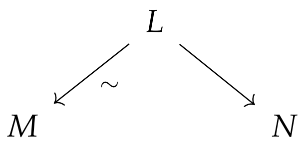
This motivates the following definition.

Definition 2. Let $M, N \in \mathrm{Ob}(\mathcal{A})$. A *left roof* (resp. *right roof*) from M to N is a diagram

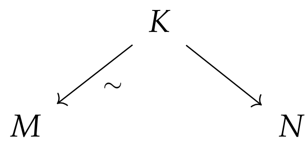


Remark. Replacing $\mathcal{A}$ with $\mathcal{A}^{\mathrm{op}}$ switches left and right roofs, so we can simply focus on left roofs.

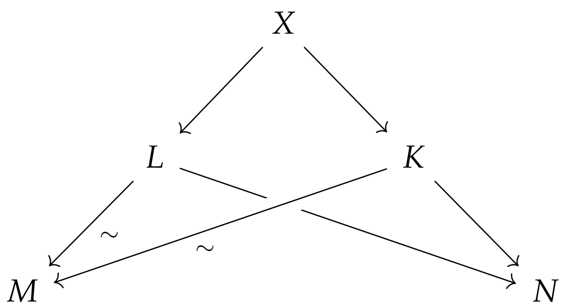
Definition 3. Let $M, N \in \mathrm{Ob}(\mathcal{A})$. Two left roofs



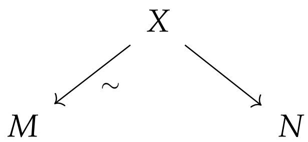
and



are *equivalent* if there is a commutative diagram



such that the induced diagram

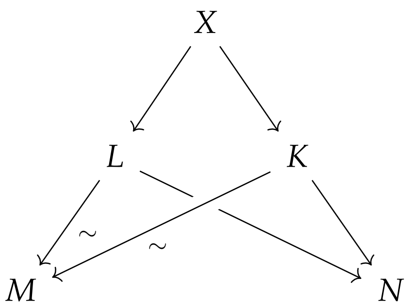


is a left roof. (The analogous notion of equivalence of right roofs is defined by reversing all arrows.)

As we will see later, equivalent roofs represent the same morphism in $\mathcal{A}[S^{-1}]$.

Lemma 4. *Equivalence of left roofs is an equivalence relation.*

Proof (transitivity). Consider two equivalences



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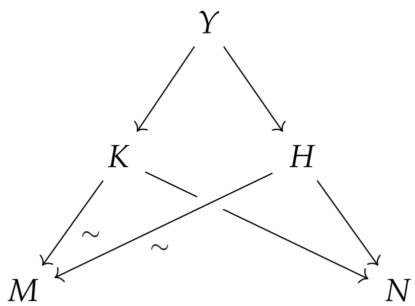

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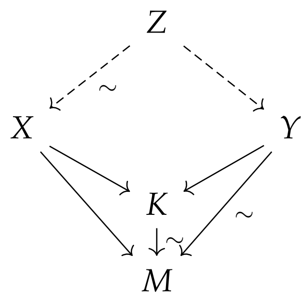
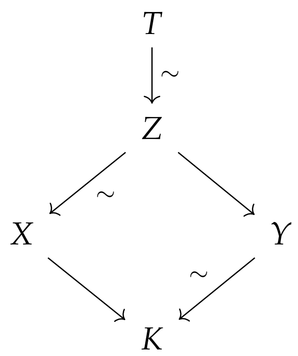
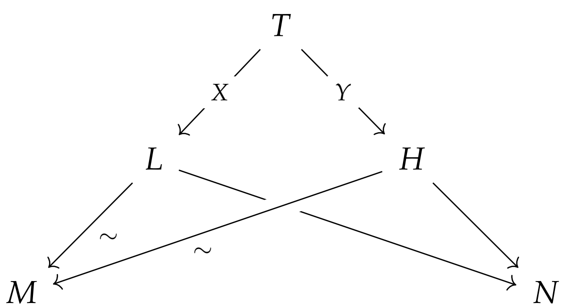
By (LC3a), we obtain a diagram



where the two paths from Z to M are the same. But now by (LC4) there exists a morphism $(T \underset{\sim}{\to} Z) \in S$ such that the two paths from T to K in



are the same. All in all, the induced diagram

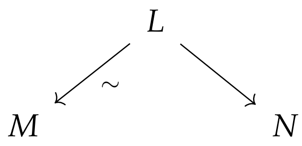


is an equivalence. (The morphism $T \to M$ is in S because it is the composition $T \to Z \to X \to L \to M$ of morphisms in S.) □

Proposition 5. *Let $M, N \in \mathrm{Ob}(\mathcal{A})$. Property (LC3) induces a bijection between the equivalence classes of left and right roofs between M and N.*

Proof. Omitted (similar to the previous proof). □

Definition 6. Let $M, N, P \in \mathrm{Ob}(\mathcal{A})$. The *composition* of two equivalence classes of left roofs



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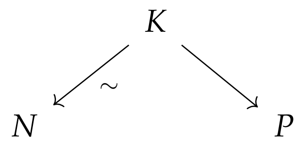
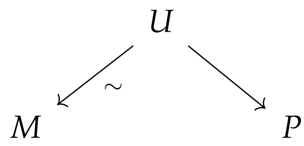
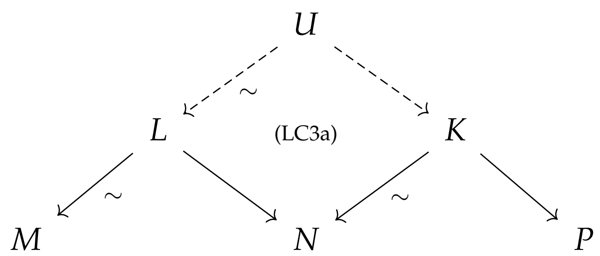


is the equivalence class of a left roof



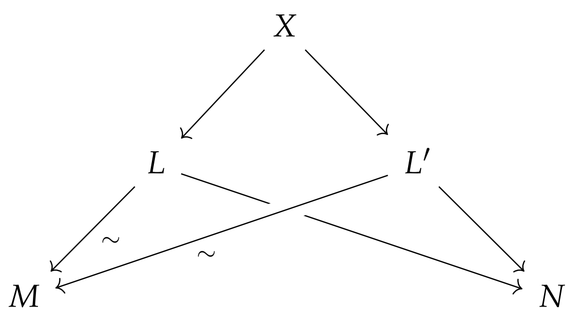
obtained from (LC3a) as follows:



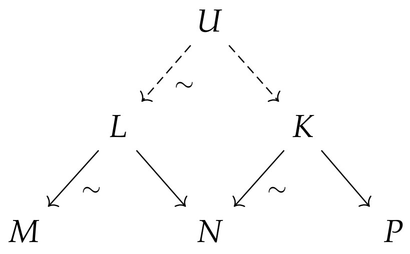
(The analogous notion of composition of equivalence classes of right roofs is defined by reversing all arrows and using (LC3b) instead of (LC3a).)

Remark. Composition is well-defined.

- If



is an equivalence and we consider two possible compositions



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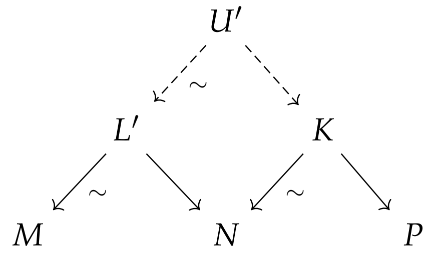

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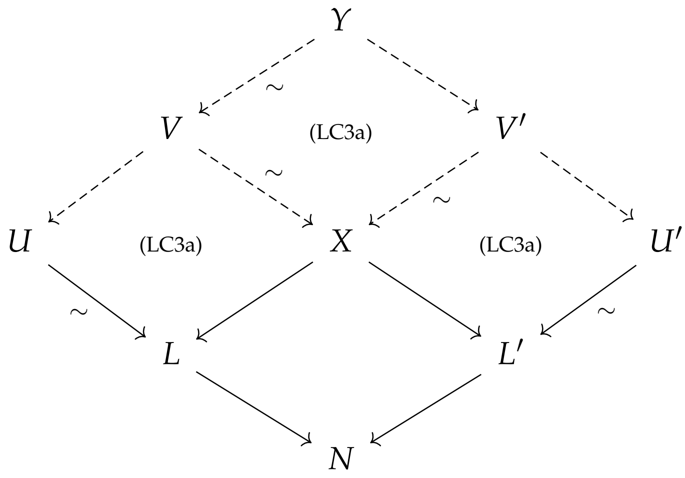
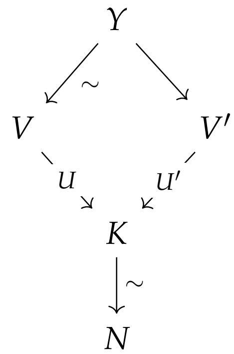
then we can use (LC3a) three times to obtain a commutative diagram as follows:

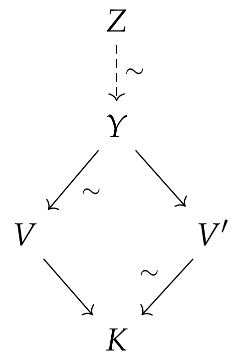


Thus, in

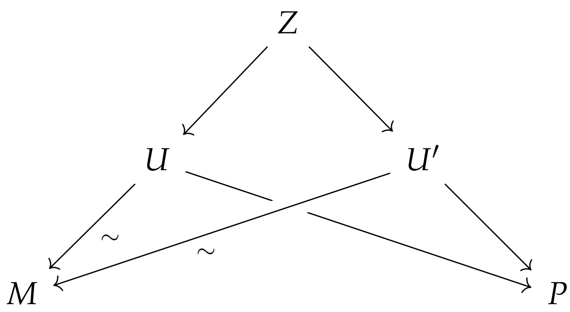


the two paths from R to N are the same and so, by (LC4), there exists

$(Z \underset{\sim}{\to} Y) \in S$ such that the two paths from Z to K in

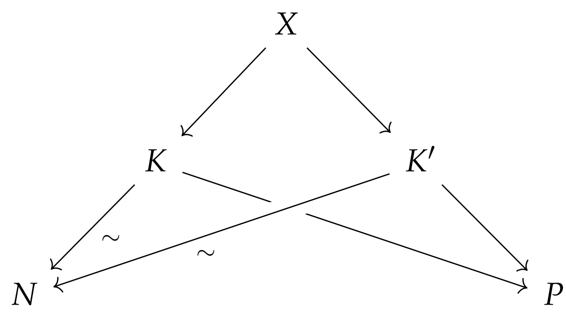


are the same. All in all,

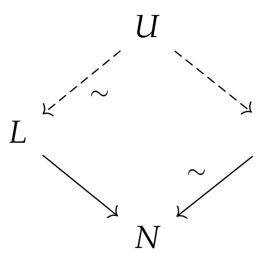


is an equivalence.

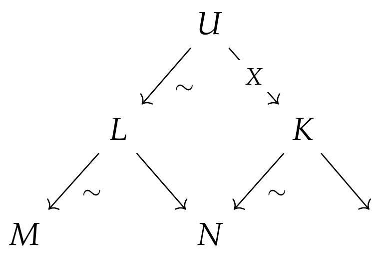
- If



is an equivalence, we can use (LC3a) to obtain a commutative diagram



and then both



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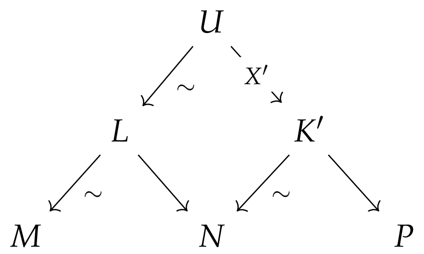


are possible compositions (in fact, the same one).

Proposition 7. *The bijection between equivalence classes of left and right roofs given by (LC3) (cf. proposition 5) is compatible with composition.*

Proof. Omitted. □

Lemma 8. *The equivalence classes of left roofs*

$$\left[\, M \xleftarrow[\sim]{\mathrm{id}_M} M \xrightarrow{\mathrm{id}_M} M \,\right]$$

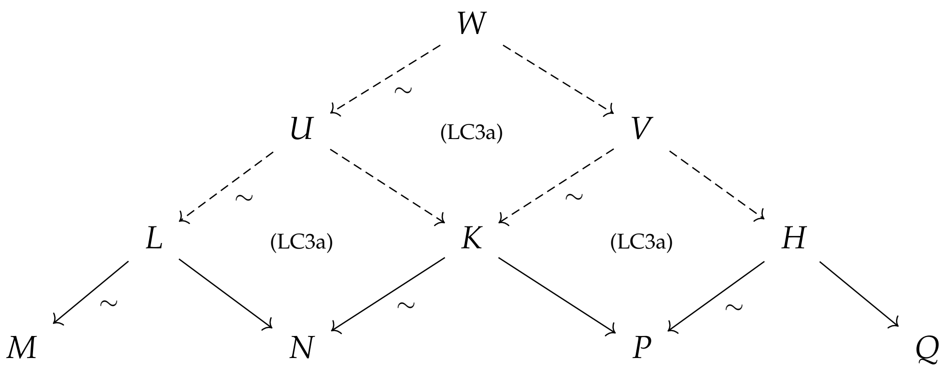
for $M \in \mathrm{Ob}(\mathcal{A})$ *are identity elements for composition.*

Proposition 9. *Composition of equivalence classes of (left) roofs is associative.*

Proof. Consider three left roofs

$$M \xleftarrow{\sim} L \to N, \qquad N \xleftarrow{\sim} K \to P \quad \text{and} \quad P \xleftarrow{\sim} H \to Q .$$

Applying (LC3a) three times, we obtain the following commutative diagram:



The roof

$$\begin{array}{ccc} & X & \\ \swarrow^{\sim} & & \searrow \\ M & & Q \end{array}$$

can be interpreted as both the compositions

$$\begin{array}{ccccc} & & W & & \\ & \swarrow^{\sim} & & \searrow & \\ & U & & H & \\ \swarrow^{\sim} & & \searrow \swarrow^{\sim} & & \searrow \\ M & & P & & Q \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} & & W & & \\ & \swarrow^{\sim} & & \searrow & \\ & L & & V & \\ \swarrow^{\sim} & & \searrow \swarrow^{\sim} & & \searrow \\ M & & N & & Q \end{array} .$$

□

Definition 10. We define a category $\mathcal{A}_S^{\mathrm{l}}$ (resp. $\mathcal{A}_S^{\mathrm{r}}$)

- whose objects are $\mathrm{Ob}(\mathcal{A})$ and
- whose morphisms are equivalence classes of left roofs (resp. right roofs).

Corollary 11. *Property (LC3) induces an equivalence of categories between $\mathcal{A}_S^l$ and $\mathcal{A}_S^r$ (cf. propositions 5 and 7).*

Remark. From now on, we simply write $\mathcal{A}_S$ (and mostly work with left roofs, taking into account that all results are valid for right roofs too).

Remember that the notion of roofs appeared naturally when trying to understand the possible morphisms in $\mathcal{A}[S^{-1}]$ (provided that S is a localizing class). Our next goal is to prove that in fact $\mathcal{A}_S$ provides an alternative (more manageable) description of the localization of $\mathcal{A}$ by S.

First, we have to construct the structure functor $Q\colon \mathcal{A} \to \mathcal{A}_S$ of localization:

- For every $M \in \mathrm{Ob}(\mathcal{A})$, we set $Q(M) = M \in \mathrm{Ob}(\mathcal{A}_S)$.
- For every $(f\colon M \to N) \in \mathrm{Mor}(\mathcal{A})$, we set

$$Q(f) = \left[\begin{array}{ccc} & M & \\ {}^{\mathrm{id}_M}\swarrow^{\sim} & & \searrow^{f} \\ M & & N \end{array} \right].$$

This is indeed a functor: given $(f\colon M\to N),(g\colon N\to P)\in\mathrm{Mor}(\mathcal{A})$,

$$Q(g)\circ Q(f)=\left[\begin{array}{ccccccc} & & & M & & & \\ & & \overset{\mathrm{id}_M}{\underset{\sim}{\swarrow}} & & \overset{f}{\searrow} & & \\ & M & & & & N & \\ \overset{\mathrm{id}_M}{\underset{\sim}{\swarrow}} & & \overset{f}{\searrow} & & \overset{\mathrm{id}_N}{\underset{\sim}{\swarrow}} & & \overset{g}{\searrow} \\ M & & & N & & & P \end{array}\right]=Q(g\circ f).$$

Theorem 12. *Suppose (as above) that S is a localizing class of morphisms of $\mathcal{A}$. The pair $(\mathcal{A}_S, Q\colon\mathcal{A}\to\mathcal{A}_S)$ is a localization of $\mathcal{A}$ by S.*

Proof. We have to check the two characterizing properties of localization.

(1) For every $(s\colon M\to N)\in S$, $Q(s)$ is an isomorphism with inverse

$$Q(s)^{-1}=\left[\begin{array}{ccc} & M & \\ \overset{s}{\underset{\sim}{\swarrow}} & & \overset{\mathrm{id}_M}{\searrow} \\ N & & M \end{array}\right].$$

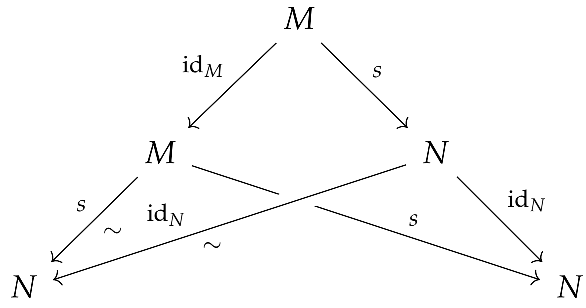
Indeed,

$$Q(s)^{-1}\circ Q(s)=\left[\begin{array}{ccccccc} & & & M & & & \\ & & \overset{\mathrm{id}_M}{\underset{\sim}{\swarrow}} & & \overset{\mathrm{id}_M}{\searrow} & & \\ & M & & & & M & \\ \overset{\mathrm{id}_M}{\underset{\sim}{\swarrow}} & & \overset{s}{\searrow} & & \overset{s}{\underset{\sim}{\swarrow}} & & \overset{\mathrm{id}_M}{\searrow} \\ M & & & N & & & M \end{array}\right]$$

and

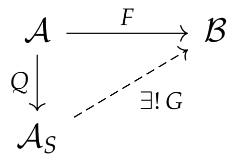
$$Q(s)\circ Q(s)^{-1}=\left[\begin{array}{ccccccc} & & & M & & & \\ & & \overset{\mathrm{id}_M}{\underset{\sim}{\swarrow}} & & \overset{\mathrm{id}_M}{\searrow} & & \\ & M & & & & M & \\ \overset{s}{\underset{\sim}{\swarrow}} & & \overset{\mathrm{id}_M}{\searrow} & & \overset{\mathrm{id}_M}{\underset{\sim}{\swarrow}} & & \overset{s}{\searrow} \\ N & & & M & & & N \end{array}\right]=\left[\begin{array}{ccc} & N & \\ \overset{\mathrm{id}_N}{\underset{\sim}{\swarrow}} & & \overset{\mathrm{id}_N}{\searrow} \\ N & & N \end{array}\right]$$

because



is an equivalence.

(2) The pair $(\mathcal{A}_S, Q\colon \mathcal{A} \to \mathcal{A}_S)$ is universal with this property. Let $F\colon \mathcal{A} \to \mathcal{B}$ be another functor such that $F(s)$ is an isomorphism for all $s \in S$. We have to prove that there exists a unique $G\colon \mathcal{A}_S \to \mathcal{B}$ making the diagram



commutative. The only possibility is to define

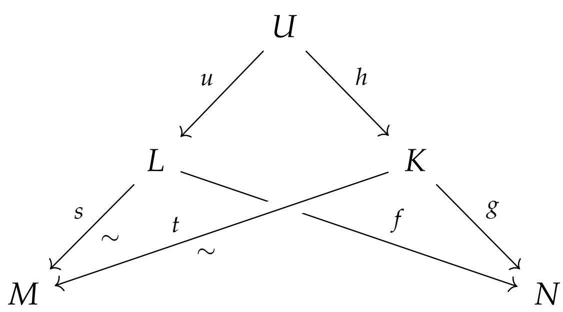
- $G(M) = F(M)$ for every $M \in \mathrm{Ob}(\mathcal{A}_S) = \mathrm{Ob}(\mathcal{A})$ and,
- for every

$$\varphi = \left[M \xleftarrow[\sim]{s} L \xrightarrow{f} N \right] \in \mathrm{Mor}(\mathcal{A}_S),$$

$$G(\varphi) = G(Q(f) \circ Q(s)^{-1}) = G(Q(f)) \circ G(Q(s))^{-1} = F(f) \circ F(s)^{-1}.$$

We obtain a well-defined functor:

- Given an equivalence



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we have

$$\begin{aligned} F(f) \circ F(s)^{-1} &= F(f) \circ F(u) \circ F(u)^{-1} \circ F(s)^{-1} = \\ &= F(f \circ u) \circ F(s \circ u)^{-1} = F(g \circ h) \circ F(t \circ h)^{-1} = \\ &= F(g) \circ F(h) \circ F(h)^{-1} \circ F(t)^{-1} = F(g) \circ F(t)^{-1}. \end{aligned}$$

Here, we used that $F(u)$ (resp. $F(h)$) is an isomorphism because both $F(s)$ and $F(s \circ u)$ (resp. $F(t)$ and $F(t \circ h)$) are.

- Given a composition

$$\psi \circ \varphi = \left[\begin{array}{ccccccccc} & & & & U & & & & \\ & & & \overset{u}{\underset{\sim}{\swarrow}} & & \overset{h}{\searrow} & & & \\ & & L & & & & K & & \\ & \overset{s}{\underset{\sim}{\swarrow}} & & \overset{f}{\searrow} & & \overset{t}{\underset{\sim}{\swarrow}} & & \overset{g}{\searrow} & \\ M & & & & N & & & & P \end{array} \right],$$

we have

$$\begin{aligned} G(\psi \circ \varphi) &= F(g \circ h) \circ F(s \circ u)^{-1} = F(g) \circ F(h) \circ F(u)^{-1} \circ F(s)^{-1} = \\ &= F(g) \circ F(t)^{-1} \circ F(f) \circ F(s)^{-1} = G(\psi) \circ G(\varphi). \end{aligned}$$

Therefore, Q satisfies the desired universal property. □

One of the main results that one can prove using roofs to characterize the localization is the following:

Theorem 13. *Suppose (as above) that S is a localizing class of morphisms of $\mathcal{A}$.*

(1) *If $\mathcal{A}$ is additive, then $\mathcal{A}_S$ is additive and the structure functor $Q\colon \mathcal{A} \to \mathcal{A}_S$ is additive.*

(2) *If $\mathcal{A}$ is abelian, then $\mathcal{A}_S$ is abelian and the structure functor $Q\colon \mathcal{A} \to \mathcal{A}_S$ is exact.*

The proof of this theorem consists of a number of tedious but straight-forward computations and diagram chases similar to what we have done. We omit all the details and just state how sums, kernels and cokernels are built. To do so, we need an intermediate result.

Lemma 14 ("common denominator"). *Let $M, N \in \mathrm{Ob}(\mathcal{A}_S)$ and consider morph-*

isms

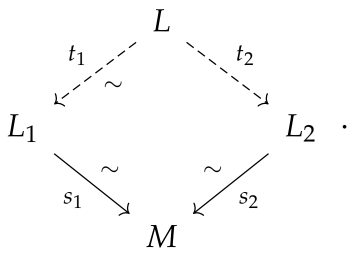
$$\varphi_i = \left[M \xleftarrow[\sim]{s_i} L_i \xrightarrow{f_i} N \right] \in \operatorname{Hom}_{\mathcal{A}_S}(M, N)$$

for $1 \le i \le n$. *There exist* $(s\colon L \to M) \in S$ *and* $(g_i\colon L \to N) \in \operatorname{Hom}_{\mathcal{A}}(L, N)$ *for* $1 \le i \le n$ *such that*

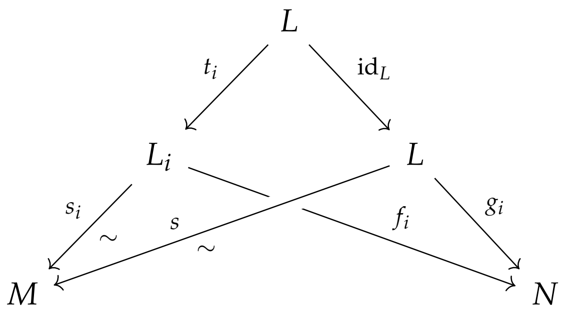
$$\varphi_i = \left[M \xleftarrow[\sim]{s} L \xrightarrow{g_i} N \right].$$

for $1 \le i \le n$.

Proof. Arguing by induction, it suffices to treat the case $n = 2$. By (LC3a), we get a commutative diagram



We can take $s = s_i \circ t_i$ and $g_i = f_i \circ t_i$, $1 \le i \le 2$. By construction, the commutative diagrams



are equivalences of roofs. □

Given two morphisms $\varphi, \psi \in \operatorname{Hom}_{\mathcal{A}_S}(M, N)$, we may choose representative left roofs

$$\varphi = \left[M \xleftarrow[\sim]{s} L \xrightarrow{f} N \right] \quad \text{and} \quad \psi = \left[M \xleftarrow[\sim]{s} L \xrightarrow{g} N \right]$$

by lemma 14. Then,

$$\varphi + \psi = \left[\begin{array}{ccccc} & & L & & \\ & \underset{\sim}{\overset{s}{\swarrow}} & & \overset{f+g}{\searrow} & \\ M & & & & N \end{array} \right].$$

Given $\varphi \in \mathrm{Hom}_{\mathcal{A}_S}(M, N)$, we may choose representative left and right roofs

$$\varphi = \left[\begin{array}{ccccc} & & L & & \\ & \underset{\sim}{\overset{s}{\swarrow}} & & \overset{f}{\searrow} & \\ M & & & & N \end{array} \right] = \left[\begin{array}{ccccc} M & & & & N \\ & \underset{g}{\searrow} & & \underset{t}{\overset{\sim}{\swarrow}} & \\ & & L & & \end{array} \right].$$

Then we can define the objects $\mathrm{Ker}(\varphi) = \mathrm{Ker}(g)$ and $\mathrm{Coker}(\varphi) = \mathrm{Coker}(f)$ together with the structure morphisms

$$\big(\mathrm{Ker}(\varphi) \longrightarrow M\big) = \left[\begin{array}{ccccc} & & \mathrm{Ker}(g) & & \\ & \underset{\sim}{//} & & \searrow & \\ \mathrm{Ker}(g) & & & & M \end{array} \right]$$

and

$$\big(N \longrightarrow \mathrm{Coker}(\varphi)\big) = \left[\begin{array}{ccccc} & & N & & \\ & \underset{\sim}{//} & & \searrow & \\ N & & & & \mathrm{Coker}(f) \end{array} \right].$$

References

[1] Gelfand, S. I. and Manin, Y. I. *Methods of homological algebra*. 2nd ed. Springer monographs in mathematics. Berlin, Germany: Springer–Verlag, 2003. Chap. III.2, pp. 144–153.

[2] Miličić, D. *Lectures on derived categories*. Lecture notes. 2010. Chap. I, pp. 4–45. URL: https://www.math.utah.edu/~milicic/Eprints/dercat.pdf (visited on 20/09/2019).