Representations of semisimple Lie algebras

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Abstract

These are the informal notes for a two-hour talk given in the CRM seminar[1] on the BGG complex. The objective of the talk is to review the classical theory of representations of (complex) semisimple Lie algebras, with $\mathfrak{sl}_2$ as the main example. The notes follow (parts of) chapters IV, VI and VII of Serre's book [1] and contain no original results.

1 Setting

We continue with the notation introduced in the previous talk by Giovanni. Namely, we consider

- a semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$,
- a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$,
- a root system Δ for $\mathfrak{g}$ (relative to $\mathfrak{h}$) and
- the set Δ^+ of positive roots in Δ.

2 Representations

To begin with, we introduce the notion of representations of $\mathfrak{g}$ and their basic properties.

Definition 1. A *representation of* $\mathfrak{g}$ is a homomorphism of Lie algebras of the form

$$\rho\colon \mathfrak{g} \to \mathrm{End}(V),$$

where V is a $\mathbb{C}$–vector space. Equivalently, we say that V is a $\mathfrak{g}$–*module*.

[1] I am grateful to Adrian Iovita for organizing the seminar for the thematic semester at the CRM.

Remark. For simplicity, we write

$$xv = \rho(x)(v) \quad \text{for all } x \in \mathfrak{g} \text{ and } v \in V.$$

The condition that ρ is a homomorphism means that

$$[x, y]v = xyv - yxv \quad \text{for all } x, y \in \mathfrak{g} \text{ and all } v \in V.$$

Examples.

(1) The *adjoint representation* $\mathrm{ad}\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is defined by

$$\mathrm{ad}(x)(y) = [x, y] \quad \text{for all } x, y \in \mathfrak{g}.$$

(2) The *standard representation* of $\mathfrak{sl}_2$ is $\mathbb{C}^2$ with the action given by

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} az_1 + bz_2 \\ cz_1 - az_2 \end{pmatrix} \quad \text{for all } \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{sl}_2 \text{ and all } \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^2.$$

(3) If $\mathfrak{g}$ arises from a Lie group G, then every representation of G induces a representation of $\mathfrak{g}$ by differentiation.

Definition 2. Let V_1 and V_2 be two $\mathfrak{g}$–modules.

(1) The *direct sum* $V_1 \oplus V_2$ is naturally a $\mathfrak{g}$–module with the action given by

$$x(v_1 + v_2) = xv_1 + xv_2 \quad \text{for all } x \in \mathfrak{g},\ v_1 \in V_1 \text{ and } v_2 \in V_2.$$

(2) The *tensor product* $V_1 \otimes_{\mathbb{C}} V_2$ is naturally a $\mathfrak{g}$–module with the (*diagonal*) action given by

$$x(v_1 \otimes v_2) = (xv_1) \otimes v_2 + v_1 \otimes (xv_2) \quad \text{for all } x \in \mathfrak{g},\ v_1 \in V_1 \text{ and } v_2 \in V_2.$$

(3) The *dual space* $V_1^* = \mathrm{Hom}_{\mathbb{C}}(V_1, \mathbb{C})$ is naturally a $\mathfrak{g}$–module with the action defined by

$$(xf)(v_1) = -f(xv_1) \quad \text{for all } x \in \mathfrak{g},\ v_1 \in V_1 \text{ and } f \in V_1^*.$$

More generally, the space $H = \mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$ is naturally a $\mathfrak{g}$–module with the action defined by

$$(xf)(v_1) = x(f(v_1)) - f(xv_1) \quad \text{for all } x \in \mathfrak{g},\ v_1 \in V_1 \text{ and } f \in H.$$

Definition 3.

(1) A $\mathfrak{g}$–module V is called *irreducible* (or *simple*) if $V \neq 0$ and it has no non-trivial $\mathfrak{g}$–submodules; i.e., the $\mathfrak{g}$–submodules of V are 0 and V.

(2) A $\mathfrak{g}$–module V is called *completely reducible* (or *semisimple*) if V is a direct sum of irreducible $\mathfrak{g}$–modules.

Remark. The name *semisimple* might be ambiguous for $\mathfrak{g}$: a general Lie algebra $\mathfrak{g}$ could be semisimple as a $\mathfrak{g}$–module (meaning that the adjoint representation of $\mathfrak{g}$

is completely reducible) but not as a Lie algebra. In these notes, we always assume that $\mathfrak{g}$ is a semisimple Lie algebra and there will be no possible confusion by the next result.

Theorem 4 (Weyl, complete reducibility). *Under our assumption that the Lie algebra $\mathfrak{g}$ is semisimple, all $\mathfrak{g}$–modules of finite dimension (over $\mathbb{C}$) are completely reducible.*

Proof. See section VI.3 of Serre's book [2]. □

3 Weights and primitive vectors

Recall that a root $\alpha \in \Delta$ encodes the eigenvalues of a *simultaneous eigenvector* for the restriction $\mathrm{ad}|_{\mathfrak{h}} \colon \mathfrak{h} \to \mathrm{End}(\mathfrak{g})$ of the adjoint representation of $\mathfrak{g}$ to $\mathfrak{h}$. Then, we saw that there is an *eigenspace decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \right).$$

Our next goal is to generalize this construction to other representations.

Definition 5. Let V be a $\mathfrak{g}$–module and let $\lambda \in \mathfrak{h}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$. Define

$$V_\lambda = \{\, v \in V : Hv = \lambda(H)v \text{ for all } H \in \mathfrak{h} \,\}.$$

If $V_\lambda \neq 0$, we say that λ is a *weight* of V of *multiplicity* $\dim_{\mathbb{C}}(V_\lambda)$ and we say that the elements of $V_\lambda \setminus \{ 0 \}$ *have weight* λ.

Lemma 6. *Let V be a representation of $\mathfrak{g}$. For every $\lambda \in \mathfrak{h}^*$ and $\alpha \in \Delta$,*

$$\mathfrak{g}_\alpha V_\lambda \subseteq V_{\alpha+\lambda}.$$

Proof. Take $X \in \mathfrak{g}_\alpha$, $H \in \mathfrak{h}$ and $v \in V_\lambda$. Then

$$HXv = [H, X]v + XHv = \alpha(H)Xv + X\lambda(H)v = \big(\alpha(H) + \lambda(H)\big)Xv.$$ □

Proposition 7. *Let V be a $\mathfrak{g}$–module. The sum of $\mathbb{C}$–vector spaces*

$$\sum_{\lambda \in \mathfrak{h}^*} V_\lambda$$

is direct and defines a $\mathfrak{g}$–submodule of V.

Remark. Without additional assumptions, this sum of eigenspaces can be a proper submodule of V.

Proof. The sum of eigenspaces with distinct eigenvalues is clearly direct. The fact that we obtain a $\mathfrak{g}$–submodule follows from lemma 6. □

The decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \right)$$

can be rewritten as $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$, where

$$\mathfrak{n} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{n}^- = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha},$$

and $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ is a Borel subalgebra with $[\mathfrak{b}, \mathfrak{b}] = \mathfrak{n}$. We also want to consider *simultaneous eigenvectors* for $\mathfrak{b}$:

Definition 8. Let V be a $\mathfrak{g}$–module and let $\lambda \in \mathfrak{h}^*$. We say that $v \in V$ is a *primitive vector of weight* λ if

(i) $v \in V_\lambda \setminus \{ 0 \}$ and

(ii) $\mathfrak{n} v = 0$.

Equivalently, as $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$, we can extend $\lambda \colon \mathfrak{h} \to \mathbb{C}$ to $\lambda \colon \mathfrak{b} \to \mathbb{C}$ by setting $\lambda(\mathfrak{n}) = 0$ and then v is a primitive vector of weight λ if $v \neq 0$ and $Bv = \lambda(B)v$ for all $B \in \mathfrak{b}$.

Remark. Since $\mathfrak{b}$ is solvable, every $\mathfrak{g}$–module $V \neq 0$ of finite dimension (over $\mathbb{C}$) contains a primitive vector (Lie's theorem).

4 The basic example: $\mathfrak{sl}_2$

For this section, consider

$$\mathfrak{g} = \mathfrak{sl}_2 = \{ A \in \mathrm{Mat}_2(\mathbb{C}) : \mathrm{tr}(A) = 0 \}.$$

Fix the basis of $\mathfrak{sl}_2$

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y_2 = X_{-2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

One checks easily that

$$[H, X_2] = 2X_2, \quad [H, Y_2] = -2Y_2, \quad [X_2, Y_2] = H.$$

Taking $\mathfrak{h} = \mathbb{C}H$, we get the set of roots $\Delta = \{ \pm 2 \}$ with $\mathfrak{g}_2 = \mathbb{C}X_2$ and $\mathfrak{g}_{-2} = \mathbb{C}Y_2$. (Here, we identify $\mathfrak{h}^*$ with $\mathbb{C}$ by evaluating at H.)

Let V be a representation of $\mathfrak{sl}_2$.

Lemma 9. *Let v_0 be a primitive vector of weight λ in V. Define $v_k = Y_2^k v_0$ for all $k \in \mathbb{Z}_{\geq 1}$ (and $v_{-1} = 0$). Then, for every $k \geq 0$,*

(1) $Hv_k = (\lambda - 2k)v_k$ *and*
(2) $X_2 v_k = k(\lambda - k + 1)v_{k-1}$.

Remark. Condition (1) says that $\lambda - 2k$ is another weight of V if $v_k \neq 0$.

Proof.

(1) This follows from lemma 6 (applied k times).
(2) We argue by induction on k. The base case $k = 0$ follows from the definition of primitive vector. Now, assuming that $k > 0$ and that the identity is true for $k - 1$, we compute

$$\begin{aligned} X_2 v_k &= X_2 Y_2 v_{k-1} = [X_2, Y_2]v_{k-1} + Y_2 X_2 v_{k-1} \\ &= Hv_{k-1} + Y_2(k-1)(\lambda - k + 2)v_{k-2} \\ &= (\lambda - 2k + 2)v_{k-1} + (k-1)(\lambda - k + 2)v_{k-1} = k(\lambda - k + 1)v_{k-1}, \end{aligned}$$

where we used both (1) and (2) for $k - 1$. □

Corollary 10. *In the situation of lemma 9, there are two possibilities:*
either (a) *the vectors $(v_k)_{k \geq 0}$ are linearly independent,*
or (b) *the weight λ is an integer $m \geq 0$, the vectors $v_0, v_1, \dots, v_m$ are linearly independent and $v_k = 0$ for all $k > m$.*
If V is finite-dimensional, only (b) can occur.

Proof. Since eigenvectors with different eigenvalues are linearly independent, we only need to consider whether some v_k, for $k \geq 0$, is 0.

Suppose that not all the vectors v_k, for $k \in \mathbb{Z}_{\geq 0}$, are non-zero (i.e., condition (a) does not hold). There must exist $m \in \mathbb{Z}_{\geq 0}$ such that the vectors $v_0, v_1, \dots, v_m$ are $\neq 0$ but $v_{m+1} = v_{m+2} = \cdots = 0$. By lemma 9,

$$0 = X_2 v_{m+1} = (m+1)(\lambda - m)v_m,$$

which is only possible if $\lambda = m$. Therefore, condition (b) holds. □

In case (b) of corollary 10, the subspace of V generated by $v_0, \dots, v_m$ is a $\mathfrak{g}$–submodule that must be irreducible, by the formulae relating these vectors (see lemma 9). In fact, these are the only such representations of $\mathfrak{sl}_2$:

Theorem 11. *Let $m \in \mathbb{Z}_{\geq 0}$ and let W_m be a $\mathbb{C}$–vector space with basis $w_0, w_1, \dots, w_m$. Define an $\mathfrak{sl}_2$–module structure on W_m by*

(1) $Hw_k = (m - 2k)w_k$,
(2) $Y_2 w_k = w_{k+1}$ *and*

(3) $X_2 w_k = k(m-k+1)w_{k-1}$

for all $k \in \{0,1,\ldots,m\}$ *(with the convention that* $w_{-1} = w_{m+1} = 0$*). If* V *is irreducible (as a* $\mathfrak{g}$*–module) and* $\dim_{\mathbb{C}}(V) = m+1$*, then* $V \cong W_m$*.*

Proof. Since V is finite-dimensional, it contains a primitive vector v_0 and we can apply lemma 9 and corollary 10. But V is irreducible; hence, the $\mathfrak{g}$–submodule generated by v_0 must be the whole V and, comparing dimensions, the weight of v_0 must be m. In this way, we obtain an isomorphism $V \cong W_m$ defined by

$$v_k \mapsto w_k \quad \text{for all } k \in \{0,1,\ldots,m\}. \qquad \square$$

Corollary 12. *If* V *is finite-dimensional, it is a direct sum of* $\mathfrak{sl}_2$*–modules of the form* W_m *for* $m \in \mathbb{Z}_{\geq 0}$*. We also have a decomposition*

$$V = \bigoplus_{n \in \mathbb{Z}} V_n$$

as $\mathbb{C}$*–vector spaces.*

Proof. This result follows combining theorems 4 and 11 and proposition 7. $\square$

It is easy to see from this description that finite-dimensional representations of $\mathfrak{sl}_2$ are classified (up to isomorphism) by the weights of their primitive elements (counted with multiplicities). We can use the following invariant too:

Definition 13. Suppose that V is a finite-dimensional representation of $\mathfrak{sl}_2$. The *formal character* of V is

$$\mathrm{ch}(V) = \sum_{n \in \mathbb{Z}} \dim_{\mathbb{C}}(V_n) t^n \in \mathbb{Z}[t, t^{-1}].$$

Theorem 14. *Two finite-dimensional representations* V_1 *and* V_2 *of* $\mathfrak{sl}_2$ *are isomorphic if and only if* $\mathrm{ch}(V_1) = \mathrm{ch}(V_2)$*.*

Proof. This result can be proved easily by induction on the dimension of V_1 using theorem 11 and corollary 12. $\square$

5 A structure theorem

Let us go back to the study of representations of a general (semisimple) Lie algebra $\mathfrak{g}$. But first we want to reinterpret the decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta^+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \right)$$

to reduce some proofs to the case of $\mathfrak{sl}_2$.

Theorem 15. *Let $\alpha, \beta \in \Delta$.*

1. *The subspace $\mathfrak{g}_\alpha$ has dimension 1 (over $\mathbb{C}$).*
2. *If $\alpha + \beta \neq 0$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.*
3. *Define $\mathfrak{h}_\alpha = [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathfrak{h}$. The subspace $\mathfrak{h}$ is 1–dimensional and there exists a unique $H_\alpha \in \mathfrak{h}_\alpha$ such that $\alpha(H_\alpha) = 2$.*
4. *Fix $X_\alpha \in \mathfrak{g}_\alpha \setminus \{ 0 \}$. There is a unique $Y_\alpha \in \mathfrak{g}_{-\alpha}$ such that $[X_\alpha, Y_\alpha] = H_\alpha$. Moreover, $[H_\alpha, X_\alpha] = 2X_\alpha$ and $[H_\alpha, Y_\alpha] = -2Y_\alpha$. Therefore, the subalgebra $\mathfrak{s}_\alpha = \mathfrak{g}_{-\alpha} \oplus \mathfrak{h}_\alpha \oplus \mathfrak{g}_\alpha$ is isomorphic to $\mathfrak{sl}_2$.*

Proof. See section VI.2 of Serre's book [1]. □

From now on, we use freely the notation introduced in theorem 15 (namely, the elements H_α, X_α and Y_α of $\mathfrak{g}$ for $\alpha \in \Delta$).

6 Classification of (finite) representations

Let V be a representation of $\mathfrak{g}$.

Proposition 16. *Let $v \in V$ be a primitive vector of weight λ and let E be the $\mathfrak{g}$–submodule of V generated by v.*

1. *If $\Delta^+ = \{ \alpha_1, \dots, \alpha_r \}$, then E is spanned (as a $\mathbb{C}$–vector space) by the vectors*
$$Y_{\alpha_1}^{k_1} \cdots Y_{\alpha_r}^{k_r} v \quad \text{for } k_1, \dots, k_r \in \mathbb{Z}_{\geq 0}.$$
2. *The weights of E are of the form*
$$\lambda - \sum_{\alpha \in \Delta^+} k_\alpha \alpha \quad \text{with } k_\alpha \in \mathbb{Z}_{\geq 0} \text{ for all } \alpha \in \Delta^+$$
and have finite multiplicity. In particular, λ has multiplicity 1.

In this situation, we say that λ is the highest weight *of E.*

Remark. This result is analogous to lemma 9. Both here and in loc. cit., the existence of a primitive vector in V is a hypothesis (it is not automatic unless, say, V is finite-dimensional).

Proof. See proposition 2 in section VII.2 of Serre's book [1]. □

Theorem 17. *For every $\lambda \in \mathfrak{h}^*$, there exists a unique (up to isomorphism) irreducible $\mathfrak{g}$–module W_λ with highest weight λ.*

Remark. This result is analogous to theorem 11. We will see these modules W_λ (known as Verma modules) in the next talk, so we omit all details here.

Corollary 18. *If V is finite-dimensional, it is a direct sum of $\mathfrak{g}$–modules of the form W_λ for $\lambda \in \mathfrak{h}^*$. We also have a decomposition*

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$$

as $\mathbb{C}$–vector spaces. Moreover, if λ is a weight of V, then

$$\lambda(H_\alpha) \in \mathbb{Z} \quad \textit{for all } \alpha \in \Delta.$$

Remark. This result is analogous to corollary 12.

Proof. The first part follows from theorems 4 and 17. The second part is now a consequence of proposition 7. For the last claim, regard V as a representation of $\mathfrak{s}_\alpha \cong \mathfrak{sl}_2$ and use the results of section 4. □

Even if we have several weights, we can still classify (isomorphism classes of) finite-dimensional representations of $\mathfrak{g}$ using Laurent polynomials.

Definition 19. Consider

$$\Lambda = \{ \lambda \in \mathfrak{h}^* : \lambda(H_\alpha) \in \mathbb{Z} \text{ for all } \alpha \in \Delta \}$$

(fact: Λ is a free abelian group) and let $\mathbb{Z}[\Lambda]$ be the corresponding group ring, with $\mathbb{Z}$–basis $(e_\lambda)_{\lambda \in \Lambda}$. Suppose that V is a finite-dimensional representation of $\mathfrak{g}$. The *formal character* of V is

$$\operatorname{ch}(V) = \sum_{\lambda \in \Lambda} \dim_{\mathbb{C}}(V_\lambda) e_\lambda \in \mathbb{Z}[\Lambda].$$

Theorem 20. *Two finite-dimensional representations V_1 and V_2 of $\mathfrak{g}$ are isomorphic if and only if* $\operatorname{ch}(V_1) = \operatorname{ch}(V_2)$.

Remark. This result is analogous to theorem 14.

Proof. The theorem follows easily by induction on $\dim_{\mathbb{C}}(V_1)$ using theorem 17 and corollary 18. See proposition 5 in section VII.7 of Serre's book [1] for more details. □

References

[1] Serre, J.-P. *Complex semisimple Lie algebras*. Trans. by Jones, G. A. Springer monographs in mathematics. Berlin, Germany: Springer-Verlag, 2001. 75 pp.

[2] Serre, J.-P. *Lie algebras and Lie groups. 1964 lectures given at Harvard University*. 2nd ed. Lecture notes in mathematics 1500. Corrected 5th printing. Berlin, Germany: Springer-Verlag, 2006. 173 pp.