Étale morphisms

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Abstract

These are the notes for two one-hour talks given in the students seminar[1] on Katz's correspondence. I introduce the basic notions of étale morphisms and a couple of results on the corresponding *Galois theory*. The notes mostly follow parts of Milne's book [1] and contain no original results.

0 Motivation

In topology, there is a kind of *Galois theory* of covering spaces. If X is a path-connected and semilocally simply connected topological space, there is a correspondence between path-connected coverings $Y \to X$ and subgroups of the fundamental group $\pi_1(X)$. Also, such coverings have nice lifting properties and the elements of π_1 can be seen as automorphisms acting on coverings, more or less like in the Galois theory of field extensions.

Our objective is to develop an analogous theory on a certain class of morphisms of rings or, rather, of (affine) schemes.

Let A be a ring. The sets

$$\mathrm{V}(S) = \{\, \mathfrak{p} \in \operatorname{Spec}(A) : S \subseteq \mathfrak{p} \,\} \quad \text{for } S \subseteq A$$

are the closed subsets of a topology on $\operatorname{Spec}(A)$, the Zariski topology. However, this topology is too coarse to give a satisfactory theory of covering spaces, so we need a different construction.

Notation In what follows, we use the symbols

- S, X, Y for schemes and $\mathcal{O}_S$, $\mathcal{O}_X$, $\mathcal{O}_Y$ for their structure sheaves,

[1] I am grateful to Giacomo Graziani for organizing the seminar.

- s, x, y for points of schemes, $\mathcal{O}_{S,s}$, $\mathcal{O}_{X,x}$, $\mathcal{O}_{Y,y}$ for the corresponding local rings, $\mathfrak{m}_s$, $\mathfrak{m}_x$, $\mathfrak{m}_y$ for their maximal ideals and $\kappa(s)$, $\kappa(x)$, $\kappa(y)$ for their residue fields,
- A, B, C for rings,
- $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$ for prime ideals and $\kappa(\mathfrak{p})$, $\kappa(\mathfrak{q})$, $\kappa(\mathfrak{r})$ for the corresponding residue fields and
- $f\colon X \to S$, $g\colon Y \to X$ and $h\colon Y \to S$ for morphisms of schemes.

1 Étale morphisms

Definition 1. A morphism of schemes $f\colon X \to S$ is *unramified* if it is locally of finite type and, for every $x \in X$,

(i) $\mathfrak{m}_{f(x)}\mathcal{O}_{X,x} = \mathfrak{m}_x$ and

(ii) the extension of fields $\kappa(x) \,/\, \kappa(f(x))$ is finite and separable.

Proposition 2. *A morphism $f\colon X \to S$ locally of finite type is unramified if and only if all the fibres $f_s\colon X_s = X \times_S \operatorname{Spec}(\kappa(s)) \to \operatorname{Spec}(\kappa(s))$, $s \in S$, are unramified. (That is, the property of being unramified can be checked fibrewise.)*

Sketch of the proof. Let $x \in X$ and consider $s = f(x)$. There are isomorphisms

$$\mathcal{O}_{X,x} \,/\, \mathfrak{m}_s\mathcal{O}_{X,x} \cong \kappa(s) \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{X,x} \cong \mathcal{O}_{X_s,x}$$

(where, by abuse of notation, we write x for the point of X_s corresponding to $x \in X$). □

Corollary 3. *If $f\colon X \to S$ is unramified, then it is locally quasi-finite.*

Proof. Working locally, we may assume that $S = \operatorname{Spec}(A)$ and $X = \operatorname{Spec}(B)$. Choose $\mathfrak{p} \in \operatorname{Spec}(A)$ and define $B(\mathfrak{p}) = B \otimes_A \kappa(\mathfrak{p})$. Take $\mathfrak{q} \in \operatorname{Spec}(B(\mathfrak{p}))$. We have a tower of extensions

$$\kappa(\mathfrak{p}) \subseteq B(\mathfrak{p}) \,/\, \mathfrak{q} \subseteq \kappa(\mathfrak{q}).$$

But, by hypothesis, $\kappa(\mathfrak{q}) \,/\, \kappa(\mathfrak{p})$ is finite. Therefore, the domain $B(\mathfrak{p}) \,/\, \mathfrak{q}$ must be a field. Since $\mathfrak{q}$ was arbitrary, this shows that $B(\mathfrak{p})$ has Krull dimension 0. All in all, $B(\mathfrak{p})$ is artinian and $\operatorname{Spec}(B(\mathfrak{p}))$ has only finitely many points. □

Proposition 4. *Let $f\colon X \to S$ be a morphism locally of finite type. The following conditions are equivalent:*

(1) f is unramified;

(2) $\Omega_{X/S} = 0$, and

(3) *the diagonal morphism* $\Delta_{X\,/\,S} \colon X \to X \times_S X$ *is an open immersion.*

Remark. The proof of this result is technical. Localization and Nakayama's lemma can be used to reduce it to the case of fields. For the details, see proposition 3.5 in chapter I of Milne's book [1].

Definition 5. A morphism $f \colon X \to S$ is called *étale* if it is flat, locally of finite presentation and unramified.

Example 6. Every open immersion is étale.

We can now state the basic permanence properties of étale morphisms. The proofs of the following results are standard arguments (see the corresponding parts of section I.3 of Milne's book [1] for the details).

Proposition 7. *If* $f \colon X \to S$ *and* $g \colon Y \to X$ *are étale morphisms, then* $g \circ f \colon Y \to S$ *is étale too.*

Proposition 8. *Let* $f \colon X \to S$ *be an étale morphism. For every cartesian square*

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{f_Y} & Y \\ \big\downarrow & \ulcorner & \big\downarrow \\ X & \xrightarrow[\text{étale}]{f} & S \end{array}$$

(i.e., a base change of f*), the morphism* f_Y *is étale too.*

Sketch of the proof. Since the property of being unramified can be checked fibrewise, we may restrict to the case in which S and Y are spectra of fields. But finite separable algebras over a field k are finite products of separable field extensions of k, so the result follows. □

Proposition 9. *Consider a commutative diagram*

$$\begin{array}{ccc} Y & \xrightarrow{g} & X \\ & {\scriptstyle h,\ \text{étale}} \searrow \quad \swarrow {\scriptstyle f,\ \text{unramified}} & \\ & S & \end{array}$$

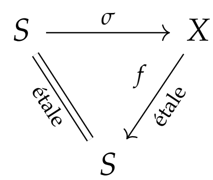
with $f \colon X \to S$ *unramified and* $h \colon Y \to S$ *étale. Then,* $g \colon Y \to X$ *is étale.*

Proposition 10. *Let* $f \colon X \to S$ *be a morphism locally of finite presentation. The set of* $x \in X$ *for which the induced map of local rings* $\mathcal{O}_{S,f(x)} \to \mathcal{O}_{X,x}$ *is flat and* $(\Omega_{X/S})_x = 0$ *is open in* X*. Hence, the locus where* f *is étale is open.*

We are finally in a position to see the first kind of lifting property of étale morphisms.

Theorem 11. *Let S be a connected scheme. If $f\colon X \to S$ is an étale (resp. étale and separated) morphism, then each section $\sigma\colon S \to X$ of f is an open immersion (resp. induces an isomorphism from S to a connected component of X). That is, there is a bijective correspondence between sections of f and open (resp. open and closed) subschemes $Y \hookrightarrow X$ such that f induces an isomorphism $Y \cong S$.*

Sketch of the proof. From the commutative diagram



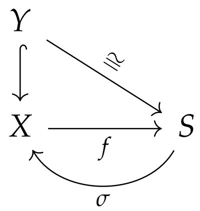
we deduce that σ is étale by proposition 9.

Moreover, if f is separated, then σ is a closed immersion because $f \circ \sigma = \mathrm{id}_S$ is. But σ is also flat and locally of finite presentation. Therefore, σ is an open immersion too.

If f is étale but not separated, we can find a covering of X such that the restrictions of f are separated and use the previous case on every restriction. □

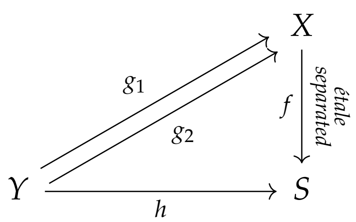
Corollary 12. *If S is connected and $f\colon X \to S$ is étale and separated, then a section $\sigma\colon S \to X$ of f is uniquely determined by the value $\sigma(s)$ at (any) one point $s \in S$.*

Proof. Let $x = \sigma(s)$ and let $Y \hookrightarrow X$ be the connected component of X containing x. The diagram



shows that σ is the composition $S \cong Y \hookrightarrow X$. □

Corollary 13. *Suppose that Y is connected. Consider a commutative diagram*



with $f\colon X \to S$ étale and separated. If there exists $y \in Y$ such that $g_1(y) = g_2(y) = x$ and such that the induced morphisms on residue fields $g_1^\sharp, g_2^\sharp\colon \kappa(x) \to \kappa(y)$ coincide, then $g_1 = g_2$. That is, a lift of h to X is uniquely determined by its value at (any) one base point.

Proof. Consider the graph $\Gamma_i\colon Y \to Y \times_S X$ of g_i, $i = 1$ and 2, and the projections $\pi_1\colon Y \times_S X \to Y$ and $\pi_2\colon Y \times_S X \to X$. Since π_1 is obtained from f by base change, it must be étale and separated. Also, $\pi_1 \circ \Gamma_i = \mathrm{id}_Y$ by definition. Corollary 12 implies that $\Gamma_1 = \Gamma_2$, whence $g_1 = \pi_2 \circ \Gamma_1 = \pi_2 \circ \Gamma_2 = g_2$. □

The previous results are a first indicator that étale morphisms might be the right analogue in algebraic geometry to coverings in topology. Following the analogy, étale morphisms should locally look all the same (as coverings are all local homeomorphisms).

Example 14. Let A be a ring and take a monic polynomial $P \in A[T]$. Define $B = A[T] \,/\, (P)$, so that $B \cong A^{\deg(P)}$ as A–modules. We observe that P is separable if and only if $(P, P') = A[T]$ or, equivalently, $P' \in B^\times$. But this can be checked locally: P' is a unit in $B = A[T] \,/\, (P)$ if and only if, for every $\mathfrak{p} \in \mathrm{Spec}(A)$, the ring $B \otimes_A \kappa(\mathfrak{p}) = \kappa(\mathfrak{p})[T] \,/\, (\overline{P})$ is unramified over $\kappa(\mathfrak{p})$. In this case, B is étale over A. More generally, for $b \in B$, B_b is étale over A if and only if $P' \in B_b^\times$.

Definition 15. A *standard étale morphism* is a morphism of affine schemes of the form $\mathrm{Spec}(B_b) \to \mathrm{Spec}(A)$, where $B = A[T] \,/\, (P)$ and $b \in B$ with $P' \in B_b^\times$ (as in example 14).

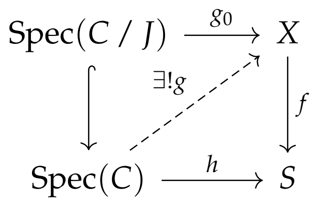
The following result shows that étale morphisms are locally standard étale.

Theorem 16. *Let $f\colon X \to S$ be an étale morphism. For every $x \in X$, there exist open affine neighbourhoods V of x and U of $f(x)$ such that the restriction $f|_V\colon V \to U$ is standard étale.*

Remark. The proof of this result is technical and quite difficult in general. See theorem 3.14 of chapter I of Milne's book [1] for a proof in the noetherian case. For the general case, see proposition 00UE in the Stacks project [3].

This characterization is useful to prove results about local properties. But there is also a *functorial* characterization of étale morphisms which will be useful in the next talk about the étale fundamental group.

Definition 17. Let $X \xrightarrow{f} S$ be an S–scheme and consider the associated functor $X_S = \mathrm{Hom}_S(\,\cdot\,, X)\colon (\mathrm{Sch}\,/\,S)^{\mathrm{op}} \to \mathrm{Set}$. We say that f (or X_S) is *formally étale* if, for every morphism $h\colon \mathrm{Spec}(C) \to S$ and every nilpotent ideal J of C, the natural map $X_S(C) \to X_S(C\,/\,J)$ is bijective. That is, for each $g_0 \in X_S(C\,/\,J)$, there is a unique $g \in X_S(C)$ making the diagram



commutative.

Remark. The schemes $\mathrm{Spec}(C)$ and $\mathrm{Spec}(C\,/\,J)$ have the same underlying topological space. Thus, the functor X_S is formally étale if and only if it only *sees* the underlying reduced subschemes of affine schemes.

Theorem 18. *A morphism $f\colon X \to S$ is étale if and only if it is formally étale and locally of finite presentation.*

Remark. Remark 3.22 of chapter I of Milne's book [1] has a proof of this result in the noetherian case. For the general case, see lemma 02HM in the Stacks project [3].

From the relation between étale and formally étale morphisms and the definition of the latter, we obtain the topological invariance of étale morphisms.

Theorem 19. *Let S_0 be a closed subscheme of S defined by a nilpotent ideal sheaf. The functor* $\mathrm{Sch}\,/\,S \to \mathrm{Sch}\,/\,S_0$ *given by*

$$X \mapsto X_0 = X \times_S S_0$$

induces an equivalence between the categories of étale S–schemes and of étale S_0–schemes.

Remark. See theorem 3.23 of chapter I of Milne's book [1] for some more details.

2 Henselian rings

We begin by recalling the definition of henselian rings, which will appear in several proofs in the following talks.

Theorem 20. *Let $(A, \mathfrak{m}, \kappa)$ be a local ring and consider $S = \mathrm{Spec}(A)$ with its closed point ξ. The following conditions are equivalent:*

(1) *if* $f \in A[T]$ *is monic and its reduction* $\overline{f} \in \kappa[T]$ *factors as* $\overline{f} = g_0 \cdot h_0$ *with* $(g_0, h_0) = \kappa[T]$*, there exist lifts* $g, h \in A[T]$ *such that* $f = g \cdot h$*,* $\overline{g} = g_0$ *and* $\overline{h} = h_0$*;*

(2) *every finite A–algebra B is of the form*

$$B \cong \prod_{\mathfrak{n} \in \mathrm{Max}(B)} B_{\mathfrak{n}},$$

and

(3) *every étale morphism* $f\colon X \to S$ *such that there is a point* $x \in X$ *satisfying that* $f(x) = \xi$ *and* $\kappa(x) = \kappa(\xi) = \kappa$ *admits a section* $\sigma\colon S \to X$*.*

Remark. See theorem 4.2 of chapter I of Milne's book [1] for these and other caracterizations of henselian rings.

Definition 21. A *henselian ring* is a local ring $(A, \mathfrak{m}, \kappa)$ satisfying conditions (1) to (3) of theorem 20.

Corollary 22. *If A is a henselian ring, every finite A–algebra B and every quotient* $A \,/\, I$ *are henselian too.*

Proposition 23. *Every complete local ring is henselian.*

Remark. This result can be proved using condition (3) and Hensel's lemma to lift sections from κ to A. See proposition 4.5 of chapter I of Milne's book [1] for the details.

In later talks, there will be some results whose proofs consist of a sequence of reductions to simpler and simpler cases (e.g., from schemes to rings, then to local noetherian rings, to complete local noetherian rings, and finally to fields). The following results are important tools for this kind of proofs.

Theorem 24. *Let* $(A, \mathfrak{m}, \kappa)$ *be a henselian ring. The functor* Sch / $A \to$ Sch / κ *given by*

$$X \mapsto X \otimes_A \kappa$$

induces an equivalence between the categories of finite étale A–schemes and of finite étale κ*–schemes.*

Remark. See proposition 4.4 of chapter I of Milne's book [1] for a proof.

Corollary 25. *Let* $(A, \mathfrak{m}, \kappa)$ *be a noetherian henselian ring and let* $\widehat{A}$ *be its* $\mathfrak{m}$*–adic completion. The functor* Sch / $A \to$ Sch / $\widehat{A}$ *given by*

$$X \mapsto X \otimes_A \widehat{A}$$

induces an equivalence between the categories of finite étale A–schemes and of finite étale $\widehat{A}$*–schemes.*

3 Galois theory

We can finally introduce a couple of results which justify the existence of a Galois theory of coverings via a certain class of étale morphisms. (This analogy with Galois theory will be even more apparent in the next talk, after the introduction of the étale fundamental group.)

Definition 26. A *finite étale covering* (or just *covering* if there is no ambiguity) is a surjective finite étale morphism of schemes.

Remark. One should think of coverings as the analogue of field extensions in algebraic geometry. Later, we will have to restrict to a narrower class of coverings to get the right *Galois theory* with the lattice of subgroups of automorphisms.

Definition 27. Let $X \xrightarrow{f} S$ be a morphism of schemes. We define the *group of automorphisms of X over S* to be

$$\operatorname{Aut}(X \,/\, S) = \{\, \sigma\colon X \to X \text{ automorphism} : f \circ \sigma = f \,\}.$$

Proposition 28. *Let $f\colon X \to S$ be a covering. If X is connected, the group $\operatorname{Aut}(X \,/\, S)$ acts freely on each geometric fibre (i.e., $\sigma \in \operatorname{Aut}(X \,/\, S) \setminus \{\, \mathrm{id}_X \,\}$ has no fixed points). In particular, since fibres are finite, $\operatorname{Aut}(X \,/\, S)$ is finite.*

Proof. Let $\sigma \in \operatorname{Aut}(X \,/\, S)$. If there exists $x \in X$ such that $\sigma(x) = x$, we see that $\sigma = \mathrm{id}_X$ by corollary 13 (applied to the two lifts σ and id_X of f). □

Proposition 29. *Let $f\colon X \to S$ be a covering with X connected. For every subgroup H of $\operatorname{Aut}(X \,/\, S)$, the quotient map $\pi\colon X \to H \setminus X$ and the induced map $\widetilde{f}\colon H \setminus X \to S$ are coverings.*

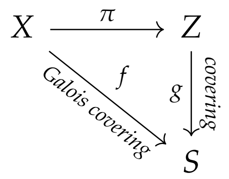
Remark. The proof of this result is based on another caracterization of finite étale coverings analogous to the existence of (local) trivializations for coverings in topology. See propositions 5.2.9 and 5.3.7 of Szamuely's book [2] for the details.

We can finally define the analogue of Galois extensions of fields.

Definition 30. A *Galois (finite étale) covering* is a covering $f\colon X \to S$ with X connected and such that $\operatorname{Aut}(X \,/\, S)$ acts transitively on each geometric fibre.

The next result should be seen as the *main theorem of Galois theory* in the setting of schemes. (In fact, one can recover the usual Galois theory from this theorem by focusing on spectra of fields.)

Theorem 31. *Let $f\colon X \to S$ be a Galois covering and set $G = \mathrm{Aut}(X \,/\, S)$. For every commutative diagram*



with Z connected and $g\colon Z \to S$ a covering, $\pi\colon X \to Z$ is a Galois covering and induces an isomorphism $Z \cong H \setminus X$ for some subgroup H of G. In addition, g is a Galois covering if and only if H is a normal subgroup of G. Thus, there is a bijective correspondence between (normal) subgroups of G and (Galois) intermediate coverings of f.

Sketch of the proof. Since f and g are finite étale, so is π. Also, π is surjective because Z is connected and the image of π must be both open and closed. Therefore, π is a covering.

Now $H = \mathrm{Aut}(X \,/\, Z)$ is a subgroup of G which can be seen to act transitively on each geometric fibre of π. Indeed, given two points x_1 and x_2 of some geometric fibre of π, they also belong to the same geometric fibre of f and so there exists $\sigma \in G$ such that $\sigma(x_1) = x_2$. By construction, $(\pi \circ \sigma)(x_1) = \pi(x_1)$ and from corollary 13 we deduce that $\pi \circ \sigma = \pi$ (i.e., $\sigma \in H$).

All in all, $\pi\colon X \to Z$ is a Galois covering and $H \setminus X \cong Z$. The other claims of the statement can be proved as in the usual Galois theory of field extensions. □

References

[1] Milne, J. S. *Étale cohomology*. Princeton mathematical series 33. Princeton, NJ, USA: Princeton University Press, 1980. Chap. I, pp. 3–45. URL: `http://www.jmilne.org/math/Books/ECpup1.pdf` (visited on 28/05/2019). Revised online version (2014).

[2] Szamuely, T. *Galois groups and fundamental groups*. Cambridge studies in advanced mathematics 117. Cambridge, UK: Cambridge University Press, 2009. Chap. 5, pp. 149–209.

[3] *The Stacks project*. URL: `https://stacks.math.columbia.edu` (visited on 28/05/2019).