The class number formula and Dirichlet's prime number theorem

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This seminar talk has two main goals. On the one hand, we are going to continue to study the Dedekind zeta function and obtain the class number formula. This formula is important because it relates certain algebraic invariants of a number field to the analytic properties of the zeta function. On the other hand, we are going to apply the theory of L–series and zeta functions in order to prove a classical result about the number of primes appearing in an arithmetic progression, namely:

Dirichlet's prime number theorem. *For any two relatively prime positive integers a and m, the arithmetic progression*

$$a,\ a+m,\ a+2m,\ a+3m,\ \ldots$$

contains infinitely many prime numbers. Equivalently, there are infinitely many prime numbers p such that $p \equiv a \pmod{m}$.

Note that the arithmetic progressions excluded from the theorem are precisely those for which all the elements have a common non-trivial divisor. Clearly, one such arithmetic progression contains at most one prime number.

As a matter of fact, we are going to prove a stronger theorem about the distribution of primes: we are going to see that the prime numbers are in some sense *equally distributed* among the congruence classes of $(\mathbb{Z} / m\mathbb{Z})^\times$ (for each $m \in \mathbb{N}$).

The class number formula

Let K be a number field with $[K : \mathbb{Q}] = n$. Let r_1 be the number of real embeddings of K and let r_2 be the number of pairs of complex conjugate embeddings of K, so that $n = r_1 + 2r_2$, and write $r = r_1 + r_2$. Let w be the number of roots of unity in K

and let d, Cl, h, R and $\mathfrak{d}$ denote the discriminant, the class group, the class number, the regulator and the different of K, respectively.

At the end of the last talk, we proved the following result:

Theorem 1. *For each $[\mathfrak{a}] \in \mathrm{Cl}$, the function*

$$Z([\mathfrak{a}], s) = Z_\infty(s)\zeta_K([\mathfrak{a}], s)$$

(defined for $\mathrm{Re}(s) > 1$) admits an analytic continuation to $\mathbb{C} \setminus \{0, 1\}$, which has simple poles at $s = 0$ and $s = 1$ with residues

$$-\frac{2^r R}{w} \quad \text{and} \quad \frac{2^r R}{w},$$

respectively. Moreover, these functions satisfy functional equations

$$Z([\mathfrak{a}], s) = Z([\mathfrak{a}^{-1}\mathfrak{d}], 1 - s).$$

Summing over all the classes of Cl, we obtain an analogous result for the completed zeta function

$$Z_K(s) = Z_\infty(s)\zeta_K(s) = \sum_{[\mathfrak{a}] \in \mathrm{Cl}} Z([\mathfrak{a}], s).$$

Corollary 2. *The completed zeta function $Z_K(s)$ (defined for $\mathrm{Re}(s) > 1$) admits an analytic continuation to $\mathbb{C} \setminus \{0, 1\}$, which has simple poles at $s = 0$ and $s = 1$ with residues*

$$-\frac{2^r hR}{w} \quad \text{and} \quad \frac{2^r hR}{w},$$

respectively. Moreover, it satisfies the functional equation

$$Z_K(s) = Z_K(1 - s).$$

As in the case of Dirichlet L–series, we can obtain more general results twisting by a character. That is to say, consider a character $\chi\colon\ \mathrm{Cl} \to S^1$ and define

$$\zeta(\chi, s) = \sum_{\mathfrak{a} \subseteq \mathcal{O}_K \text{ ideal}} \frac{\chi(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^s}$$

(where we write $\chi(\mathfrak{a})$ for $\chi([\mathfrak{a}])$) and

$$Z(\chi, s) = Z_\infty(s)\zeta(\chi, s) = \sum_{[\mathfrak{a}] \in \mathrm{Cl}} \chi(\mathfrak{a}) Z([\mathfrak{a}], s).$$

By theorem 1, we obtain the functional equation

$$Z(\chi, s) = \chi(\mathfrak{d}) Z(\overline{\chi}, 1 - s)$$

Also, if χ is not the trivial character, we have that

$$\sum_{[\mathfrak{a}] \in \mathrm{Cl}} \chi(\mathfrak{a}) = 0$$

and, as the $Z([\mathfrak{a}],s)$ for $[\mathfrak{a}] \in \mathrm{Cl}$ have all the same residues at the two simple poles 0 and 1, this implies that $Z(\chi,s)$ is an entire function.

Since we had already studied the factor $Z_\infty(s)$ in a previous talk, the results about the completed zeta function give us a similar description of the behaviour of the original Dedekind zeta function.

Theorem 3. *The Dedekind zeta function $\zeta_K(s)$ (defined for $\mathrm{Re}(s) > 1$) admits an analytic continuation to $\mathbb{C} \setminus \{1\}$, which has a simple pole at $s = 1$ with residue*

$$\frac{2^{r_1}(2\pi)^{r_2} hR}{w\sqrt{|d|}}\,.$$

Moreover, it satisfies the functional equation

$$\zeta_K(1-s) = A(s)\zeta_K(s)\,,$$

where

$$A(s) = |d|^{s-1/2}\Big(\cos\frac{\pi s}{2}\Big)^{r}\Big(\sin\frac{\pi s}{2}\Big)^{r_2} L_{\mathbb{C}}(s)^n\,.$$

Proof. By the definition of the completed zeta function,

$$\zeta_K(s) = \frac{Z_K(s)}{Z_\infty(s)} = \frac{Z_K(s)}{|d|^{s/2}L_{\mathbb{R}}(s)^{r_1}L_{\mathbb{C}}(s)^{r_2}}\,,$$

where we use the functions

$$L_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big) \quad \text{and} \quad L_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$$

defined in a previous talk. Recall that Euler's gamma function has no zeros and has a simple pole at 0. Also, a straight-forward computation yields that

$$Z_\infty(1) = \frac{\sqrt{|d|}}{\pi^{r_2}}\,.$$

This, together with corollary 2, implies that $\zeta_K(s)$ admits an analytic continuation to $\mathbb{C} \setminus \{1\}$ having a simple pole at $s = 1$ with

$$\mathrm{Res}_{s=1}\,\zeta_K(s) = \pi^{r_2}|d|^{-1/2}\,\mathrm{Res}_{s=1}\,Z_K(s) = \frac{2^{r_1}(2\pi)^{r_2}hR}{w\sqrt{|d|}}\,.$$

The functional equation for ζ_K follows immediately from the functional equations $Z_K(s) = Z_K(1-s)$ and $L_{\mathbb{R}}(s)^{r_1}L_{\mathbb{C}}(s)^{r_2} = A(s)L_{\mathbb{R}}(1-s)^{r_1}L_{\mathbb{C}}(1-s)^{r_2}$. □

The formula for $\mathrm{Res}_{s=1}\,\zeta_K(s)$ is known as the class number formula.

Dirichlet's prime number theorem

In this section, we fix a positive integer m and write $G = (\mathbb{Z}\,/\,m\mathbb{Z})^\times$. Recall that the order of G is $\varphi(m)$. Also, let $\mathbb{P}$ be the set of prime numbers.

The key point of the proof of Dirichlet's theorem on arithmetic progressions uses the behaviour of the Dirichlet L–series at 1. Thus, we first relate these series to Dedekind's zeta functions.

Proposition 4. *Let $K = \mathbb{Q}(\mu_m)$ be the m–th cyclotomic field. One has the identity*

$$\zeta_K(s) = F(s) \prod_{\chi \in \widehat{G}} L(\chi, s)\,,$$

where

$$F(s) = \prod_{\mathfrak{p} \mid m} \frac{1}{1 - \mathfrak{N}(\mathfrak{p})^{-s}}\,.$$

Proof. Let $p \in \mathbb{P}$ and let $\mathfrak{p} \subset \mathcal{O}_K$ be a prime ideal over p. We know that the ramification index and the inertia degree are given by

$$e(p) = e(\mathfrak{p} \mid p) = \varphi(p^{v_p(m)})$$

and

$$f(p) = f(\mathfrak{p} \mid p) = [(\mathbb{Z} / m'\mathbb{Z})^\times : \langle p + m'\mathbb{Z} \rangle]\,, \quad \text{where } m' = \frac{m}{p^{v_p(m)}}\,,$$

and that $\mathfrak{N}(\mathfrak{p}) = p^{f(p)}$. Write $r(p)$ for the number of prime ideals over p. Using Euler's product formula, we obtain that

$$\zeta_K(s) = \prod_{\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_K} \frac{1}{1 - \mathfrak{N}(\mathfrak{p})^{-s}} = \prod_{p \in \mathbb{P}} \prod_{\mathfrak{p} \mid p} \frac{1}{1 - \mathfrak{N}(\mathfrak{p})^{-s}} = \prod_{p \in \mathbb{P}} \left(\frac{1}{1 - p^{-f(p)s}} \right)^{r(p)}.$$

On the other hand,

$$\prod_{\chi \in \widehat{G}} L(\chi, s) = \prod_{\chi \in \widehat{G}} \prod_{p \in \mathbb{P}} \frac{1}{1 - \chi(p) p^{-s}} = \prod_{p \nmid m} \prod_{\chi \in \widehat{G}} \frac{1}{1 - \chi(p) p^{-s}}\,,$$

where we used that $\chi(p) = 0$ if $p \mid m$. Now, for $p \nmid m$, write G_p for the subgroup of G generated by p, so that $|G_p| = f(p)$ and $[G : G_p] = r(p)$. There is a short exact sequence

$$1 \longrightarrow \widehat{(G / G_p)} \longrightarrow \widehat{G} \longrightarrow \widehat{G_p} \longrightarrow 1$$

(the only non-obvious part is the surjectivity, which can be checked by induction on $[G : G_p]$ and using that $\mathbb{C}^\times$ contains all roots of unity). As a character on G_p is uniquely determined by its value at p and this can be any $f(p)$–th root of unity, this implies that for each $\xi \in \mu_{f(p)}$ there are exactly $r(p)$ characters $\chi \in \widehat{G}$ such that $\chi(p) = \xi$. Therefore,

$$\prod_{\chi \in \widehat{G}} \frac{1}{1 - \chi(p) p^{-s}} = \prod_{\xi \in \mu_{f(p)}} \left(\frac{1}{1 - \xi p^{-s}} \right)^{r(p)} = \left(\frac{1}{1 - p^{-f(p)s}} \right)^{r(p)}.$$

Putting everything together, we obtain the required formula. □

Corollary 5. *For every non-trivial character $\chi \in \widehat{G}$,*

$$L(\chi, 1) \neq 0\,.$$

Proof. Let $\chi^0 \in \widehat{G}$ be the trivial character. By proposition 4, we can write

$$\zeta_K(s) = F(s) L(\chi^0, s) \prod_{\chi \neq \chi^0} L(\chi, s) = F(s)\zeta(s) \prod_{p|m} (1 - p^{-s}) \prod_{\chi \neq \chi^0} L(\chi, s)\,.$$

But we know that both $\zeta_K(s)$ and $\zeta(s)$ have a simple pole at $s = 1$ and that the remaining factors in the right-hand side are holomorphic at $s = 1$. In conclusion, $L(\chi, s) \neq 0$ for $\chi \neq \chi^0$. □

With this, we have the main ingredient of the proof of Dirichlet's theorem. Next, we introduce the concept of density which appears in the (stronger version) of the theorem.

Lemma 6. *When $s \to 1$ (in the half-plane $\mathrm{Re}(s) > 1$), one has that*

$$\sum_{p \in \mathbb{P}} \frac{1}{p^s} \sim \log \frac{1}{s-1}$$

and that the series

$$\sum_{p \in \mathbb{P}} \sum_{k \geq 2} \frac{1}{p^{ks}}$$

remains bounded.

Proof. Using Euler's product formula for the Riemann zeta function and the power series expansion of the logarithm, we get that

$$\log \zeta(s) = \sum_{p \in \mathbb{P}} \log \frac{1}{1 - p^{-s}} = \sum_{p \in \mathbb{P}} \sum_{k \geq 1} \frac{1}{k p^{ks}} = \sum_{p \in \mathbb{P}} \frac{1}{p^s} + \psi(s)\,,$$

where

$$\psi(s) = \sum_{p \in \mathbb{P}} \sum_{k \geq 2} \frac{1}{k p^{ks}}\,.$$

But $\psi(s)$ (for $\mathrm{Re}(s) > 1$) is majorized by the series

$$\sum_{p \in \mathbb{P}} \sum_{k \geq 2} |p^{-s}|^k = \sum_{p \in \mathbb{P}} \frac{|p^{-s}|^2}{1 - |p^{-s}|} = \sum_{p \in \mathbb{P}} \frac{1}{|p^s|(|p^s| - 1)} \leq \sum_{p \in \mathbb{P}} \frac{1}{p(p-1)}$$
$$\leq \sum_{n \geq 2} \frac{1}{n(n-1)} = \sum_{n \geq 2} \left(\frac{1}{n-1} - \frac{1}{n} \right) = 1\,.$$

The other part of the lemma follows from the fact that $\zeta(s)$ has a simple pole at $s = 1$ with residue 1. □

Definition 7. We say that a subset A of $\mathbb{P}$ has *analytic density* (or *Dirichlet density*) $\delta \in [0,1] \subset \mathbb{R}$ if

$$\lim_{s\to 1}\left[\left(\sum_{p\in A}\frac{1}{p^s}\right)\Big/\left(\log\frac{1}{s-1}\right)\right] = \delta$$

(provided that this limit, taken in the half-plane $\mathrm{Re}(s) > 1$, exists).

We are finally in a position to state the refinement of Dirichlet's theorem which we are going to prove.

Theorem 8. *Let $a \in \mathbb{Z}$ such that $(a,m) = 1$. The set*

$$\mathbb{P}_a = \{\, p \in \mathbb{P} : p \equiv a \pmod{m}\,\}$$

has analytic density $\varphi(m)^{-1}$.

This theorem clearly implies the theorem on arithmetic progressions from the beginning, as any finite set has analytic density zero. For the proof of the theorem, we need a previous lemma.

For a character $\chi \in \widehat{G}$, set

$$f_\chi(s) = \sum_{p\in\mathbb{P}}\frac{\chi(p)}{p^s}\,,$$

the series being convergent for $\mathrm{Re}(s) > 1$.

Lemma 9. *If $\chi^0 \in \widehat{G}$ is the trivial character, then*

$$f_{\chi^0}(s) \sim \log\frac{1}{s-1}$$

as $s \to 1$ (in the half-plane $\mathrm{Re}(s) > 1$). In contrast, for every non-trivial character $\chi \in \widehat{G}$, $f_\chi(s)$ remains bounded as $s \to 1$ (in the half-plane $\mathrm{Re}(s) > 1$).

Proof. The result for $f_{\chi^0}(s)$ follows from lemma 6, as $f_{\chi^0}(s)$ differs from the series

$$\sum_{p\in\mathbb{P}}\frac{1}{p^s}$$

by a finite number of terms.

For $\chi \neq \chi^0$, we do as in the proof of lemma 6 and express

$$\log L(\chi,s) = \sum_{p\in\mathbb{P}}\log\frac{1}{1-\chi(p)p^{-s}} = \sum_{p\in\mathbb{P}}\sum_{k\geq 1}\frac{\chi(p)^k}{kp^{ks}} = f_\chi(s) + F_\chi(s)\,,$$

where

$$F_\chi(s) = \sum_{p\in\mathbb{P}}\sum_{k\geq 2}\frac{\chi(p)^k}{kp^{ks}}\,.$$

It is clear that $F_\chi(s)$ is majorized by the series we used in the proof of lemma 6 for $\psi(s)$, so that $F_\chi(s)$ remains bounded as $s \to 1$. And the fact that $L(\chi, 1) \neq 0$ implies that $\log L(\chi, s)$ also remains bounded as $s \to 1$. Therefore, the same holds for $f_\chi(s)$. □

Proof of theorem 8. We have to study the behaviour of the function

$$g_a(s) = \sum_{p \in \mathbb{P}_a} \frac{1}{p^s}$$

as $s \to 1$ (in the half-plane $\mathrm{Re}(s) > 1$). We claim that

$$g_a(s) = \frac{1}{\varphi(m)} \sum_{\chi \in \widehat{G}} \chi(a)^{-1} f_\chi(s) \,.$$

Indeed, we can write

$$\sum_{\chi \in \widehat{G}} \chi(a)^{-1} f_\chi(s) = \sum_{\chi \in \widehat{G}} \sum_{p \in \mathbb{P}} \frac{\chi(a)^{-1}\chi(p)}{p^s} = \sum_{p \in \mathbb{P}} \Bigg(\sum_{\chi \in \widehat{G}} \chi(a^{-1}p) \Bigg) \frac{1}{p^s}$$

and, using that

$$\sum_{\chi \in \widehat{G}} \chi(a^{-1}p) = \begin{cases} \varphi(m) & \text{if } a^{-1}p \equiv 1 \pmod{m}, \\ 0 & \text{otherwise,} \end{cases}$$

we obtain the function $\varphi(m) g_a(s)$.

Thus, lemma 9 implies that

$$g_a(s) \sim \frac{1}{\varphi(m)} \log \frac{1}{s-1}$$

as $s \to 1$, as desired. □

We conclude with some remarks about densities. Instead of the analytic density introduced in definition 7, one usually considers another notion of density: we say that a subset A of $\mathbb{P}$ has *natural density* (or *asymptotic density*) $\delta \in [0,1] \subset \mathbb{R}$ if

$$\lim_{N \to \infty} \frac{|\{\, p \in A : p \leq N \,\}|}{|\{\, p \in \mathbb{P} : p \leq N \,\}|} = \delta$$

(provided that this limit exists). One can prove that, if A has natural density δ, then A has analytic density δ as well. However, the converse is not true in general. That is, there exist subsets A of $\mathbb{P}$ which have a well-defined analytic density but no natural density.

In any case, the subsets $\mathbb{P}_a$ (with $(a, m) = 1$) of primes which we considered are "nice enough" to have a natural density, which must then agree with their analytic density. To prove that the subset $\mathbb{P}_a$ has natural density $\varphi(m)^{-1}$, one can proceed as in the proof of the prime number theorem, which states that

$$|\{ p \in \mathbb{P} : p \leq N \}| \sim \frac{N}{\log N}$$

as $N \to \infty$. The proof of this result is also based on analytic methods. The argument, together with the techniques about Dirichlet series we have studied, can be modified to show that

$$|\{ p \in \mathbb{P}_a : p \leq N \}| \sim \frac{N}{\varphi(m) \log N}$$

as $N \to \infty$. See Kedlaya's notes [1] for more details.

These results about densities of subsets of prime numbers can be further generalized to the case of number fields. More precisely, one can compute the densities of certain subsets of prime ideals of a number field using the theory of Dedekind zeta functions and L–series. A remarkable example of such generalizations is Chebotarev's density theorem, which was one of the first results in class field theory.

References

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