Algebraic de Rham cohomology and the Gauss–Manin connection

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Abstract

These are the notes for a two-hour-talk given virtually (through a web conferencing tool) in the students seminar[1] of Montréal's number theory group on Lawrence–Venkatesh's new proof of Faltings's theorem. The talk itself is independent and introduces the background material related to algebraic de Rham cohomology, including relevant tools in homological algebra. The text here is admittedly insufficient to learn everything from scratch. I prepared it in the hope that it would be a good review for people acquainted with the topic and, more importantly, that it could convey a few important ideas to newcomers. Generality and comprehensiveness were not among my goals. The notes are a summary of various sources and contain no original results.

0 Motivation

One can compute cohomology of a smooth manifold X via its differential structure: let $\Omega_{X/\mathbb{R}}$ be the sheaf of smooth differentials on X, set $\Omega^n_{X/\mathbb{R}} = \bigwedge^n \Omega_{X/\mathbb{R}}$ for each $n \in \mathbb{Z}_{\geq 1}$ and consider the complex

$$\Omega^\bullet_{X/\mathbb{R}}\colon \quad 0 \longrightarrow \mathscr{O}_X = \Omega^0_{X/\mathbb{R}} \xrightarrow{d} \Omega^1_{X/\mathbb{R}} \xrightarrow{d} \Omega^2_{X/\mathbb{R}} \longrightarrow \cdots$$

Using Poincaré's lemma and a standard argument with partitions of unity, one can prove that $\Omega^\bullet_{X/\mathbb{R}}$ is an acyclic resolution of the constant sheaf $\underline{\mathbb{R}}_X$. Therefore, for each $n \in \mathbb{Z}_{\geq 0}$,

$$\mathrm{H}^n(X, \underline{\mathbb{R}}_X) = \mathrm{H}^n\big(\Gamma(X, \Omega^\bullet_{X/\mathbb{R}})\big)$$

[1] I thank Henri Darmon, Adrian Iovita, Jackson Morrow and Marc-Hubert Nicole for organizing the seminar.

and this last group is usually called *the n–th (real) de Rham cohomology group of X*, usually written as $\mathrm{H}^n_{\mathrm{dR}}(X/\mathbb{R})$.

We want an algebraic analogue of this situation. As an intermediate step, consider now a complex analytic manifold X (which, under suitable hypotheses, will be an algebraic variety over $\mathbb{C}$). Let $\Omega_{X/\mathbb{C}}$ be the sheaf of holomorphic differentials on X. The de Rham complex $\Omega^\bullet_{X/\mathbb{C}}$, defined as before, is again a resolution of the constant sheaf $\underline{\mathbb{C}}_X$ but in this case is *not* acyclic (note that partitions of unity do not make sense in a complex analytic setting). Thus, the group $\mathrm{H}^n\big(\Gamma(X, \Omega^\bullet_{X/\mathbb{C}})\big)$ does not seem a good candidate for the definition of $\mathrm{H}^n_{\mathrm{dR}}(X/\mathbb{C})$, as de Rham cohomology should compute the sheaf cohomology of X. In fact, we need to use a generalization of sheaf cohomology.

1 Hypercohomology

Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Suppose that $\mathcal{A}$ has enough injectives. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left-exact functor.

Let $\mathrm{Com}(\mathcal{A})$ denote the category of cochain complexes in $\mathcal{A}$ that are concentrated in degrees ≥ 0.[2]

Definition 1. Let $C^\bullet \in \mathrm{Ob}\big(\mathrm{Com}(\mathcal{A})\big)$.

(1) An *injective resolution* of $C^\bullet$ is a quasi-isomorphism from $C^\bullet$ to a complex of injective objects $I^\bullet$; i.e., a sequence of morphisms

$$\begin{array}{cccccccccc} C^\bullet\colon & 0 & \longrightarrow & C^0 & \longrightarrow & C^1 & \longrightarrow & C^2 & \longrightarrow & \cdots \\ \downarrow & & & \downarrow & & \downarrow & & \downarrow & & \\ I^\bullet\colon & 0 & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & I^2 & \longrightarrow & \cdots \end{array}$$

(vertical arrows) which induce isomorphisms on cohomology.

(2) Let $n \in \mathbb{Z}_{\geq 0}$. The *n–th right hyperderived functor* $\mathbb{R}^n F\colon \mathrm{Com}(\mathcal{A}) \to \mathcal{B}$ is given by

$$\mathbb{R}^n F(C^\bullet) = \mathrm{H}^n\big(F(I^\bullet)\big)$$

(for a choice of injective resolution $C^\bullet \to I^\bullet$).

Remark. By hypothesis on $\mathcal{A}$, we can always find injective resolutions. Moreover, by the properties of injectives, the object $\mathbb{R}^n F(C^\bullet)$ is well-defined (up to unique isomorphism).

[2] Some hypotheses can be weakened and one could use other conventions. I made choices for the sake of simplicity.

Definition 2. Let $n \in \mathbb{Z}_{\geq 0}$.

(1) Let X be a smooth variety over a field K. Let $\Omega_{X/K}$ be the sheaf of Kähler differentials of X and consider the *de Rham complex*

$$\Omega^\bullet_{X/K}\colon \qquad 0 \longrightarrow \mathscr{O}_X \longrightarrow \Omega^1_{X/K} \longrightarrow \Omega^2_{X/K} \longrightarrow \cdots,$$

where $\Omega^n_{X/K} = \bigwedge^n \Omega_{X/K}$. The *n–th de Rham cohomology of X/K* is the group

$$\mathrm{H}^n_{\mathrm{dR}}(X/K) = \mathbb{H}^n(X, \Omega^\bullet_{X/K}) = \mathbb{R}^n\Gamma(X,-)(\Omega^\bullet_{X/K}).$$

(2) More generally, let $\pi\colon X \to Y$ be a smooth morphism of schemes. Let $\Omega_{X/Y}$ be the sheaf of Kähler differentials of X relative to Y and consider the *relative de Rham complex* $\Omega^\bullet_{X/Y}$. The *n–th (relative) de Rham cohomology of X/Y* is the sheaf (on Y)

$$\mathscr{H}^n_{\mathrm{dR}}(X/Y) = \mathbb{R}^n\pi_*(\Omega^\bullet_{X/Y}).$$

2 Computations

Often *the only hope* to explicitly compute sheaf cohomology is Čech cohomology.

Suppose that $\pi\colon X \to Y$ is a separated smooth morphism and, up to shrinking, that Y is affine, say $Y = \mathrm{Spec}(R)$. Let $\mathcal{U} = (U_i)_{i \in I}$ be an affine open covering of X with a total order on I. We restrict to quasi-coherent sheaves of modules, which over affines are acyclic (Serre). We write $\mathrm{QCoh}(X)$ for the category of quasi-coherent $\mathscr{O}_X$–modules.

For every finite $J \subseteq I$, write

$$U_J = \bigcap_{j \in J} U_j$$

and let $\iota_J\colon U_J \hookrightarrow X$ be the natural inclusion. Since X is separated, both U_J and ι_J are affine. Then $(\iota_J)_*$ and $(\iota_J)^*$ are exact functors on $\mathrm{QCoh}(U_J)$ and $\mathrm{QCoh}(X)$, respectively.

For $\mathscr{F} \in \mathrm{Ob}\big(\mathrm{QCoh}(X)\big)$ and $n \in \mathbb{Z}_{\geq 0}$, define sheaves

$$\mathscr{C}^n(\mathcal{U}, \mathscr{F}) = \prod_{|J| = n+1} (\iota_J)_*(\mathscr{F}|_{U_J})$$

and morphisms $\delta^n\colon \mathscr{C}^n(\mathcal{U}, \mathscr{F}) \to \mathscr{C}^{n+1}(\mathcal{U}, \mathscr{F})$ given by $\delta^n\big((x_J)_J\big) = (y_{J'})_{J'}$ on sections, where

$$y_{J'} = \sum_{i=0}^{n+1} (-1)^i x_{J' \setminus \{j_i\}} \quad \text{if } J' = \{\, j_0 < j_1 < \cdots < j_{n+1} \,\}.$$

(Here, we omit restrictions to $U_{J'}$ by abuse of notation.)

In this way, given $\mathscr{F}^\bullet \in \mathrm{Ob}\big(\mathrm{Com}(\mathrm{QCoh}(X))\big)$, we get a double complex $\mathscr{C}^{\bullet,\bullet}$ with $\mathscr{C}^{i,j} = \mathscr{C}^j(\mathcal{U}, \mathscr{F}^i)$:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \iddots \\
 & & | & & | & & | & & \\
 & & \mathscr{C}^{0,2} & \longrightarrow & \mathscr{C}^{1,2} & \longrightarrow & \mathscr{C}^{2,2} & \text{---} & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \mathscr{C}^{0,1} & \longrightarrow & \mathscr{C}^{1,1} & \longrightarrow & \mathscr{C}^{2,1} & \text{---} & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \mathscr{C}^{0,0} & \longrightarrow & \mathscr{C}^{1,0} & \longrightarrow & \mathscr{C}^{2,0} & \text{---} & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathscr{F}^{0} & \longrightarrow & \mathscr{F}^{1} & \longrightarrow & \mathscr{F}^{2} & \text{---} & \cdots
\end{array}$$

(where the vertical arrows $d_{\mathrm{ver}}^{\bullet,\bullet}$ are the ones defined in the previous paragraph and the horizontal arrows $d_{\mathrm{hor}}^{\bullet,\bullet}$ are naturally induced by the connecting morphisms of the complex $\mathscr{F}^\bullet$).

Theorem 3. *With the assumptions above, the total complex $T^\bullet = \mathrm{Tot}(\mathscr{C}^{\bullet,\bullet})$, defined by*

$$T^n = \bigoplus_{i+j=n} \mathscr{C}^{i,j} \quad \textit{and} \quad d_T^n = \sum_{i+j=n} \Big[d_{\mathrm{hor}}^{i,j} + (-1)^i \, d_{\mathrm{ver}}^{i,j} \Big] \colon T^n \to T^{n+1}$$

for each $n \in \mathbb{Z}_{\geq 0}$, is an acyclic resolution of $\mathscr{F}^\bullet$ and

$$\mathbb{R}^n \pi_*(\mathscr{F}^\bullet) = \mathrm{H}^n\big(\pi_*(T^\bullet)\big) \quad \textit{for all } n \in \mathbb{Z}_{\geq 0}.$$

Example 4. Let E be an elliptic curve over a field K given by an equation

$$E\colon Y^2 Z = X(X - Z)(X - \lambda Z) \quad \text{with } \lambda \in K \setminus \{\, 0, 1 \,\}.$$

Take the open covering $\mathcal{U} = \{\, U, V \,\}$, where $U = \{\, Z \neq 0 \,\}$ and $V = \{\, Y \neq 0 \,\}$.

We can compute the cohomology groups $\mathrm{H}^n_{\mathrm{dR}}(E/K)$ from the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathscr{O}_E(U\cap V) & \xrightarrow{d} & \Omega^1_E(U\cap V) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle(+,-)} & & \uparrow{\scriptstyle(-,+)} & & \\
0 & \longrightarrow & \mathscr{O}_E(U)\oplus\mathscr{O}_E(V) & \xrightarrow{d} & \Omega^1_E(U)\oplus\Omega^1_E(V) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathscr{O}_E(E) & \longrightarrow & \Omega^1_E(E) & \longrightarrow & 0
\end{array}$$

(where the arrows are the obvious ones except for some signs that need to be accounted for). In particular,

$$\mathrm{H}^1_{\mathrm{dR}}(E/K) = \frac{\left\{ \begin{array}{r} (f,\alpha,\beta) \in \mathscr{O}_E(U\cap V)\oplus\Omega^1_E(U)\oplus\Omega^1_E(V)\colon \\ df-\alpha+\beta = 0 \ \text{ in } \Omega^1_E(U\cap V) \end{array} \right\}}{\{\, (g-h, dg, dh)\colon (g,h)\in\mathscr{O}_E(U)\oplus\mathscr{O}_E(V)\,\}}.$$

Using that

$$\mathrm{H}^1(E,\mathscr{O}_E) = \check{\mathrm{H}}^1(\mathcal{U},\mathscr{O}_E) = \frac{\mathscr{O}_E(U\cap V)}{\{\, g-h\colon (g,h)\in\mathscr{O}_E(U)\oplus\mathscr{O}_E(V)\,\}},$$

one can check that

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{H}^0(E,\Omega^1_E) & \longrightarrow & \mathrm{H}^1_{\mathrm{dR}}(E/K) & \longrightarrow & \mathrm{H}^1(E,\mathscr{O}_E) & \longrightarrow & 0 \\
 & & \omega & \longmapsto & (0,\omega,\omega) & & & & \\
 & & & & (f,\alpha,\beta) & \longmapsto & f & &
\end{array}$$

is a short exact sequence (known as the *Hodge filtration* of E). A basis of $\mathrm{H}^1_{\mathrm{dR}}(E/K)$ is

$$\omega = \Big(0, \frac{dx}{y}, \frac{dx}{y}\Big), \quad \eta = \Big(\frac{2y}{x}, \frac{x\,dx}{y}, \frac{x\,dx}{y}\Big).$$

(Here, x and y are the usual affine coordinates in U.)

3 Spectral sequences

We need a tool in homological algebra generalizing long exact sequences.

Definition 5. A *spectral sequence* is a sequence $(E_r, d_r)_{r \geq r_0}$, where each *page* E_r is a bigraded group

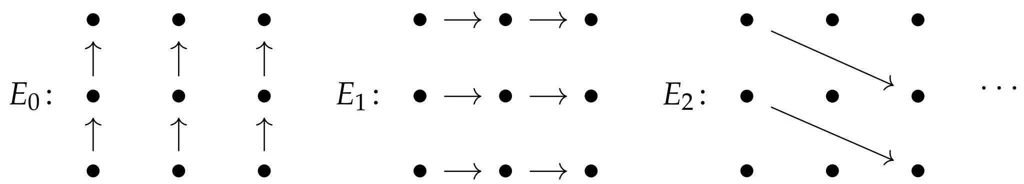
$$E_r = \bigoplus_{i,j \geq 0} E_r^{i,j}$$

with connecting morphisms $d_r^{i,j} : E_r^{i,j} \to E_r^{i+r, j-r+1}$, such that $d_r \circ d_r = 0$ and

$$E_{r+1}^{i,j} = \mathrm{H}^{i,j}(E_r^{\bullet,\bullet}) = \frac{\mathrm{Ker}\big(E_r^{i,j} \xrightarrow{d_r} E_r^{i+r,j-r+1}\big)}{\mathrm{Im}\big(E_r^{i-r,j+r-1} \underset{d_r}{\longrightarrow} E_r^{i,j}\big)} \quad \text{for all } r \geq r_0.$$

If at some $s \geq r_0$ we have $E_s = E_{s+1} = E_{s+2} = \cdots$, we write $E_\infty = E_s$ and call it the *limit* of the spectral sequence.

Remark. Viewing the superindices as coordinates in the first quadrant of the cartesian plane, we have sequences of complexes with arrows in the following patterns:



(From one page to the next, the tip of each arrow *moves one position to the right and one position downwards*.)[3]

Theorem 6. *Let* $C^\bullet = \mathrm{Fil}^0 C^\bullet \supset \mathrm{Fil}^1 C^\bullet \supset \cdots \supset \mathrm{Fil}^{n+1} C^\bullet = 0$ *be a complex with a bounded (decreasing) filtration. There exists a spectral sequence* $(E_r, d_r)_{r \geq 0}$ *with*

$$\begin{aligned} E_0^{i,j} &= \mathrm{Gr}^i C^{i+j}, \\ E_1^{i,j} &= \mathrm{H}^{i+j}(\mathrm{Gr}^i C^\bullet) \quad \textit{and} \\ E_\infty^{i,j} &= \mathrm{Gr}^i \mathrm{H}^{i+j}(C^\bullet) \end{aligned}$$

for all $i, j \geq 0$*, where the filtration on* $\mathrm{H}^n(C^\bullet)$ *is the following:*

$$\mathrm{Fil}^i \mathrm{H}^n(C^\bullet) = \mathrm{Im}\big(\mathrm{H}^n(\mathrm{Fil}^i C^\bullet) \to \mathrm{H}^n(C^\bullet)\big).$$

(In this situation, we write

$$E_r^{i,j} \implies \mathrm{H}^{i+j}(C^\bullet)$$

and say that the spectral sequence converges *or* abuts *to* $\mathrm{H}^n(C^\bullet)$*.)*

[3] Of course, there are other conventions that make as much sense.

In the setting of section 1, take

$$T^n = \bigoplus_{i+j=n} \mathscr{C}^{i,j} \quad \text{and} \quad \operatorname{Fil}^k T^n = \bigoplus_{\substack{i+j=n \\ i\geq k}} \mathscr{C}^{i,j}.$$

The corresponding filtration on

$$\mathrm{H}^n\big(\pi_*(T^\bullet)\big) = \mathscr{H}^n_{\mathrm{dR}}(X/Y)$$

is called the *Hodge filtration* and we get the *Hodge–de Rham spectral sequence*

$$E_1^{i,j} = \mathrm{R}^j\pi_*(\Omega^i_{X/Y}) \implies \mathscr{H}^{i+j}_{\mathrm{dR}}(X/Y)$$

from theorem 6.

4 Connections

Let $Y \to S$ be a smooth morphism of schemes and let $\mathscr{E} \in \operatorname{Ob}\big(\operatorname{QCoh}(Y)\big)$.

Definition 7.

(1) A *connection* on $\mathscr{E}$ is a morphism $\nabla\colon \mathscr{E} \to \Omega^1_{Y/S} \otimes_{\mathscr{O}_Y} \mathscr{E}$ of abelian sheaves such that

$$\nabla(fe) = df \otimes e + f\nabla(e) \quad \text{on sections } f \in \mathscr{O}_Y(U) \text{ and } e \in \mathscr{E}(U)$$

(for any open subset U of Y).

(2) A section $e \in \mathscr{E}(U)$ is called *horizontal* if $\nabla(e) = 0$.

(3) For $i \in \mathbb{Z}_{\geq 1}$, define $\nabla_i\colon \Omega^i_{Y/S} \otimes_{\mathscr{O}_Y} \mathscr{E} \to \Omega^{i+1}_{Y/S} \otimes_{\mathscr{O}_Y} \mathscr{E}$ by

$$\nabla_i(\omega \otimes e) = d\omega \otimes e + (-1)^i \omega \wedge \nabla(e)$$

on sections $\omega \in \Omega^i_{Y/S}(U)$ and $e \in \mathscr{E}(U)$ (for any open subset U of Y).

(4) We say that ∇ is *integrable* if $\nabla_1 \circ \nabla = 0$.

A connection can be interpreted as a tool to "differentiate along a direction (in the tangent space)". Consider the sheaf of derivations

$$\mathscr{D}er_{\mathscr{O}_S}(\mathscr{O}_Y, \mathscr{O}_Y) \cong \mathscr{H}om_{\mathscr{O}_Y}(\Omega^1_{Y/S}, \mathscr{O}_Y).$$

To the connection ∇ we attach a morphism

$$\begin{aligned} \mathscr{D}er_{\mathscr{O}_S}(\mathscr{O}_Y,\mathscr{O}_Y) &\longrightarrow \mathscr{H}om_{\mathscr{O}_S}(\mathscr{E},\mathscr{E}) \\ D &\longmapsto \Big(\nabla_D\colon \mathscr{E} \xrightarrow{\nabla} \Omega^1_{Y/S}\otimes_{\mathscr{O}_Y}\mathscr{E} \xrightarrow{D\otimes 1} \mathscr{O}_Y\otimes_{\mathscr{O}_Y}\mathscr{E}\cong\mathscr{E}\Big) \end{aligned}$$

and every ∇_D satisfies the *Leibniz rule*

$$\nabla_D(fe) = D(f)\cdot e + f\cdot\nabla_D(e) \quad \text{on sections } f\in\mathscr{O}_Y(U) \text{ and } e\in\mathscr{E}(U)$$

(for any open subset U of Y).

We use ∇ to give isomorphisms between fibres of $\mathscr{E}$ (locally in open subsets of Y) via *parallel transport*: for $y_0\in Y$ and y in a "small enough" neighbourhood U of y_0, we define an isomorphism $\mathscr{E}_{y_0}\cong\mathscr{E}_y$ by choosing a path $\gamma\colon [0,1]\to U$ with $\gamma(0)=y_0$ and $\gamma(1)=y$ and solving the differential equations

$$\begin{cases} \nabla_{\gamma'}(e)=0 \\ e(0)=e_0\in\mathscr{E}_{y_0} \end{cases}$$

to obtain a map

$$e_0\mapsto e(1)\in\mathscr{E}_y.$$

If ∇ is integrable and U is simply connected, this map is independent of the choice of γ and we obtain a canonical isomorphism $\mathscr{E}_{y_0}\cong\mathscr{E}_y$.[4]

5 The Gauss–Manin connection

Let S be a scheme and consider two smooth S–schemes X and Y. Let $\pi\colon X\to Y$ be a smooth S–morphism. We want to define a canonical integrable connection ∇ on the sheaf $\mathscr{H}^n_{\mathrm{dR}}(X/Y)$. To do so, we can use the exact sequence

$$0\longrightarrow \pi^*(\Omega^1_{Y/S})\longrightarrow \Omega^1_{X/S}\longrightarrow\Omega^1_{X/Y}\longrightarrow 0$$

to define the *Koszul filtration*

$$\mathrm{Fil}^i\,\Omega^\bullet_{X/S} = \mathrm{Im}\big(\pi^*(\Omega^i_{Y/S})\otimes_{\mathscr{O}_X}\Omega^{\bullet-i}_{X/S}\to\Omega^\bullet_{X/S}\big).$$

[4]As the reader will have noted, I am cheating in the sense that I only gave an idea of the situation in *differential* geometry. To actually understand how connections give isomorphisms on stalks in *algebraic* geometry, one needs to study the Riemann–Hilbert correspondence. Adrian Iovita will explain that in the next talk.

It is easy to check (by locally freeness) that

$$\operatorname{Gr}^i \Omega^\bullet_{X/S} \cong \pi^*(\Omega^i_{Y/S}) \otimes_{X/S} \Omega^{\bullet-i}_{X/Y}.$$

Consider the functor $\mathbb{R}^0\pi_* \colon \operatorname{Com}(\operatorname{QCoh}(X)) \to \operatorname{QCoh}(Y)$, whose right derived functors are the functors $\mathbb{R}^i\pi_*$ for $i \geq 0$. Choose an injective resolution

$$0 \longrightarrow \Omega^\bullet_{X/S} \longrightarrow I^\bullet_0 \longrightarrow I^\bullet_1 \longrightarrow \cdots$$

(in the category $\operatorname{Com}(\operatorname{QCoh}(X))$) with filtrations on the complexes $I^\bullet_n$ for $n \in \mathbb{Z}_{\geq 0}$ that are compatible with the Koszul filtration in the following sense: for every $i \in \mathbb{Z}_{\geq 0}$,

$$0 \longrightarrow \operatorname{Fil}^i \Omega^\bullet_{X/S} \longrightarrow \operatorname{Fil}^i I^\bullet_0 \longrightarrow \operatorname{Fil}^i I^\bullet_1 \longrightarrow \cdots$$

is an injective resolution.[5] For each $n \in \mathbb{Z}_{\geq 0}$, we obtain an induced filtration on $\mathbb{R}^0\pi_*(I^\bullet_n) \in \operatorname{Ob}\big(\operatorname{QCoh}(Y)\big)$. Now we can apply theorem 6 to the filtered complex

$$0 \longrightarrow \mathbb{R}^0\pi_*(I^\bullet_0) \longrightarrow \mathbb{R}^0\pi_*(I^\bullet_1) \longrightarrow \cdots$$

to obtain a spectral sequence $(E_r, d_r)_{r\geq 0}$ converging to the relative hypercohomology $\mathbb{R}^n\pi_*(\Omega^\bullet_{X/S})$, $n \in \mathbb{Z}_{\geq 0}$, with

$$\begin{aligned} E_1^{i,j} &= \mathbb{R}^{i+j}\pi_*(\operatorname{Gr}^i \Omega^\bullet_{X/S}) \cong \mathbb{R}^{i+j}\pi_*\big(\pi^*(\Omega^i_{Y/S}) \otimes_{\mathscr{O}_X} \Omega^{\bullet-i}_{X/Y}\big) \\ &\cong \Omega^i_{Y/S} \otimes_{\mathscr{O}_Y} \mathbb{R}^j\pi_*(\Omega^\bullet_{X/Y}) = \Omega^i_{Y/S} \otimes_{\mathscr{O}_Y} \mathscr{H}^j_{\mathrm{dR}}(X/Y) \end{aligned}$$

(where the second isomorphism comes from the facts that the differential acts only on the second piece and that $\Omega_{Y/S}$ is locally free).

Definition 8. Consider the construction explained in the previous paragraphs and let $n \in \mathbb{Z}_{\geq 0}$. The *Gauss–Manin connection* is the composition

$$\nabla \colon \mathscr{H}^n_{\mathrm{dR}}(X/Y) \cong E_1^{0,n} \xrightarrow{d_1^{0,n}} E_1^{1,n} \cong \Omega^1_{Y/S} \otimes_{\mathscr{O}_Y} \mathscr{H}^n_{\mathrm{dR}}(X/Y).$$

Remark. We can compute the maps $d_1^{i,j}$ as the connecting homomorphisms of the

[5] Such an injective resolution with compatible filtrations exists and can be constructed by induction on the length of the filtration. See lemma 13.6.2 of EGA chap. 0_{III}.

functors $\mathbb{R}^j\pi_*$ applied to the short exact sequences

$$0 \longrightarrow \mathrm{Gr}^{i+1}\,\Omega^\bullet_{X/S} \longrightarrow \frac{\mathrm{Fil}^i\,\Omega^\bullet_{X/S}}{\mathrm{Fil}^{i+2}\,\Omega^\bullet_{X/S}} \longrightarrow \mathrm{Gr}^i\,\Omega^\bullet_{X/S} \longrightarrow 0.$$

Example 9. Let K be a field and let $E \xrightarrow{\pi} Y = \mathbb{P}^1_K \setminus \{0,1,\infty\} \to \mathrm{Spec}(K)$ be the Legendre family (i.e., the fibre E_λ over each closed point λ in Y is the curve from example 4). Since E/Y is a curve, the Koszul filtration has $\mathrm{Fil}^2 = 0$. We can apply the functor $\mathbb{R}^0\pi_*$ to the short exact sequence of the previous remark, which takes the form

$$0 \longrightarrow \pi^*(\Omega^1_{Y/K}) \otimes_{\mathscr{O}_E} \Omega^{\bullet-1}_{E/Y} \longrightarrow \Omega^\bullet_{E/K} \longrightarrow \Omega^\bullet_{E/Y} \longrightarrow 0,$$

to find the connecting homomorphism

$$\begin{array}{ccc} \mathbb{R}^1\pi_*(\Omega^\bullet_{E/Y}) & \xrightarrow{\delta} & \mathbb{R}^2\pi_*\big(\pi^*(\Omega^1_{Y/K}) \otimes_{\mathscr{O}_E} \Omega^{\bullet-1}_{E/Y}\big) \\ \| & & \wr\| \\ \mathscr{H}^1_{\mathrm{dR}}(E/Y) & \overset{\nabla}{\dashrightarrow} & \Omega^1_{Y/K} \otimes_{\mathscr{O}_Y} \mathscr{H}^1_{\mathrm{dR}}(E/Y) \end{array}$$

(which, as the diagram shows, is essentially the Gauss–Manin connection on $\mathscr{H}^1_{\mathrm{dR}}(E/Y)$). It is an interesting (although too long for this talk) exercise to compute this map locally in terms of a basis ω, η of $\mathscr{H}^1_{\mathrm{dR}}(E/Y)$.

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