ABELIAN VARIETIES
The theorem of the cube

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Throughout this document, we consider always algebraic varieties over a fixed algebraically closed field k. (As a matter of fact, the results explained here hold for an arbitrary field k after replacing closed points with k–rational points wherever necessary, as the proofs can be performed after base change to an algebraic closure of k.) In particular, varieties are assumed to be irreducible, as in the first talk.

We are going to give a proof of a result which will be useful in the study of line bundles over abelian varieties, namely:

Theorem of the cube. *Let X and Y be two complete varieties and let Z be any variety. Let L be a line bundle over $X \times Y \times Z$. If there exist closed points x_0, y_0 and z_0 of X, Y and Z such that the restrictions $L|_{\{x_0\}\times Y\times Z}$, $L|_{X\times\{y_0\}\times Z}$ and $L|_{X\times Y\times\{z_0\}}$ are trivial, then L is trivial.*

This theorem can be interpreted in the following way. Let $\mathcal{P}_k^+$ be the category of pointed complete varieties over k, whose objects are complete varieties X together with a base (closed) point $x_0 \in X(k)$. Consider $n+1$ objects $X_0, \dots, X_n$ of $\mathcal{P}_k^+$ and a contravariant functor $F\colon \mathcal{P}_k^+ \rightsquigarrow \mathrm{Ab}$. For each i, we have a canonical projection

$$\pi_i\colon X_0 \times \cdots \times X_n \longrightarrow X_0 \times \cdots \times \widehat{X_i} \times \cdots \times X_n$$

and a canonical inclusion

$$\sigma_i\colon X_0 \times \cdots \times \widehat{X_i} \times \dots X_n \longrightarrow X_0 \times \cdots \times X_n$$

(mapping the missing coordinate to the base point). Using these, we define two morphisms

$$\alpha^n = \sum_{i=0}^{n} F(\pi_i)\colon \bigoplus_{i=0}^{n} F(X_0 \times \cdots \times \widehat{X_i} \times \cdots \times X_n) \longrightarrow F(X_0 \times \cdots \times X_n)$$

and

$$\beta^n = (F(\sigma_0), \dots, F(\sigma_n)) \colon F(X_0 \times \cdots \times X_n) \longrightarrow \bigoplus_{i=0}^{n} F(X_0 \times \cdots \times \widehat{X}_i \times \cdots \times X_n)$$

in opposite directions.

Lemma 1. *In the above setting, $F(X_0 \times \cdots \times X_n) = \operatorname{Im}(\alpha^n) \oplus \operatorname{Ker}(\beta^n)$.*

Proof. We argue by induction on n. For $\varnothing \subsetneq I \subseteq \{0, \dots, n\}$, define

$$\beta^I \colon F\Big(\prod_{i \in I} X_i\Big) \longrightarrow \bigoplus_{j \in I} F\Big(\prod_{i \in I \setminus \{j\}} X_i\Big)$$

to be the map induced by the canonical inclusions, by analogy with β^n. We are also going to prove that

$$F\Big(\prod_{i=0}^{n} X_i\Big) = F(\operatorname{Spec}(k)) \oplus \Big[\bigoplus_{\varnothing \subsetneq I \subseteq \{0,\dots,n\}} \operatorname{Ker}(\beta^I)\Big]$$

(where the groups in the right-hand side are regarded as subgroups of the group $F(X_0 \times \cdots \times X_n)$ through the morphisms induced by the canonical projections).

For $n = 0$, we have

$$\alpha^0 = F(\pi_0) \colon F(X_0) \longrightarrow F(\operatorname{Spec}(k))$$

and

$$\beta^0 = F(\sigma_0) \colon F(\operatorname{Spec}(k)) \longrightarrow F(X_0)\,.$$

Since $\pi_0 \circ \sigma_0 = \mathrm{id}_{\operatorname{Spec}(k)}$, we have that $\beta^0 \circ \alpha^0 = \mathrm{id}_{F(\operatorname{Spec}(k))}$ and we obtain a split short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Ker}(\beta^0) \arrow[r] & F(X_0) \arrow[r, "\beta^0"] & F(\operatorname{Spec}(k)) \arrow[r] \arrow[l, bend left, dashed, "\alpha^0"] & 0 \end{tikzcd}$$

which yields the decomposition $F(X_0) = \operatorname{Ker}(\beta^0) \oplus \operatorname{Im}(\alpha^0)$.

Now consider $n \geq 1$ and assume the statement holds for $n-1$. By the induction hypothesis, we can express

$$\beta^n \colon F\Big(\prod_{i=0}^{n} X_i\Big) \longrightarrow \bigoplus_{j=0}^{n} F\Big(\prod_{\substack{i=0 \\ i \neq j}}^{n} X_i\Big) = \bigoplus_{j=0}^{n} \Big\{F(\operatorname{Spec}(k)) \oplus \Big[\bigoplus_{\varnothing \subsetneq I \subseteq \{0,\dots,n\} \setminus \{j\}} \operatorname{Ker}(\beta^I)\Big]\Big\}$$

and this factors over the canonical morphism

$$\widetilde{\beta}^n \colon F\Big(\prod_{i=0}^{n} X_i\Big) \longrightarrow F(\operatorname{Spec}(k)) \oplus \Big[\bigoplus_{\varnothing \subsetneq I \subsetneq \{0,\dots,n\}} \operatorname{Ker}(\beta^I)\Big]$$

in the obvious way. In particular, $\operatorname{Ker}(\beta^n) = \operatorname{Ker}(\widetilde{\beta}^n)$. Moreover, the morphism $\widetilde{\beta}^n$ has a section induced by the canonical projections, from which we obtain a split short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker}(\beta^n) \arrow[r] & F\Big(\prod_{i=0}^{n} X_i\Big) \arrow[r, "\widetilde{\beta}^n"] & F(\operatorname{Spec}(k)) \oplus \Big[\bigoplus_{\varnothing \subsetneq I \subsetneq \{0,\dots,n\}} \operatorname{Ker}(\beta^I)\Big] \arrow[r] \arrow[l, dashed, bend left, "\widetilde{\alpha}^n"] & 0
\end{tikzcd}$$

We observe that, for every $\varnothing \subsetneq I \subsetneq \{0, \dots, n\}$, the projection

$$\prod_{i=0}^{n} X_i \longrightarrow \prod_{i \in I} X_i$$

factors over π_j for some $j \notin I$. As $\widetilde{\alpha}^n$ is defined in terms of these projections, we deduce that $\operatorname{Im}(\widetilde{\alpha}^n) \subseteq \operatorname{Im}(\alpha^n)$. Hence, $\operatorname{Im}(\alpha^n) + \operatorname{Ker}(\beta^n) = F(X_0 \times \cdots \times X_n)$ and it only remains to prove that $\operatorname{Im}(\alpha^n) \cap \operatorname{Ker}(\beta^n) = 0$. Indeed, we observe that, for each $i \in \{0, \dots, n\}$,

$$F(\sigma_i) \circ F(\pi_i) = F(\pi_i \circ \sigma_i) = \mathrm{id}_{F(X_0 \times \cdots \times \widehat{X_i} \times \cdots \times X_n)}$$

and so $\operatorname{Im}(F(\pi_i)) \cap \operatorname{Ker}(F(\sigma_i)) = 0$. By the definitions of α^n and of β^n, this implies that $\operatorname{Im}(\alpha^n) \cap \operatorname{Ker}(\beta^n) = 0$. □

In this situation, we say that F is *of order n* (or *linear* if $n = 1$, *quadratic* if $n = 2$, etc.) if β^n is injective or, equivalently, α^n is surjective. In particular, the theorem of the cube implies that the functor $\operatorname{Pic}\colon \mathcal{P}_k^+ \rightsquigarrow \mathrm{Ab}$ which gives the Picard group of a complete variety is quadratic. (The theorem is slightly stronger than this because we do not assume Z to be complete.)

Let us now see some easy consequences of the theorem of the cube.

Corollary 2. *Let X, Y and Z be complete varieties. Every element of* $\operatorname{Pic}(X \times Y \times Z)$ *is of the form*

$$p_{12}^* L_{12} \otimes p_{13}^* L_{13} \otimes p_{23}^* L_{23}$$

for some line bundles L_{12}, L_{13} and L_{23} over $X \times Y$, $X \times Z$ and $Y \times Z$, respectively, where p_{12}, p_{13} and p_{23} are the projections from $X \times Y \times Z$ to $X \times Y$, $X \times Z$ and $Y \times Z$.

Proof. Since $\operatorname{Pic}\colon \mathcal{P}_k^+ \rightsquigarrow \mathrm{Ab}$ is quadratic, the map

$$\alpha^2\colon \operatorname{Pic}(X \times Y) \times \operatorname{Pic}(X \times Z) \times \operatorname{Pic}(Y \times Z) \longrightarrow \operatorname{Pic}(X \times Y \times Z)$$

is surjective. □

Corollary 3. *Let Y be a variety and let X be an abelian variety. For every three morphisms $f, g, h\colon Y \to X$ and every line bundle L over X, the line bundle*

$$(f+g+h)^* L \otimes (f+g)^* L^{-1} \otimes (f+h)^* L^{-1} \otimes (g+h)^* L^{-1} \otimes f^* L \otimes g^* L \otimes h^* L$$

over Y is trivial.

Proof. It suffices to prove the result in the case when $Y = X$ and f, g and h are the three projections $p_1, p_2, p_3 \colon X \times X \times X \to X$; the general case then follows from taking the pull-back under $(f, g, h) \colon Y \to X \times X \times X$.

Consider the morphism $i \colon X \times X \to X \times X \times X$ given by $i(x_1, x_2) = (0, x_1, x_2)$. The pull-back under i of the line bundle which we want to be trivial is

$$(p_2 + p_3)^* L \otimes p_2^* L^{-1} \otimes p_3^* L^{-1} \otimes (p_2 + p_3)^* L^{-1} \otimes 0^* L \otimes p_2^* L \otimes p_3^* L$$

and this is trivial in $\operatorname{Pic}(\{0\} \times X \times X)$. Analogously, we obtain that the restrictions to $X \times \{0\} \times X$ and $X \times X \times \{0\}$ are also trivial. Therefore, we can apply the theorem of the cube to conclude. ☐

Corollary 4. *Let X be an abelian variety. For any integer n, write $n_X \colon X \to X$ for the morphism given by multiplication by n. For all line bundles L over X and all integers n,*

$$n_X^* L \cong L^{(n^2+n)/2} \otimes (-1)_X^* L^{(n^2-n)/2} .$$

Proof. We apply corollary 3 to n_X, 1_X and $(-1)_X$ and obtain that

$$(n+1)_X^* L \cong n_X^* L^2 \otimes (n-1)_X^* L^{-1} \otimes L \otimes (-1)_X^* L .$$

Since the required result is obvious for $n = 0$ and $n = 1$, we can prove it for $n \in \mathbb{N}$ by induction, using the following identities:

$$[(n+1)^2 + (n+1)]/2 = (n^2 + n) - [(n-1)^2 + (n-1)]/2 + 1 ,$$
$$[(n+1)^2 - (n+1)]/2 = (n^2 - n) - [(n-1)^2 - (n-1)]/2 + 1 .$$

A similar induction argument proves the result for the negative integers. ☐

Corollary 5 (theorem of the square). *Let X be an abelian variety. For all line bundles L over X and all closed points x and y of X,*

$$t_{x+y}^* L \otimes L \cong t_x^* L \otimes t_y^* L .$$

Consequently, for each line bundle L over X, the map

$$\begin{aligned} \phi_L \colon X(k) &\longrightarrow \operatorname{Pic}(X) \\ x &\longmapsto t_x^* L \otimes L^{-1} \end{aligned}$$

is a homomorphism of groups.

Proof. This is an application of corollary 3 to id_X and the two morphisms which are constant equal to x and y (described in terms of closed points). ☐

We now turn to the proof of the theorem of the cube. But, first, we need a couple of lemmata.

Lemma 6 (see-saw principle). *Let X be a complete variety and let T be any variety. Let L be a line bundle over $X \times T$ with the property that the restrictions $L|_{X\times\{t\}}$ are trivial for all closed points t of T. If there exists a closed point x_0 of X such that $L|_{\{x_0\}\times T}$ is trivial, then L is trivial.*

Proof. By the version of the see-saw principle which was proved in the previous talk, we know that $L \cong p_2^* M$ for some line bundle M over T, where $p_2 \colon X \times T \to T$ is the canonical projection. We consider the inclusion $i \colon T \to X \times T$ given in terms of closed points by $t \mapsto (x_0, t)$. Then,

$$L|_{\{x_0\}\times T} \cong i^* L \cong i^* p_2^* M = \mathrm{id}_T^* M = M$$

and so M is trivial by hypothesis. Therefore, L is also trivial. $\square$

Lemma 7. *Let X be a complete variety. For any two distinct closed points x_0 and x_1 of X, there exists a complete curve C on X (i.e., a subvariety of dimension 1) passing through both x_0 and x_1.*

Proof. If X itself is a curve, there is nothing to prove, so we assume that $\dim X > 1$. Since X is complete, Chow's lemma yields a surjective birational morphism from a projective variety X' to X. Thus, it suffices to prove the lemma in the case of a projective variety X. Moreover, by induction on $\dim X$, we only need to find a closed subvariety Y of codimension ≥ 1 in X containing both x_0 and x_1.

Let $\pi \colon \widetilde{X} \to X$ be the blow-up of X along $\{ x_0, x_1 \}$ (in particular, $\widetilde{X}$ is also a projective variety, π is a surjective birational morphism and $\pi^{-1}(x_0)$ and $\pi^{-1}(x_1)$ are effective Cartier divisors). We choose a closed immersion $\widetilde{X} \hookrightarrow \mathbb{P}_k^n$ with n minimal. By Bertini's theorem, for *almost all* hyperplanes H in $\mathbb{P}_k^n$, the intersection $H \cap \widetilde{X} = \widetilde{Y}$ is irreducible (hence, defines a variety): we pick one such H. Since $\dim \widetilde{X} > 1$ and $\pi^{-1}(x_i)$ is an effective Cartier divisor (for $i \in \{ 0, 1 \}$), we obtain that $\dim \pi^{-1}(x_i) \geq 1$, whence $\dim(H \cap \pi^{-1}(x_i)) \geq 0$ and so $H \cap \pi^{-1}(x_i) \neq \emptyset$. In conclusion, $Y = \pi(\widetilde{Y})$ is a proper closed subvariety of X which contains both x_0 and x_1. $\square$

Proof of the theorem of the cube. Since $L|_{X\times\{y_0\}\times Z}$ is trivial, by lemma 6 it suffices to prove that $L|_{\{x\}\times Y\times\{z\}}$ is trivial for all closed points (x, z) of $X \times Z$. Actually, it is enough to show this for a dense subset of $X \times Z$, as the locus

$$T_1 = \{ (x, z) \in X(k) \times Z(k) : L|_{\{x\}\times Y\times\{z\}} \text{ is trivial} \}$$

is closed.

Lemma 7 states that, for every closed point x of X, we can find a complete curve C passing through x_0 and x. Let $i \colon C \hookrightarrow X$ be the inclusion. Consider the normalization $\pi\colon \widetilde{C} \to C$ and define $\varphi = (i \circ \pi, \mathrm{id}_Y, \mathrm{id}_Z)\colon \widetilde{C} \times Y \times Z \to X \times Y \times Z$. It suffices to prove that $\varphi^* L$ is trivial. Thus, up to replacing X with $\widetilde{C}$, x_0 with a point of $\widetilde{C}$ lying over x_0 and L with $\varphi^* L$, we assume further that X is a complete normal curve, and so smooth too. In this situation, we are going to prove that there is a non-empty open (and so dense) subset Z' of Z such that $L|_{X\times Y\times Z'}$ is trivial.

Let Ω^1 be the canonical bundle of X and let $g = \dim H^0(X, \Omega^1)$. We can find g closed points $P_1, \dots, P_g$ of X such that, for the Weil divisor $D = P_1 + \cdots + P_g$, $\dim H^0(X, \Omega^1 \otimes \mathcal{O}_X(-D)) = 0$ (that is, $\Omega^1 \otimes \mathcal{O}_X(-D)$ is a non-trivial line bundle over X of degree 0 and so has no non-zero global sections). Let $p_1\colon X \times Y \times Z \to X$ be the canonical projection and consider $L' = L \otimes p_1^* \mathcal{O}_X(D)$. Since $L|_{X\times Y\times\{z_0\}}$ is trivial, $L'|_{X\times\{y\}\times\{z_0\}} \cong \mathcal{O}_X(D)$ for every closed point y of Y. In particular,

$$\dim H^1(X, L'|_{X\times\{y\}\times\{z_0\}}) = \dim H^0(X, \Omega^1 \otimes \mathcal{O}_X(-D)) = 0\,.$$

Therefore, the subset $F = \{\, (y,z) \in Y(k) \times Z(k) : \dim H^1(X, L'|_{X\times\{y\}\times\{z\}}) > 0 \,\}$ of $Y \times Z$, which is closed by upper-semicontinuity, is disjoint from $Y \times \{\, z_0 \,\}$. But, since Y is complete, the projection $p_2'\colon Y \times Z \to Z$ is closed, so $p_2'(F)$ is closed and we find an open neighbourhood $Z' = Z \setminus p_2'(F)$ of z_0 such that $(Y \times Z') \cap F = \varnothing$. After replacing Z with Z', we may assume that $\dim H^1(X, L'|_{X\times\{y\}\times\{z\}}) = 0$ for all closed points (y,z) of $Y \times Z$. Let $p_{23}\colon X \times Y \times Z \to Y \times Z$ be the canonical projection. By the semicontinuity theorem applied to the morphism p_{23} and the sheaf L', the Euler characteristic of $L'|_{X\times\{y\}\times\{z\}}$ is locally constant as a function of (y,z) and so, after restriction to a smaller open neighbourhood of z_0, we may assume that this Euler characteristic is constant as a function of z. Therefore, using the Riemann–Roch theorem, we obtain that

$$\begin{aligned}\dim H^0(X, L'|_{X\times\{y\}\times\{z\}}) &= \chi(L'|_{X\times\{y\}\times\{z\}}) = \chi(L'|_{X\times\{y\}\times\{z_0\}}) \\ &= \chi(\mathcal{O}_X(D)) = 1 - g + \deg(D) = 1\end{aligned}$$

for all closed points (y,z) of $Y \times Z$. In this situation, the theorem of cohomology and base from the last talk yields isomorphisms

$$(p_{23})_* L' \otimes_{\mathcal{O}_{Y\times Z}} \kappa(y,z) \to H^0(X, L'|_{X\times\{y\}\times\{z\}})$$

for all closed points (y,z) of $Y \times Z$.

Let U be any open subset of $Y \times Z$ over which $(p_{23})_* L'$ is trivial and take a generating section $\sigma_U \in \Gamma(U, (p_{23})_* L') = \Gamma(p_{23}^{-1}(U), L')$. We define $\widetilde{D}_U$ to be the divisor of zeros of σ_U in $p_{23}^{-1}(U)$. Since two such sections differ (wherever both are defined) by an invertible function (by reason of dimension), these divisors can be glued together to a well-defined effective divisor $\widetilde{D}$ on $X \times Y \times Z$. By definition, for each closed point (y, z) of $Y \times Z$, the restriction of $\widetilde{D}$ to $X \times \{ y \} \times \{ z \}$ is the divisor of zeros of a generating section of $L'|_{X \times \{y\} \times \{z\}}$. In particular, $\widetilde{D}$ restricted to either $X \times \{ y \} \times \{ z_0 \}$ for $y \in Y(k)$ or $X \times \{ y_0 \} \times \{ z \}$ for $z \in Z(k)$ must coincide with D, as the corresponding restrictions of L' are isomorphic to $\mathcal{O}_X(D)$ and all these divisors are effective. That is to say, for $P \in X(k) \setminus \{ P_1, \ldots, P_g \}$, the support S of $\widetilde{D}|_{\{P\} \times Y \times Z}$ intersects neither $\{ P \} \times Y \times \{ z_0 \}$ nor $\{ P \} \times \{ y_0 \} \times Z$. Hence, the projection of S on Z is a proper closed subset T of Z (it does not contain z_0) and, since S is of pure codimension 1 in $\{ P \} \times Y \times Z$, it must be of the form

$$S = \bigsqcup_{i=1}^{m} (\{ P \} \times Y \times T_i)$$

for some closed subsets T_i of codimension 1 in Z. But we have already seen that $S \cap (\{ P \} \times \{ y_0 \} \times Z) = \emptyset$, and this is only possible for $S = \emptyset$. This shows that the support of $\widetilde{D}$ does not intersect $\{ P \} \times Y \times Z$ for $P \neq P_i$, $1 \leq i \leq g$. All in all,

$$\widetilde{D} = \sum_{i=1}^{g} n_i(\{ P_i \} \times Y \times Z)$$

for some $n_i \in \mathbb{Z}$. Moreover, restricting $\widetilde{D}$ to $X \times \{ y_0 \} \times \{ z_0 \}$ (where it coincides with D), we see that $n_i = 1$ for each $i \in \{ 1, \ldots, g \}$. All in all,

$$\widetilde{D} = \sum_{i=1}^{g} (\{ P_i \} \times Y \times Z) .$$

This means that, for every closed point (y, z) of $Y \times Z$,

$$L|_{X \times \{y\} \times \{z\}} \otimes \mathcal{O}_X(D) \cong (L \otimes p_1^* \mathcal{O}_X(D))|_{X \times \{y\} \times \{z\}} = L'|_{X \times \{y\} \times \{z\}} \cong \mathcal{O}_X(D) ,$$

whence $L|_{X \times \{y\} \times \{z\}}$ must be trivial. Finally, lemma 6 implies that L is trivial. $\square$

References

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