The Brauer group

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Seminar talks on 19th December 2016 and 9th January 2017

Throughout this document, K always denotes a field. In addition, A and B will be K–algebras and D will be a division K–algebra.

In the last talk, the Brauer group $\operatorname{Br} K$ was introduced as the set of similarity classes of central-simple K–algebras. Moreover, as the name suggests, this set can be endowed with a group structure. Thus, the main objective of these two talks is to prove that $\operatorname{Br} K$ together with the tensor product is an abelian group.

Tensor products

Definition 1. The K–algebra $A \otimes_K B$ is the tensor product of A and B (as K–vector spaces) together with the product defined by

$$(a \otimes b)(a' \otimes b') = aa' \otimes bb' \quad \text{for all } a, a' \in A \text{ and } b, b' \in B\,.$$

We regard A and B as subalgebras of $A \otimes_K B$ through the natural maps $a \mapsto a \otimes 1$ and $b \mapsto 1 \otimes b$.

Since we want the tensor product to induce a well-defined operation on $\operatorname{Br} K$, we would like to prove that the tensor product of two central-simple K–algebras is also a central-simple K–algebra. To this aim, we first study some basic properties of the tensor product algebra and its centre and how these relate to the two components.

Definition 2. The K–algebra $A \otimes_K A^\circ$ is called the *enveloping algebra* of A. There is a natural morphism

$$\begin{aligned} A \otimes_K A^\circ &\longrightarrow \operatorname{End}_K(A) \\ a \otimes a' &\longmapsto a_A \circ {}_A a' \end{aligned}$$

(as left multiplication and right multiplication commute) and so A can be viewed as an $(A \otimes_K A^\circ)$–module with the scalar multiplication given by

$$(a \otimes a')x = axa' .$$

Proposition 3. *The $(A \otimes_K A^\circ)$–module A is simple if and only if the K–algebra A is simple.*

Proof. This is immediate from the fact that the $(A \otimes_K A^\circ)$–submodules of A are precisely its (two-sided) ideals as a ring. □

Proposition 4. $\mathrm{End}_{A \otimes_K A^\circ}(A) \cong \mathrm{Z}(A)$.

Proof. $\mathrm{End}_{A \otimes_K A^\circ}(A)$ is the subalgebra of $\mathrm{End}_{A^\circ}(A) \cong A$ consisting of those elements which commute with multiplication by elements of A. □

Definition 5. If B is a K–subalgebra of A, we define the *centralizer* of B in A to be

$$\mathrm{Z}_A(B) = \{\, a \in A : ab = ba \text{ for all } b \in B \,\} .$$

We call $\mathrm{Z}_A(\mathrm{Z}_A(B))$ the *bicentralizer* of B in A.

Proposition 6. $\mathrm{End}_{B \otimes_K A^\circ}(A) \cong \mathrm{Z}_A(B)$.

Proof. Analogous to the proof of proposition 4. □

Proposition 7. *If A' and B' are K–subalgebras of A and B, respectively,*

$$\mathrm{Z}_{A \otimes_K B}(A' \otimes_K B') = \mathrm{Z}_A(A') \otimes_K \mathrm{Z}_B(B')$$

as subalgebras of $A \otimes_K B$. In particular, $\mathrm{Z}(A \otimes_K B) \cong \mathrm{Z}(A) \otimes_K \mathrm{Z}(B)$.

Proof. It is clear that $\mathrm{Z}_A(A') \otimes_K \mathrm{Z}_B(B') \subseteq \mathrm{Z}_{A \otimes_K B}(A' \otimes_K B')$, so we only need to prove the converse.

Let $(a_i)_{i \in I}$ be a K–basis of A. Take $z \in \mathrm{Z}_{A \otimes_K B}(A' \otimes_K B')$ and express it as

$$z = \sum_{i \in I} a_i \otimes b_i$$

for some elements $b_i \in B$, all but finitely many of which are 0. For every $b' \in B'$, we have that $z(1 \otimes b') = (1 \otimes b')z$ by definition of z and so $b_i b' = b' b_i$ for all $i \in I$. This implies that $z \in A \otimes_K \mathrm{Z}_B(B')$. Analogously, $z \in \mathrm{Z}_A(A') \otimes_K B$. Now, if we use a K–basis $(a_i)_{i \in I}$ of A extending a K–basis of $\mathrm{Z}_A(A')$ in the previous argument, we see that the terms $a_i \otimes b_i$ with $b_i \neq 0$ must belong to $\mathrm{Z}_A(A') \otimes_K \mathrm{Z}_B(B')$. □

Lemma 8. *If A is simple and central over K, every (two-sided) ideal T of $A \otimes_K B$ is of the form $A \otimes_K I$ for the ideal $I = T \cap B$ of B.*

Proof. Since, $I \subseteq T$, it is clear that $A \otimes_K I \subseteq T$. We need to prove the other inclusion.

Fix a K–basis $(x_i)_{i \in I}$ of A. Every $t \in T$ can be expressed in the form

$$t = \sum_{j=1}^{n} x_{i_j} \otimes y_j \,.$$

On the other hand,

$$\mathrm{End}_{A \otimes_K A^\circ}(A) \cong \mathrm{Z}(A) \cong K$$

and, since A is a simple (and so semisimple too) $(A \otimes_K A^\circ)$–module, Jacobson's density theorem states that for every $f \in \mathrm{End}_{\mathrm{End}_{A \otimes_K A^\circ}(A)}(A) = \mathrm{End}_K(A)$ there exists $a \in A \otimes_K A^\circ$ such that $ax_{i_j} = f(x_{i_j})$. In particular, for each $j \in \{1, \ldots, n\}$, we can apply this to the endomorphism f_j defined by $f_j(x_{i_j}) = 1$ and $f_j(x_l) = 0$ for $l \neq i_j$ and obtain thus the corresponding $a_j \in A \otimes_K A^\circ$. Since T is a two-sided ideal and multiplying by a_j corresponds to taking a sum of terms obtained after multiplying by elements of A (both to the left and to the right),

$$\sum_{l=1}^{n} a_j x_{i_l} \otimes y_l = 1 \otimes y_j \in T \,.$$

Then $y_j = 1 \otimes y_j \in T \cap B = I$ for each $j \in \{1, \ldots, n\}$. Therefore, $t \in A \otimes_K I$. □

Theorem 9. *If A and B are simple and one of them is central over K, $A \otimes_K B$ is simple.*

Proof. Suppose that A is central over K. Since the only ideals of B are (0) and B, the ideals of $A \otimes_K B$ are $A \otimes_K (0) = (0)$ and $A \otimes_K B$ (by the previous lemma). □

Proposition 10. *If $A \otimes_K B$ is simple (resp. artinian), so are A and B.*

Proof. It is obvious from the fact that A and B are K–subalgebras of $A \otimes_K B$. □

Lemma 11. *If $A : K = n < \infty$ and B is artinian, $A \otimes_K B$ is artinian.*

Proof. We have that $A \otimes_K B \cong K^n \otimes_K B \cong B^n$ as B–modules and B^n is artinian because B is. □

Corollary 12. *If both A and B are simple and artinian, at least one of them is finite-dimensional over K and at least one of them is central over K, then $A \otimes_K B$ is also simple and artinian.*

Proof. This follows from theorem 9 and lemma 11. □

We are finally in a position to prove the previously hinted theorem (which is now just a corollary of the results we have seen.)

Theorem 13. *If A and B are central-simple, so is $A \otimes_K B$.*

Proof. $A \otimes_K B$ is simple by theorem 9. Moreover,

$$\mathrm{Z}(A \otimes_K B) \cong \mathrm{Z}(A) \otimes_K \mathrm{Z}(B) \cong K \otimes_K K \cong K$$

and $(A \otimes_K B) : K = (A : K)(B : K) < \infty$. □

The Brauer group

The associativity and commutativity (up to isomorphism) of the tensor product of K–algebras are clear from the same properties of the tensor product of K–vector spaces. Next we see what the inverses in the Brauer group are going to be.

Proposition 14. *If A is central-simple, the natural map*

$$\phi \colon A \otimes_K A^\circ \longrightarrow \mathrm{End}_K(A)$$

is an isomorphism of K–algebras. Consequently, if $A : K = n$,

$$A \otimes_K A^\circ \cong \mathrm{M}_n(K)\,.$$

Proof. $A \otimes_K A^\circ$ is simple because it is the tensor product of two central-simple algebras. Hence, by Schur's lemma, ϕ is injective. But looking at the dimensions as K–vector spaces we see that $(A \otimes_K A^\circ) : K = n^2 = \mathrm{End}_K(A) : K$, so that ϕ is surjective as well. □

With this, we are almost ready to prove that $\mathrm{Br}\, K$ admits an abelian group structure induced by the tensor product of K–algebras. Using Wedderburn's theorem, we are going to reduce ourselves to the case of matrix algebras, where the computations can be made explicit.

Lemma 15. *For every natural number n, $A \otimes_K \mathrm{M}_n(K) \cong \mathrm{M}_n(A)$.*

Proof. Let $(E_{ij})_{1 \le i,j \le n}$ be the canonical K–basis of $\mathrm{M}_n(K)$. Then $(1 \otimes E_{ij})_{1 \le i,j \le n}$ is clearly an A–basis of $A \otimes_K \mathrm{M}_n(K)$ satisfying that

$$(1 \otimes E_{ij})(1 \otimes E_{kl}) = \begin{cases} 1 \otimes E_{il} & \text{if } j = k\,, \\ 0 & \text{otherwise.} \end{cases}$$

These relations define the product in $\mathrm{M}_n(A)$ in terms of its canonical A–basis. □

Theorem 16. *The set $\mathrm{Br}\, K$ together with the operation induced by the tensor product of K–algebras is an abelian group.*

Proof. First, we have to prove that the group operation is well-defined. We already know that the tensor product of two central-simple K–algebras is a central-simple K–algebra (by theorem 13), so we only need to prove that this tensor product is compatible with the similarity equivalence relation. Indeed, if $A \cong \mathrm{M}_p(D_1)$ and $B \cong \mathrm{M}_q(D_2)$ for some central-simple division K–algebras D_1 and D_2, we can use the previous lemma to compute

$$\begin{aligned} A \otimes_K B &\cong D_1 \otimes_K \mathrm{M}_p(K) \otimes_K \mathrm{M}_q(K) \otimes_K D_2 \cong D_1 \otimes_K \mathrm{M}_p(\mathrm{M}_q(K)) \otimes_K D_2 \\ &\cong D_1 \otimes_K D_2 \otimes_K \mathrm{M}_{pq}(K) \cong \mathrm{M}_{pq}(D_1 \otimes_K D_2) \end{aligned}$$

so that the similarity class of $A \otimes_K B$ depends only on the similarity classes of A and B (i.e., on the isomorphism classes of D_1 and D_2). Here, we have already been using that the tensor product of K–algebras is associative and commutative, as mentioned above.

Thus, the tensor product operation on $\operatorname{Br} K$ is well-defined, associative and commutative. Moreover, $A \otimes_K K \cong A$ and so $[K]$ is the neutral element. Finally, proposition 14 implies that the inverse of $[A]$ is $[A^\circ]$. $\square$

Example 17. In some cases, the structure of the Brauer group is very simple.

(i) If K is algebraically closed, $\operatorname{Br} K$ is trivial because every finite-dimensional simple K–algebra is isomorphic to $\mathrm{M}_n(K)$ for some n.

(ii) If K is finite, $\operatorname{Br} K$ is trivial because every finite division K–algebra is a field.

(iii) If K is real-closed (i.e., K is not algebraically closed but $K(\sqrt{-1})$ is), $\operatorname{Br} K$ is cyclic of order 2.

Base change and splitting fields

To conclude this part of the seminar, we study how the Brauer group behaves with respect to field extensions. To this aim, we need to introduce a way to extend a K–algebra to a larger field and study some properties of this construction.

In what follows, let L be an extension field of K.

Definition 18. The L–algebra

$$A_L = A \otimes_K L$$

is said to arise from A by *base change* to L.

Proposition 19. *Base change satisfies the following properties.*

(i) $A_L : L = A : K$.

(ii) *If A is central over K, A_L is central over L.*

(iii) *If A is simple and central over K, A_L is simple and central over L.*

(iv) *If A is a central-simple K–algebra, A_L is a central-simple L–algebra.*

Proof. A K–basis $(x_i)_{i\in I}$ of A is also an L–basis of A_L by the definition of A_L. This proves (i). For (ii), if A is central over K, $\mathrm{Z}(A_L) \cong \mathrm{Z}(A) \otimes_K \mathrm{Z}(L) \cong K \otimes_K L \cong L$. Now (iii) follows from theorem 9. Finally, (iv) is a consequence of the other three properties. $\square$

Corollary 20. *If A is central-simple, $A : K$ is a perfect square.*

Proof. Consider an algebraic closure $\overline{K}$ of K. $A : K = A_{\overline{K}} : \overline{K}$ and this last number is a square because every finite-dimensional simple $\overline{K}$–algebra is isomorphic to $\mathrm{M}_n(\overline{K})$ for some n. $\square$

Corollary 21. *If the division K–algebra D is central over K, $D : K$ is either infinite or a perfect square.*

Proof. Since D is a division algebra, it is obviously simple. Therefore, $D : K < \infty$ implies that D is central-simple and the previous corollary applies. $\square$

Lemma 22. *Base change to L induces a group homomorphism* $\operatorname{Br} K \to \operatorname{Br} L$.

Proof. If $[A] \in \operatorname{Br} K$, then $A \cong \mathrm{M}_r(D)$ for some central-simple division K–algebra D. In this situation,

$$A_L \cong D \otimes_K \mathrm{M}_r(K) \otimes_K L \cong D \otimes_K \mathrm{M}_r(L) \cong D \otimes_K L \otimes_L \mathrm{M}_r(L) \cong \mathrm{M}_r(D \otimes_K L)\,,$$

which implies that $[A_L]$ depends only on the isomorphism class of D. Therefore, the map $[A] \mapsto [A_L]$ is well-defined. Moreover, this is a morphism. Indeed, if $[A], [B] \in \operatorname{Br} K$,

$$(A \otimes_K B)_L \cong A \otimes_K L \otimes_K B \cong A \otimes_K L \otimes_L L \otimes_K B \cong A_L \otimes_L B_L\,. \qquad \square$$

Definition 23. The *restriction map* of the Brauer group with respect to $L\,/\,K$ is the group homomorphism

$$\begin{aligned} \operatorname{res}_{L/K}\colon \operatorname{Br} K &\longrightarrow \operatorname{Br} L \\ [A] &\longmapsto [A_L] \end{aligned}$$

and its kernel is

$$\operatorname{Br}(L\,/\,K) = \{\, [A] \in \operatorname{Br} K : A_L \sim L \,\}\,.$$

If $[A] \in \operatorname{Br}(L\,/\,K)$, we say that L is a *splitting field* of A (or of $[A]$) or that A *splits over L*.

Corollary 24. *Suppose that A is central-simple. L is a splitting field of A if and only if $A \otimes_K L \cong \mathrm{M}_n(L)$ with $n^2 = A : K$.*

The name *restriction* is due to the following property, which is basically the functoriality of the Brauer group construction.

Proposition 25. *If F is an intermediate field of L / K,*

$$\mathrm{res}_{L/F} \circ \mathrm{res}_{F/K} = \mathrm{res}_{L/K} \, .$$

Proof. It is immediate that $A \otimes_K F \otimes_F L \cong A \otimes_K L$. □

Definition 26. If D is central-simple, we can define the *Schur index* of D to be the natural number $s = \sqrt{D : K}$. More generally, if A is central-simple and $A \sim D$, we say that s is the *Schur index* of A or even of $[A]$ and write $s = s(A) = s([A])$. In this situation, we also define the *reduced degree* of A to be the natural number $n = \sqrt{A : K}$.

From now on, suppose that A is central-simple.

Proposition 27. *A splits over L if and only if $s(A_L) = 1$.*

Proof. By the definition of the Schur index, $s(A_L) = 1$ is equivalent to $A_L \sim L$. □

Proposition 28. *Let r, s and n denote the length, the Schur index and the reduced degree of A, respectively. Then*

$$n = rs \, .$$

In particular, $A : K = s^2$ if and only if A is a division algebra.

Proof. Wedderburn's theorem yields an isomorphism of K–algebras $A \cong \mathrm{M}_r(D)$. Therefore, $n^2 = A : K = (A : D)(D : K) = r^2 s^2$. □

Corollary 29. *$s(A_L)$ divides $s(A)$.*

Proof. As the Schur index depends only on the similarity class, it suffices to prove this result in the case that $A = D$ is a division algebra. Now, by the previous proposition, $s(D_L)$ divides $\sqrt{D_L : L} = \sqrt{D : K} = s(D)$. □

Thus, in some sense, the Schur index measures how far we are from a splitting field. These facts are going to be discussed in more detail in the following talk.

References

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