ABELIAN VARIETIES
Definition and first properties

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Throughout this document, let k be a field. By an algebraic variety over k, we mean a geometrically integral and separated scheme of finite type over k. (Note that, in Milne's notes [1], an algebraic variety is not assumed to be geometrically irreducible but only geometrically reduced.)

The main objective of this talk is to define abelian varieties. An abelian variety is roughly a complete algebraic variety endowed with a group structure on its sets of points. We are also going to prove that the existence of such a group structure forces the underlying variety to be smooth and that the group law must be commutative (i.e., abelian).

Group varieties

If X is an algebraic variety over k, a group structure on X must give a group law on the set $X(R)$ of R–valued points of X for every k–algebra R. Moreover, the assignment of such a group law must be functorial in the following sense: given a morphism $R \to R'$ of k–algebras, the corresponding map $X(R) \to X(R')$ must be a group homomorphism.

But a group structure on a set is defined by a multiplication map, an inverse map and an identity element satisfying certain conditions. Thus, by the Yoneda lemma, defining such functorial group structures on the sets $X(R)$ is equivalent to giving a multiplication map, an inverse map and an identity element on X. (In fact, since we are dealing with varieties, this is also equivalent to having a group structure on $X(\bar{k})$ for an algebraic closure $\bar{k}$ of k.)

Definition 1. A *group variety* over k is an algebraic variety X over k together with two k–morphisms $m\colon X \times_k X \to X$ (multiplication) and $i\colon X \to X$ (inversion) and a k–rational point $e \in X(k)$ (identity element) such that the following diagrams are commutative:

(i) (associativity)

$$\begin{array}{ccc} X \times_k X \times_k X & \xrightarrow{m \times \mathrm{id}_X} & X \times_k X \\ {\scriptstyle \mathrm{id}_X \times m}\downarrow & & \downarrow{\scriptstyle m} \\ X \times_k X & \xrightarrow{\quad m \quad} & X \end{array} \quad ;$$

(ii) (identity element)

$$\begin{array}{ccc} X \times_k \mathrm{Spec}(k) & \xrightarrow{\mathrm{id}_X \times e} & X \times_k X \\ \| & & \downarrow{\scriptstyle m} \\ X & \xrightarrow{\quad \mathrm{id}_X \quad} & X \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{Spec}(k) \times_k X & \xrightarrow{e \times \mathrm{id}_X} & X \times_k X \\ \| & & \downarrow{\scriptstyle m} \\ X & \xrightarrow{\quad \mathrm{id}_X \quad} & X \end{array}$$

(where we identify $X \times_k \mathrm{Spec}(k)$ and $\mathrm{Spec}(k) \times_k X$ with X through the canonical isomorphisms given by the projections), and

(iii) (inverses)

$$\begin{array}{ccc} X & \xrightarrow{(\mathrm{id}_X, i)} & X \times_k X \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle m} \\ \mathrm{Spec}(k) & \xrightarrow{\quad e \quad} & X \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \xrightarrow{(i, \mathrm{id}_X)} & X \times_k X \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle m} \\ \mathrm{Spec}(k) & \xrightarrow{\quad e \quad} & X \end{array}$$

(where $\pi\colon X \to \mathrm{Spec}(k)$ is the structure morphism).

This definition is easily generalized to schemes over some other scheme S, obtaining what are known as *group schemes* (over S).

Definition 2. Let (X, m_X, i_X, e_X) and (Y, m_Y, i_Y, e_Y) be two group varieties over k. A morphism of k–varieties $f\colon X \to Y$ is called a *homomorphism of group varieties* if

$$f \circ m_X = m_Y \circ (f \times f)\,,$$

in which case $f \circ e_X = e_Y$ and $f \circ i_X = i_Y \circ f$ as well.

Definition 3. Let X be a group variety over k. For every k–rational point $x \in X(k)$, we define the *right translation* $t_x\colon X \to X$ to be the composition

$$t_x\colon X \cong X \times_k \mathrm{Spec}(k) \xrightarrow{\mathrm{id}_X \times x} X \times_k X \xrightarrow{\quad m \quad} X\,.$$

Similarly, the *left translation* $t'_x\colon X \to X$ is

$$t'_x\colon X \cong \mathrm{Spec}(k) \times_k X \xrightarrow{x \times \mathrm{id}_X} X \times_k X \xrightarrow{\quad m \quad} X\,.$$

More generally, let T be a k–scheme and let $x\colon T \to X$ be a T–valued point of X. We define the *right translation* $t_x\colon X_T \to X_T$ and the *left translation* $t'_x\colon X_T \to X_T$ (where $X_T = X \times_k T$) as follows:

$$t_x\colon X_T \cong X_T \times_T T \xrightarrow{\mathrm{id}_{X_T} \times x_T} X_T \times_T X_T \cong X \times_k X \times_k T \xrightarrow{m \times \mathrm{id}_T} X \times_k T = X_T\,,$$
$$t'_x\colon X_T \cong T \times_T X_T \xrightarrow{x_T \times \mathrm{id}_{X_T}} X_T \times_T X_T \cong X \times_k X \times_k T \xrightarrow{m \times \mathrm{id}_T} X \times_k T = X_T\,,$$

where $x_T = (x, \mathrm{id}_T)\colon T \to X \times_k T = X_T$.

One can check that, for $x, y \in X(T)$, $t_y \circ t_x = t_{m\circ(x,y)}$. Also, if $e_T\colon T \to X$ is the composition of the structural morphism $T \to \mathrm{Spec}(k)$ and e, $t_{e_T} = \mathrm{id}_{X_T}$. Therefore, $t_{i\circ x} = t_x^{-1}$. The analogous identities for the left translations also hold.

In particular, for any $x, y \in X(\bar{k})$, there is a translation sending x to y (the translation defined by $m \circ (i \circ x, y)$) and this translation is an automorphism of X. That is, X "looks everywhere the same". This fact allows us to extend a property which holds locally at a point to all points of our variety.

Proposition 4. *Every group variety X over k is smooth over k. Furthermore, if $T_{X,e}$ is the tangent space of X at e, there are natural isomorphisms $\mathcal{T}_{X/k} \cong T_{X,e} \otimes_k \mathcal{O}_X$ and $\Omega^n_{X/k} \cong (\bigwedge^n T^\vee_{X,e}) \otimes_k \mathcal{O}_X$. In particular, if $\dim X = g$, then $\Omega^g_{X/k} \cong \mathcal{O}_X$.*

Sketch of the proof. Since X is a variety, the smooth locus X_{sm} of X over k is open and dense. But the translates of X_{sm} cover X and the smooth locus is stable under translation, so $X_{\mathrm{sm}} = X$.

For the remainder of the proposition, we observe that tangent spaces, tangent sheaves and sheaves of differentials behave well under base change. Therefore, we may assume that k is algebraically closed (up to replacing k with an algebraic closure $\bar{k}$ and X with the corresponding pull-back).

Given a tangent vector $\tau\colon\ \mathrm{Spec}(k[\varepsilon]\ /\ (\varepsilon^2)) \to X$ at e, composing τ with the translation t_x yields a tangent vector at x for all $x \in X(k)$ and these tangent vectors can be glued to a section of the tangent sheaf $\mathcal{T}_{X/k} = \mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$. Thus, we obtain a k–linear map $T_{X,e} \to \Gamma(X, \mathcal{T}_{X/k})$ which in turn induces a morphism of $\mathcal{O}_X$–modules $\alpha\colon T_{X,e} \otimes_k \mathcal{O}_X \to \mathcal{T}_{X/k}$.

We want to prove that α is an isomorphism. Indeed, since both $T_{X,e} \otimes_k \mathcal{O}_X$ and $\mathcal{T}_{X/k}$ are locally free $\mathcal{O}_X$–modules of the same rank (because X is a non-singular variety), it suffices to prove that α is surjective. But, for every $x \in X(k)$, the map

$$(\alpha_x \mod \mathfrak{m}_x)\colon T_{X,e} = T_{X,e} \otimes_k k \longrightarrow (\mathcal{T}_{X/k})_x \otimes_{\mathcal{O}_{X,x}} k = T_{X,x}$$

is the isomorphism given by t_x. Hence, by Nakayama's lemma, the maps on stalks $\alpha_x\colon T_{X,e} \otimes_k \mathcal{O}_{X,x} \to (\mathcal{T}_{X/k})_x = T_{X,x}$ are surjective for all closed points $x \in X$ and so α is surjective.

Now the other isomorphism follows immediately from the duality relation $\mathcal{T}_{X/k} = \mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_{X/k}, \mathcal{O}_X)$ and from the identity $\Omega^n_{X/k} = \bigwedge^n \Omega^1_{X/k}$. □

With this, we can already move on to our object of study: abelian varieties.

Abelian varieties

Definition 5. An *abelian variety* over k is a complete group variety over k. (Recall that an algebraic variety X over k is called complete if the structure morphism $X \to \mathrm{Spec}(k)$ is proper.)

The name of abelian varieties comes from the fact that their group structure must be commutative (i.e., they are equipped with abelian group structures), as we shall see next.

Lemma 6 (rigidity theorem). *Let X, Y and Z be three algebraic varieties over k and assume that X is complete. Let $f\colon X \times Y \to Z$ be a morphism of k–varieties. If there exist k–rational points $y_0 \in Y(k)$ and $z_0 \in Z(k)$ such that $f(X \times \{ y_0 \}) = \{ z_0 \}$, then f factors through the projection $\pi_Y\colon X \times Y \to Y$.*

Proof. Since all the hypotheses continue to hold after base change to $\overline{k}$, we may assume that k is algebraically closed. We choose a point $x_0 \in X(k)$ and define $g\colon Y \to Z$ given by $g(y) = f(x_0, y)$. We want to prove that $f = g \circ \pi_Y$. Since $X \times Y$ is reduced and all our morphisms are separated, it suffices to prove that these two morphisms coincide on closed points.

Let $U \subseteq Z$ be an open affine neighbourhood of z_0. Since X is complete, π_Y is closed and so $T = \pi_Y(f^{-1}(Z \setminus U))$ is a closed subset of Y. Also, by definition, T consists of the second coordinates of the points of $X \times Y$ which are mapped outside of U under f. Therefore, a point y of Y lies outside of T if and only if $f(X \times \{ y \}) \subseteq U$. But, as $X \times \{ y \}$ is complete and U is affine, $f(X \times \{ y \})$ must be a single point. That is, $f(x, y) = f(x_0, y) = g(y)$ for all x, which means that f and $g \circ \pi_Y$ coincide in the non-empty open set $X \times (Y \setminus T)$. In fact, f and $g \circ \pi_Y$ coincide everywhere because $X \times Y$ is a variety. □

Proposition 7. *Every morphism of k–varieties $f\colon X \to Y$ between abelian varieties over k is the composition of a homomorphism of abelian varieties (i.e., of group varieties) and a translation. More precisely, $f = t_{f(e_X)} \circ h$, where $h\colon X \to Y$ is a homomorphism of abelian varieties.*

Proof. Let $y = i_Y(f(e_X))$ and define $h = t_y \circ f$, so that $h(e_X) = e_Y$. Consider the two morphisms

$$g_1 \colon X \times X \xrightarrow{m_X} X \xrightarrow{h} Y,$$
$$g_2 \colon X \times X \xrightarrow{h \times h} Y \times Y \xrightarrow{m_Y} Y;$$

these two morphisms should agree, as we want to prove that h is a homomorphism. Thus, we define another morphism

$$g \colon X \times X \xrightarrow{(g_1, i_Y \circ g_2)} Y \times Y \xrightarrow{m_Y} Y$$

(which is given on k–valued points by $g(x, x') = h(x + x') - (h(x) + h(x'))$, using additive notation for the group law). But we observe that

$$g(\{e_X\} \times X) = \{e_Y\} = g(X \times \{e_X\}).$$

Therefore, by the rigidity lemma, g factors both through the first and through the two projections $X \times X \to X$. That is to say, g is constant equal to e_Y. From this, we conclude that h is a homomorphism of abelian varieties. □

Corollary 8. *The group law on an abelian variety is commutative. That is, every abelian variety (X, m, i, e) satisfies that $m \circ s = m$, where $s \colon X \times X \to X \times X$ is the morphism which swaps the two coordinates of $X \times X$.*

Proof. Abelian groups are precisely those groups for which the map sending an element to its inverse is a group homomorphism. And the previous proposition implies that $i \colon X \to X$ is a homomorphism of abelian varieties, as it sends the identity element to the identity element. □

Now that we know that abelian varieties are indeed abelian as group varieties, it is justified to use additive notation for the group law on an abelian variety. Also, there is no need to distinguish between right and left translations on an abelian variety.

The group law on abelian varieties defines in the obvious way (pointwise) a group law on homomorphisms of abelian varieties. That is to say, for any two abelian varieties X and Y, the set $\mathrm{Hom}_{(\mathrm{AV})}(X, Y)$ of homomorphisms of abelian varieties from X to Y has a natural structure of abelian group. By proposition 7, $\mathrm{Hom}_{(\mathrm{AV})}(X, Y)$ is the subgroup of $\mathrm{Hom}_{(\mathrm{Sch}\,/k)}(X, Y) = Y(X)$ consisting of the morphisms $f \colon X \to Y$ such that $f(0) = 0$.

Another interesting consequence of the rigidity of abelian varieties is that the group structure of an abelian variety is uniquely determined by its identity element, in the same way as for elliptic curves.

Proposition 9. *Let X be a complete algebraic variety over k and let $e \in X(k)$. There is at most one structure of abelian variety on X for which e is the identity element.*

Proof. Suppose that (X, m, i, e) and (X, n, j, e) are two abelian varieties with the same underlying algebraic variety X and the same identity element e. Since both m and n are equal when restricted to $X \times \{ e \}$ and to $\{ e \} \times X$, the morphism

$$g \colon X \times X \xrightarrow{(m, i \circ n)} X \times X \xrightarrow{m} X$$

is constant equal to e when restricted to $X \times \{ e \}$ and to $\{ e \} \times X$. Now the rigidity lemma implies that $m = n$. From this, we obtain that $i = j$ as well. □

References

[1] Milne, J. S. *Abelian varieties (v2.00)*. Course notes. 2008. Chap. I.1, pp. 7–10. URL: http://www.jmilne.org/math/CourseNotes/av.html (visited on 13/04/2017).

[2] Van der Geer, G. and Moonen, B. *Abelian varieties*. Preliminary chapters for a book. 2014. Chap. I, pp. 5–16. URL: https://www.math.ru.nl/~bmoonen/BookAV/DefBasEx.pdf (visited on 13/04/2017).